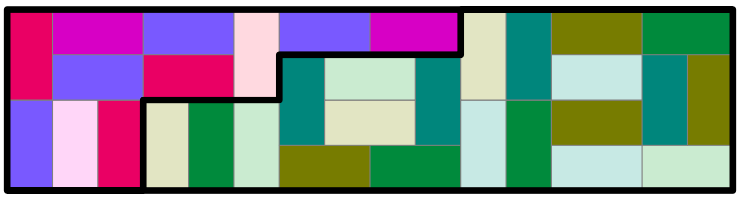
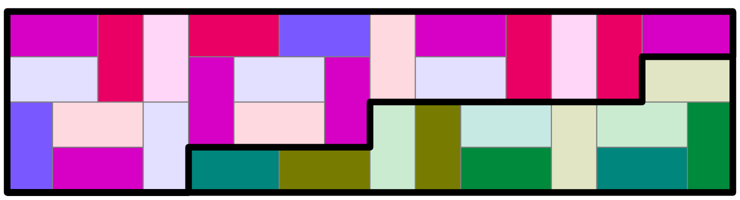
MOP 2025 Homework Problems

Thug Raiders

May 2025





Congratulations on your excellent performance on the USA(J)MO, which has earned you an invitation to attend the Math Olympiad Program! This program will be an intense and challenging opportunity for you to learn a tremendous amount of mathematics.

To celebrate your achievement, and give you something to talk about on the way to Aurora, we invite you to work on the following homework problems. (These are optional and just for your interest; you are not expected to formally submit any solutions.)

You're welcome to share these problems with friends as there is nothing confidential in here. Happy solving!

Algebra

A1. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that, for all x and y,

$$f(x + f(y)) \in \{x + f(y), f(f(x)) + y\}.$$

A2. Consider real-coefficient polynomials of degree 2025 with all distinct real roots. Find the minimum number of nonzero coefficients that such a polynomial could have.

A3. Determine whether or not there exists a function $f\colon \mathbb{Z} \to \mathbb{Z}$ for which there are infinitely many integers c such that the function $g(x) = f(x) + cx$ is bijective.

A4. Let $f(x) = x^2 + bx + c$, where $b, c \in \mathbb{R}$ and $b \geq 0$. Is it possible to partition the interval $[0, 1]$ into two disjoint sets A and B such that $f(A) = B$?

Combinatorics

C1. For positive integers p, q and r we are given $p \cdot q \cdot r$ unit cubes. We drill a hole along a space diagonal of each of these cubes and then tie them to a very thin thread of length $p \cdot q \cdot r \cdot \sqrt{3}$ like a string of pearls. We now want to construct a rectangular prism of side lengths p, q and r out of the cubes, without tearing the thread.

(a) For which numbers p, q and r is this possible?

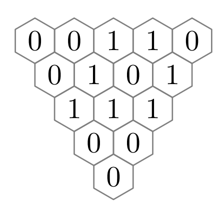
(b) For which numbers p, q and r is this possible in a way such that both ends of the thread coincide?

C2. Let $n \geq 2$ be an even integer. Given n points in the plane, no three collinear, what is the maximum possible number of right triangles with vertices on these points?

C3. Let $G = K_{45,45,\ldots,45}$ be the complete 45-partite graph on 2025 vertices. A. Mo and J. Mo play a game, starting with the elder A. Mo. Each player takes turns removing exactly one edge from a graph G, and the first player to make a move that leaves no cycles loses. Who has the winning strategy?

C4. The following two problems are from an upcoming sequel to Leningrad Math Olympiads (1961–1991), but for which the jury's solution is lost. If you find and submit a full solution to either, the author will credit you in the book. (There are actually eight problems on the bounty list, but I picked the two which I suspect to be the most doable. If you or your friends want to see the other six problems, email Evan.)

(a) Consider a hexagonal grid shaped like a downwards equilateral triangle of side length n. We wish to fill each cell with 0 or 1 such that so that in every row except the first, each cell has the same parity as the sum of the two cells above it. Show that, for every $n \geq 1$, there's a way to do this where the number of 0's and 1's used differ by at most 1. An example is given for $n = 5$.



(b) Solve **C2** for odd n.

Geometry

G1. Six square mirrors are put together to form a cube $ABCDEFGH$ with a mirrored interior. At each of the eight vertices, there is a tiny hole through which a laser beam can enter and leave the cube. A laser beam enters the cube at vertex A in a direction not parallel to any of the cube's sides. If the beam hits a side, it is reflected; if it hits an edge, the light is absorbed, and if it hits a vertex, it leaves the cube.

For each positive integer n, determine the set of vertices where the laser beam can leave the cube after exactly n reflections.

G2. (a) Let ABC be a triangle. Points D, E and F are on lines BC, CA, and AB such that $ABDE$ and $ACDF$ are cyclic quadrilaterals. Suppose that the circumcircle of AEF meets BC at two distinct points X and Y. Prove that $DX = DY$.

(b) Let H be the orthocenter of an acute triangle ABC, let F be the foot of the altitude from C to AB, and let P be the reflection of H across BC. Prove that the lines tangent to the circumcircle of AFP at F and P intersect on line BC.

G3. Two lines ℓ_1 and ℓ_2 intersect the sides BC, CA, AB of triangle ABC at A_1, B_1, C_1 and A_2, B_2, C_2, respectively. The circumcircles of AB_1C_1 and AB_2C_2 meet at point A'; define B' and C' analogously. Prove that $\overline{AA'}$, $\overline{BB'}$, $\overline{CC'}$ are concurrent.

G4. Let ADE be a triangle with circumcircle O. Points P and Q are on line DE with $AP = PD$ and $AQ = EQ$. The circumcircles of $\triangle ADE$ and $\triangle APQ$ intersect at another point $K \neq A$. Points B and C are on sides AE and AD respectively with

$$\angle AKB = \angle AKC = 90^\circ - \angle DAE.$$

Prove that B, O, C are collinear.

G5. Let ABC be a triangle with incenter I and circumcenter O. The incircle touches side BC at D, and line ℓ is parallel to line BC and tangent to the incircle (at the antipode of D). Let E and F be points on ℓ such that $\angle EIF = 90^\circ$. Lines EI and FI meet the circumcircle of $\triangle AEF$ again at E' and F'. Prove that O lies on $E'F'$.

Number Theory

N1. Prove that if p is a prime number, then $7p + 3^p - 4$ is not a perfect square.

N2. Find all $f\colon \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ which satisfy $f(f(m)f(n)) = mn$ and $f(2024a+1) = 2024a+1$.

N3. If p is prime and $n \geq p$ is an integer, prove that p divides

$$\sum_{pi+j=n} \frac{n!}{p^i i! j!}.$$

N4. Let n be a positive integer other than 3 or 5. Prove that the number of positive integer divisors of $n!$ is itself a divisor of $n!$.