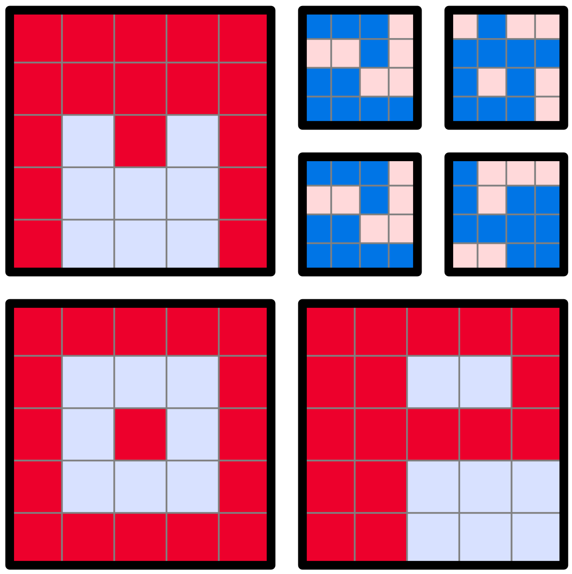
MOP 2024 Homework Problems

Thug Raiders

May 2024



Congratulations on your excellent performance on the USA(J)MO, which has earned you an invitation to attend the Math Olympiad Program! This program will be an intense and challenging opportunity for you to learn a tremendous amount of mathematics.

To celebrate your achievement, and give you something to talk about on the way to Pittsburgh, we invite you to work on the following homework problems. (These are optional and just for your interest; you are not expected to formally submit any solutions.)

You're welcome to share these problems with friends as there is nothing confidential in here. Happy solving!

Algebra

A1. Let f be the cubic polynomial

$$f(x) = x^3 + bx^2 + cx + d,$$

where b, c, and d are real numbers. Let x_1, x_2, ..., x_n be nonnegative numbers, and let m be their average. Suppose that $m \geq -\frac{b}{2}$. Prove that

$$\sum_{i=1}^{n} f(x_i) \geq nf(m).$$

A2. Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that $f(x+1) = f(x) + 1$ and $f(x^2) = f(x)^2$ for all real x.

A3. Let $P(x)$ be a polynomial with real coefficients, such that $P(x) \geq 0$ for all $x \in \mathbb{R}$. Show that there exist polynomials $Q_1(x), Q_2(x), \ldots, Q_k(x)$, also with real coefficients, such that $P = Q_1^2 + Q_2^2 + \cdots + Q_k^2$. Can we always have $k = 2$?

Combinatorics

C1. Suppose there are many people at a party, but each person knows at most 7 other people at the party. Show that it's possible to separate the people into two rooms such that in each room, every person knows at most 3 of the people in their room.

C2. Let n be a positive integer and let σ be a random $2n$-cycle on $\{1, 2, \ldots, 2n\}$ and τ be a random fixed-point-free involution on $\{1, 2, \ldots, 2n\}$. What is the maximum number of cycles $\sigma \circ \tau$ can have and what is the probability that this value is achieved?

C3. There is a hidden infinite binary string $a_1a_2a_3 \cdots \in \{0,1\}^\infty$ for which

$$\lim_{n\to\infty} \frac{a_1 + \cdots + a_n}{n} = 0$$

(that is, the set of indices i with $a_i = 1$ has density zero). The digits a_t are revealed in order. For each $t \geq 1$, before a_t is revealed, a frog decides to wait or cross the road. Then a_t is revealed. If the frog crossed the road and $a_t = 1$, then the frog is squashed. If the frog crossed the road and $a_t = 0$ then the frog has safely crossed the road. Otherwise the process repeats with a_{t+1}.

Find a randomized strategy for the frog such that, *regardless of which string* $a_1a_2a_3\ldots$ *is chosen*, the frog successfully crosses the road with probability at least 99%.

Geometry

G1. Let ABC be an acute triangle for which $AB > AC$, and let D and E denote the midpoints of AB, AC respectively. The circumcircle of ADE intersects the circumcircle of BCE again at P. The circumcircle of ADE intersects the circumcircle BCD again at Q. Prove that $AP = AQ$.

G2. In the exterior of $\triangle ABC$ we construct the regular pentagons $ABDEF$, $BCGHI$, $CALKJ$. Show that the perpendicular bisectors of FL, DI, GJ are concurrent or parallel.

G3. Let ABC be an acute triangle, and let M denote the midpoint of arc $\widehat{BAC}$ of its circumcircle. Let X be a point lying in the interior of the acute triangle ABC such that

$$\angle BAX = 2\angle XBA \quad \text{and} \quad \angle XAC = 2\angle ACX.$$

Prove that $XM = XA$.

Number Theory

N1. Let F_n be the Fibonacci sequence, i.e., $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$ for $n > 1$. Prove that there exist infinitely many prime numbers p such that p divides F_{p-1}.

N2. Decide whether there are infinitely many triples of positive integers (a, b, c) such that all prime factors of $a! + b! + c!$ are less than 2024.

N3. For any positive integer n, let $\Omega(n)$ be the unique positive integer such that n is a product of $\Omega(n)$ prime numbers. Prove that there exist infinitely many positive integers n such that

$$\Omega(n+1) < \Omega(n+2) < \cdots < \Omega(n+2024).$$

If you finish early...

Some of these questions may not have reasonable solutions.

X1. Find all functions $f\colon \mathbb{Z} \to \mathbb{R}$ such that for all $x, y, z \in \mathbb{Z}$ with $x + y + z = 0$, we have

$$f(x)^2 + f(y)^2 + f(z)^2 = 1 + 2f(x)f(y)f(z).$$

X2. In the setup of problem **C2**, find the distribution of the cycle count of $\sigma \circ \tau$.

X3. Let $ABCD$ be a quadrilateral and I_A, I_B, I_C, I_D be the incenters of BCD, CDA, DAB, ABC respectively. Show that if three of the lines AI_A, BI_B, CI_C, DI_D are concurrent, then all four are concurrent.

X4. Solve problem **N3** but with $\Omega(n)$ replaced with $\omega(n)$, the number of distinct prime factors of n.