Mock IMO

Day: **1**

Saturday, June 9, 2018

Ready or not, here we go!

Problem 1. Let p be a fixed prime number. Ankan and Ryan play the following turn-based game, with Ankan moving first. On their turn, each player selects an index $i \in \{0, \ldots, p-1\}$ not chosen on a previous turn, and a digit $a_i \in \{0, \ldots, 9\}$. This continues until all indices have been chosen (hence for p turns). Then, Ankan wins if the number

$$N = a_0 + a_1 \cdot 10 + a_2 \cdot 10^2 + \cdots + a_{p-1} 10^{p-1}$$

is divisible by p; otherwise Ryan wins. For each prime p, determine which player has the winning strategy.

Problem 2. A sequence of real numbers $a_1, a_2, \ldots,$ satisfies

$$a_n = -\max_{i+j=n} (a_i + a_j) \qquad \text{for all } n > 2017.$$

Prove that this sequence is bounded.

Problem 3. There are 2017 mutually disjoint circles drawn on a blackboard, such that no two are tangent and no three share a common tangent. A *tangent segment* is a line segment that is a common tangent to two circles, starting at one tangent point and ending at the other one. Luciano is drawing tangent segments on the blackboard, one at a time, so that no tangent segment intersects any other circles or previously drawn tangent segments. Luciano keeps drawing tangent segments until no more can be drawn. Find all possible numbers of tangent segments when he stops.

Time limit: 4 hours and 30 minutes.
Each problem is worth seven points.

IMO

Mock IMO

Day: **2**

Saturday, June 16, 2018

Stay determined!

Problem 4. Let ABC be an acute triangle with orthocenter H and circumcenter O. Let $P = \overline{BH} \cap \overline{AO}$ and $Q = \overline{CH} \cap \overline{AO}$. Prove that the circumcenter of $\triangle HPQ$ lies on the A-median.

Problem 5. Let p be a prime number, and let $s_1, \ldots, s_p$ be pairwise distinct positive integers for which $s_1 + \cdots + s_p$ is not divisible by p. Fred the frog starts at the point 0 on a number line. He picks a permutation $(t_1, \ldots, t_p)$ of $(s_1, \ldots, s_p)$, then makes p jumps to the right with lengths $t_1, \ldots, t_p$ in that order. Show that there are at least $(p-1)!$ permutations $(t_1, \ldots, t_p)$ such that Fred the frog never jumps on any positive multiple of p.

Problem 6. Let $f\colon \mathbb{Z}_{>0} \times \mathbb{Z}_{>0} \to \{0, 1\}$ be a function such that $f(2,1) = f(1,1) = 0$. Assume that for any relatively prime positive integers (a, b) not both equal to 1, we have

$$f(a,b) = 1 - f(b,a) = f(a+b,b).$$

Let p be an odd prime. Prove that

$$\sum_{n=1}^{p-1} f(n^2, p) \geq \sqrt{2p} - 2.$$

Time limit: 4 hours and 30 minutes.
Each problem is worth seven points.