Mock IMO Day 1

MOP 2021

Thursday, June 17, 2021

Time limit: 4.5 hours. Yes, there were more inequalities on the shortlist.

1. In a regular 100-gon, 41 vertices are colored black and the other 59 vertices are colored white. A quadrilateral is *weird* if it has three vertices of one color and one vertex of the other color.

   Prove that there exist 24 pairwise disjoint weird quadrilaterals. (Two quadrilaterals are disjoint if they have no common vertices and their interiors do not intersect.)

2. Let a, b, c, d be positive real numbers satisfying $(a+c)(b+d) = ac + bd$. Find the smallest possible value of
$$\frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a}.$$

3. Let S be a set of $n \geq 3$ positive integers, none of which is the sum of two different numbers in S. Prove that there exists a permutation of S in which none of the middle $n-2$ integers divides the sum of its neighbors.

Mock IMO Day 2

MOP 2021

Friday, June 18, 2021

Time limit: 4.5 hours. Determination.

4. Let $ABCD$ be a convex quadrilateral with $\min\{\angle B, \angle D\} > 90^\circ$ and $\angle A = \angle C$. Points E and F are the reflections of A in $\overline{BC}$ and $\overline{CD}$. Segments AE and AF meet line BD at K and L.

   Prove that the circumcircles of $\triangle BEK$ and $\triangle DFL$ are tangent to each other.

5. Determine all functions $f\colon \{1, 2, \dots\} \to \{0, 1, 2, \dots\}$ such that
   - $f(xy) = f(x) + f(y)$ for all positive integers x and y, and
   - there exists an infinite set S of positive integers such that $f(a) = f(b)$ whenever $a + b \in S$.

6. Anastasia and Bananastasia play a game on the board as follows. Initially, the board contains 2020 copies of the number 1. Each round proceeds as follows:
   1. Anastasia erases two numbers x and y from the board.
   2. Bananastasia writes one of $x + y$ and $|x - y|$ on the board.

   After each round, the game ends if one of the following holds:
   - one number on the board is larger than the sum of all other numbers on the board, or
   - all numbers on the board are zeroes.

   After the game ends, Bananastasia must give Anastasia one slice of banana bread for every number remaining on the board. How many slices of banana bread can Anastasia guarantee, assuming optimal play from both players?