Mock IMO Day 1

MOP 2023

Saturday, June 10, 2023

Time limit: 4.5 hours. We hope you're looking forward to your first free day tomorrow!

1. Let $k \geq 2$ be an integer. A nonempty set S of real numbers has the property that every element $s \in S$ can be written as the sum of k distinct elements of S that are not equal to s. Find the smallest possible value of $|S|$, in terms of k.

2. Find all rational numbers q for which there exists a function $f\colon \mathbb{R} \to \mathbb{R}$ satisfying

$$f(x + f(y)) = f(x) + f(y) \quad \text{and} \quad f(z) \neq qz$$

for all real numbers x, y, z.

3. Let $AA'BCC'B'$ be a convex cyclic hexagon such that line AC is tangent to the incircle of $\triangle A'B'C'$ and line $A'C'$ is tangent to the incircle of $\triangle ABC$. Let lines AB and $A'B'$ intersect at X and lines BC and $B'C'$ intersect at Y. Prove that if $XBYB'$ is a convex quadrilateral, then it has an incircle.

Mock IMO Day 2

MOP 2023

Sunday, June 18, 2023

Time limit: 4.5 hours. Remember to wish happy birthday to Po!

4. Point P lies in the interior of acute triangle ABC such that lines AP and BC are perpendicular. Points D and E on side BC satisfy $PD \parallel AC$ and $PE \parallel AB$, and points $X \neq A$ and $Y \neq A$ lie on the circumcircles of $\triangle ABD$ and $\triangle ACE$, respectively, such that $DA = DX$ and $EA = EY$. Prove that points B, C, X, and Y are concyclic.

5. For each $1 \leq i \leq 9$ and positive integer T, let $d_i(T)$ denote the total number of times the digit i appears when all multiples of 2023 between 1 and T inclusive are written out in base 10. Prove that there are infinitely many positive integers T such that there are exactly two distinct values among $d_1(T)$, $d_2(T)$, …, $d_9(T)$.

6. Let s be a positive integer. Lucy and Lucky play the following game on a blackboard. Lucy initially writes s integer-valued 2023-tuples on the board. Lucky then gives Lucy an integer-valued 2023-tuple. Afterwards, Lucy can repeatedly take any two (not necessarily distinct) tuples $(v_1, \ldots, v_{2023})$ and $(w_1, \ldots, w_{2023})$ on the blackboard and write the tuples

$$(v_1 + w_1, \ldots, v_{2023} + w_{2023}) \quad \text{and} \quad (\max(v_1, w_1), \ldots, \max(v_{2023}, w_{2023}))$$

on the board. Lucy wins if she can write Lucky's tuple on the board in a finite number of steps.

Determine the smallest value of s for which Lucy has a winning strategy.

Mock IMO Day 3

MOP 2023

Whenever

Time limit: ∞. Only if you want to see some leftover shortlist problems.

7. Let ABC be a triangle, and let ℓ_1 and ℓ_2 be parallel lines. For $i \in \{1, 2\}$, let ℓ_i meet lines BC, CA, and AB at X_i, Y_i, and Z_i respectively. Suppose that the line through X_i perpendicular to $\overline{BC}$, the line through Y_i perpendicular to $\overline{CA}$, and the line through Z_i perpendicular to $\overline{AB}$ determine a non-degenerate triangle Δ_i. Prove that the circumcircles of Δ_1 and Δ_2 are tangent to each other.

8. Let n be a positive integer and $X_1, \ldots, X_m$ be distinct nonempty subsets of $\{1, \ldots, n\}$. Prove that there are at least n^n functions $f\colon \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n+1\}$ such that there exists an index k satisfying
$$\sum_{x \in X_k} f(x) > \sum_{x \in X_i} f(x)$$
for all $i \neq k$.

9. Prove that $2^n + 65$ does not divide $5^n - 3^n$ for any positive integer n.