Mock IMO Day 1

MOP 2024

Saturday, June 8, 2024

Time limit: 4.5 hours. Insert funny caption here.

1. Let n be a positive integer. Suppose that $a_1, \ldots, a_n, b_1, \ldots, b_n$ are positive integers such that the $n+1$ products

$$a_1a_2a_3\cdots a_n,\ b_1a_2a_3\cdots a_n,\ b_1b_2a_3\cdots a_n,\ \ldots,\ b_1b_2b_3\cdots b_n$$

form a strictly increasing arithmetic progression in that order. Determine the smallest possible value of its common difference.

2. Let n be a fixed positive integer. Determine the largest possible length L of a sequence $a_1, \ldots, a_L$ of positive integers such that
   - every term of the sequence is at most n, and
   - for any integers i and j with $1 \le i \le j \le L$ and integers $\varepsilon_i, \ldots, \varepsilon_j \in \{\pm 1\}$, we have

$$\varepsilon_i a_i + \cdots + \varepsilon_j a_j \neq 0.$$

3. Let N be a positive integer. Prove that there exist three permutations $a_1, \ldots, a_N$, $b_1, \ldots, b_N$, and $c_1, \ldots, c_N$ of $1, \ldots, N$ such that

$$\left|\sqrt{a_k} + \sqrt{b_k} + \sqrt{c_k} - 2\sqrt{N}\right| < 2023$$

for all k.

Mock IMO Day 2

MOP 2024

Saturday, June 15, 2024

Time limit: 4.5 hours. You all think Resnik will have a more diverse playlist than Schatz?

4. Let $ABCDE$ be a convex pentagon such that $\angle ABC = \angle AED = 90^\circ$. Suppose that the midpoint of $\overline{CD}$ is the circumcenter of $\triangle ABE$. Let O be the circumcenter of $\triangle ACD$. Prove that line AO bisects $\overline{BE}$.

5. Let $a_1 < a_2 < \cdots$ be positive integers such that a_{k+1} divides $2(a_1 + \cdots + a_k)$ for all k. Suppose that there are infinitely many prime numbers which divide some term of the sequence. Prove that every positive integer divides some term of the sequence.

6. Elisa has 2023 treasure chests, all of which are unlocked and empty at first. Each day, Elisa adds a gem to one unlocked chest of her choice. Afterwards, a fairy does exactly one of the following actions:

   - If at least two chests are unlocked, it locks one of the unlocked chests.
   - If only one chest is unlocked, it unlocks all of the locked chests.

   This process goes on forever. Find the smallest constant C such that Elisa can ensure the difference between the numbers of gems in any two chests never exceeds C, regardless of how the fairy acts.

Mock IMO Day 3

MOP 2024

Whenever

Time limit: ∞. Only if you want to see some leftover shortlist problems.

7. Let ABC be an acute scalene triangle with circumcircle ω. Circle Ω is internally tangent to ω at A and is also tangent to $\overline{BC}$ at D. Lines AB and AC meet Ω again at P and Q. Let M and N be points on line BC such that B is the midpoint of $\overline{DM}$ and C is the midpoint of $\overline{DN}$. Lines MP and NQ meet at K, and intersect Ω again at I and J. Ray KA meets the circumcircle of $\triangle IJK$ at $X \neq K$. Prove that $\angle BXP = \angle CXQ$.