Mock IMO Day 1

MOP 2022

Saturday, June 11, 2022

Time limit: 4.5 hours. Enjoy your weekend!

1. Let S be an infinite set of positive integers. Assume there exist pairwise distinct $a, b, c, d \in S$ satisfying $\gcd(a, b) \neq \gcd(c, d)$. Prove that there exist pairwise distinct $x, y, z \in S$ satisfying $\gcd(x, y) = \gcd(y, z) \neq \gcd(z, x)$.

2. Show that $n! = a^{n-1} + b^{n-1} + c^{n-1}$ has only finitely many solutions in positive integers.

3. Find all integers $n \geq 3$ for which every convex equilateral n-gon of side length 1 contains an equilateral triangle of side length 1. (Here polygons contain their boundaries.)

Mock IMO Day 2

MOP 2022

Sunday, June 19, 2022

Time limit: 4.5 hours. Not a significant source of saturated fat, protein, or combinatorics.

4. Which positive integers n make the equation

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\left\lfloor\frac{ij}{n+1}\right\rfloor=\frac{n^2(n-1)}{4}$$

true?

5. Consider a 100×100 square unit lattice $\mathbf{L}$ (hence $\mathbf{L}$ has 10000 points). Suppose $\mathcal{F}$ is a set of polygons such that all vertices of polygons in $\mathcal{F}$ lie in $\mathbf{L}$ and every point in $\mathbf{L}$ is the vertex of exactly one polygon in $\mathcal{F}$. Find the maximum possible sum of the areas of the polygons in $\mathcal{F}$.

6. Determine all integers $n \geq 2$ with the following property: every n pairwise distinct integers whose sum is not divisible by n can be arranged in some order $a_1, a_2, \ldots, a_n$ so that n divides $1 \cdot a_1 + 2 \cdot a_2 + \cdots + n \cdot a_n$.

Mock IMO Day 3

MOP 2022

Whenever

Time limit: ∞. Not a real exam, just if you want to see the rest of the IMO shortlist.

7. Let $n \geq 2$ be an integer and let $a_1, a_2, \ldots, a_n$ be positive real numbers with sum 1. Prove that
$$\sum_{k=1}^{n} \frac{a_k}{1-a_k}(a_1 + a_2 + \cdots + a_{k-1})^2 < \frac{1}{3}.$$

8. Let $a_1, a_2, a_3, \ldots$ be an infinite sequence of positive integers such that a_{n+2m} divides $a_n + a_{n+m}$ for all positive integers n and m. Prove that this sequence is eventually periodic.

9. Consider a checkered $3m \times 3m$ square, where $m > 1$ is an integer. A frog sits on the lower left corner cell S and wants to get to the upper right corner cell F. The frog can hop one cell right or cell up.

   Some cells can be sticky, and the frog gets trapped if it hops on such a cell. A set X of cells is *blocking* if the frog cannot reach F from S when all the cells of X are sticky. A blocking set is *minimal* if it does not contain a smaller blocking set.

   (a) Prove that there exists a minimal blocking set containing at least $3m^2 - 3m$ cells.

   (b) Prove that every minimal blocking set contains at most $3m^2$ cells.