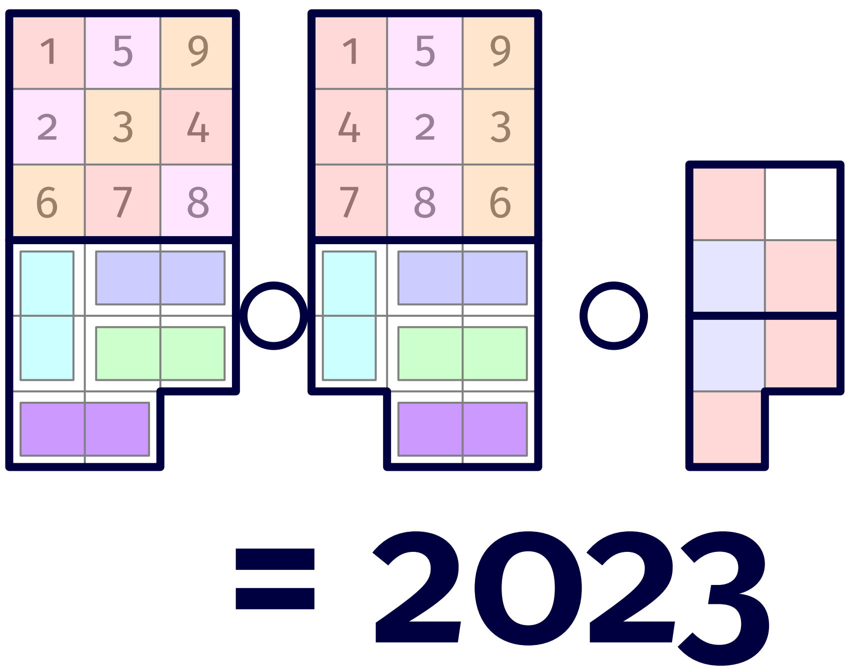
MOP 2023 Homework Problems

Thug Raiders

May 2023



Hello hello hey,

Congratulations on your excellent performance on the USA(J)MO, which has earned you an invitation to attend the Math Olympiad Summer Program! This program will be an intense and challenging opportunity for you to learn a tremendous amount of mathematics.

To celebrate your achievement, and give you something to talk about on the way to Pittsburgh, we invite you to work on the following homework problems. (These are optional and just for your interest; you are not expected to formally submit any solutions.)

You're welcome to share these problems with friends as there is nothing confidential in here. Happy solving!

§1 Problems

The problems in this section should be approachable for all students.

1. Let $P(x, y)$ be a polynomial with real coefficients. What are the possibilities for the range of P?

2. Exhibit a polynomial $f(x, y)$ with integer coefficients satisfying the following properties:
   - For any integers a and b, we have $f(a, b) \neq 0$.
   - For any nonzero integer n, there exist integers a and b satisfying $f(a, b) = n$.

3. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that, for all real numbers x, we have $f(x+1) = f(x) + 1$ and $f(x^2) = f(x)^2$.

4. For which positive integers n is it possible to split the integers $1, \ldots, 2n$ into n pairs such that whenever one integer is selected from each pair, the sum of the n chosen integers is not a multiple of $2n$?

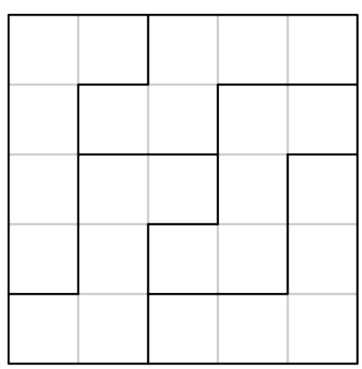
5. Let $p \geq 3$ be prime. Find the smallest possible degree of a polynomial f with integer coefficients for which $f(0)$, $f(1)$, ..., $f(p-1)$ leave exactly three distinct remainders upon division by p.

6. Let f be a function from the set of rational numbers to the set of real numbers. Suppose that for all rational numbers r and s, the expression $f(r+s) - f(r) - f(s)$ is an integer. Prove that there is a positive integer q and an integer p such that
$$\left| f\left(\frac{1}{q}\right) - p \right| \leq \frac{1}{2023}.$$

7. Prove that a triangle cannot be dissected into finitely many concave quadrilaterals.

8. Let n be a positive integer and consider an $n \times n$ square grid. A *ribbon* is a sequence of n cells, such that every cell is above or to the right of the previous cell.

   How many ways are there to partition the grid into n ribbons? For example, for $n = 5$, one such way is shown below.



9. Triangle ABC has incenter I and circumcircle γ. Let $D = BI \cap AC$, $E = CI \cap AB$. Ray DE intersects γ at P. Line IP intersects γ again at Q and intersects BC at X. Prove that $IQ = QX$.

10. Ten points are given in the plane. Show that they may be simultaneously covered by non-overlapping unit disks (a disk may cover multiple points).

§2 Problems for returning students

The problems in this section are targeted at students who have previous MOP experience, but some of them should be accessible to new students too.

11. Let ABC be a triangle with $AB \neq AC$. The incircle of triangle ABC is tangent to BC, CA, AB at D, E, F, respectively. Let I_A be the A-excenter of triangle ABC and let the internal $\angle A$-bisector meet BC at L. Set $G = BE \cap CF$, $P = EF \cap BC$, $X = GL \cap DI_A$. Let the A-excircle of triangle ABC be tangent to BC at T. Prove that PX is perpendicular to AT.

12. Let $n \geq 2$ be an integer. Construct finite pairwise disjoint sets of integers A_1, ..., A_n such that whenever σ is any permutation of $\{1, \dots, n\}$ and elements $a_k \in A_k$ are selected uniformly randomly, the probability that

$$a_{\sigma(1)} < \cdots < a_{\sigma(n)}$$

equals $\frac{1}{n!}$. (Informally, show that we can construct n dice so that whenever they're rolled, the $n!$ possible relative orders of the rolls occur equally often.)

13. Given $2^d + 1$ points in d-dimensional space with no three collinear, show that some three of them form an obtuse angle.

14. Are there infinitely many integers $n \geq 1$ for which $(2023n)!$ is divisible by $n! + 1$?

15. An ordered pair (a, b) of positive integers is *organic* if there do not exist positive integers x and y such that $(a^2+1)x^2 + y^2$ and $(b^2+1)y^2 + x^2$ are both perfect squares. Let j and k be odd positive integers. Prove that there exists an odd positive integer d with the following property: for any positive integers J and K such that $J \equiv j \pmod{d}$ and $K \equiv k \pmod{d}$, the ordered pair (J, K) is organic.

16. Let $n \geq 2$ be an integer. Alice wishes to simulate a fair n-sided die, using a (possibly unfair) coin, as follows. First, she chooses a positive integer k, and flips the coin k times. Next, depending on which of the 2^k sequences of coin flips she obtains, she chooses one of the integers 1, ..., n.
    (a) Show that she can choose the bias of the coin, the integer k, and the scheme for converting coin flip sequences to integers so that each of the integers 1, ..., n is chosen with equal probability.
    (b) Show that she can do the above under the restriction that the probability of flipping heads be between 0.499 and 0.501.

17. Let $\aleph_0$ and $\mathfrak{c} = 2^{\aleph_0}$ denote the cardinalities of $\mathbb{Z}$ and $\mathbb{R}$, respectively.
    (a) Let A be a set of cardinality $\aleph_0$. Let $\mathcal{P}(A)$ denote the power set of A. Demonstrate a subset $F \subseteq \mathcal{P}(A)$ with $|F| = \mathfrak{c}$, such that for any $S, T \in F$, we have either $S \subseteq T$ or $T \subseteq S$.
    (b) Demonstrate a subset $F \subseteq \mathcal{P}(A)$ with $|F| = \mathfrak{c}$, such that for each $S \in F$, we have $|S| = \aleph_0$, and for each distinct $S, T \in F$, we have that $|S \cap T|$ is finite.
    (c) Let G be the complete graph on $\mathfrak{c}$ many vertices. Demonstrate a coloring of the edges of G with the colors red and blue, such that there's no uncountable monochromatic clique.

§3 Advanced problems

The problems in this section are more difficult or may require more background.

18. Let p be a fixed prime. Which integers arise as determinants of $p \times p$ circulant matrices with integer entries?

19. Let p be prime and let A be a matrix over the integers mod p such that for $0 \le j, k \le p-1$, the (j,k) entry is $\binom{j+k}{j}$. Prove that A^3 is the identity matrix.

20. Given a positive integer d, and a finite family F of subsets of a finite set S such that every element of S is in at most d sets of F, show that there is a function f from S to $\{1, -1\}$ such that each set T in F has $|\sum_{s \in T} f(s)| \le 2d-1$.

21. Let ABC be an acute scalene triangle with orthocenter H and circumcenter O. The circle with diameter HO intersects circle (BOC) again at P and line AO at Q. Prove that line PQ passes through the Kosnita point of triangle ABC. (The Kosnita point is the point at which the line through A and the circumcenter of BOC and the other two analogous lines concur; it is the isogonal conjugate of the nine-point center.)

22. Is it possible to color the points of the Euclidean plane with countably infinitely many colors such that there aren't three vertices of the same color forming the vertices of an axis-aligned right triangle? What about four vertices of the same color forming the vertices of an axis-aligned rectangle?

23. Fix positive integers n, d and consider n points $p_1, \dots, p_n \in \mathbb{R}^d$. What is the maximum possible number of permutations σ of $\{1, 2, \dots, n\}$ such that there exists a point $q \in \mathbb{R}^d$ with
$$|q - p_{\sigma(1)}| < |q - p_{\sigma(2)}| < \cdots < |q - p_{\sigma(n)}|?$$
(Here $|-|$ denotes Euclidean distance.)

24. Let m be a positive integer and define $n = 4m$. We are given real numbers $a_1, a_2, \dots, a_n \in [0,1]$. Show that there exist real numbers $y_0, y_1, y_2, \dots, y_n \in [0, 2 - \frac{1}{2^{m+3}}]$, such that $|y_i - y_{i-1}| = a_i$ for each $1 \le i \le n$.

25. Let $A_1, \ \dots, \ A_{2n}$ be $n \times n$ matrices. Show that
$$\sum_{\sigma \in S_{2n}} \operatorname{sgn} \sigma \, A_{\sigma(1)} A_{\sigma(2)} \cdots A_{\sigma(2n)} = 0.$$

26. Let n and k be positive integers. Given a tournament graph on n vertices, prove that there are at most $n \left(\frac{n-1}{2}\right)^k$ sequences $(v_0, \dots, v_k)$ of (not necessarily distinct) vertices such that there is an edge from v_i to v_{i+1} for all i (in particular, we require $v_i \ne v_{i+1}$).

§4 Miscellaneous

27. Give artistic commentary on the MOP server logo on the first page.

28. In front of you is a single sheet of printer paper (8.5 by 11 inches), a paper clip, a rubber band, and a single matchstick. Using only these materials, construct the tallest freestanding structure that you can.

29. Explain and justify the joke in the following.

    > One day, Eve and Anne were chatting when Anne mentioned that she was really hungry. Eve suggested that they go out to buy some corn (Anne's favorite snack), but Anne proclaimed that she needed all the corn in the world.
    >
    > Unbeknownst to Anne, Eve had been waiting for this moment her whole life. "You know what? I can do better than that. I'll give you all the corn in the world, and then some!" she exclaimed. Anne was intrigued, and so she decided to play along.
    >
    > Eventually, the pair reached a fenced corn field. "Yeah, this is a lot of corn, but you said you'll give me all of the corn in the world. What gives?" Anne inquired. "Let's go inside. You'll see soon enough," Eve replied, with a smirk on her face.
    >
    > They entered the corn field, with Eve pushing open the slightly-ajar door in the fence. "Are you ready?" she asked. But Anne had already started munching on raw corn. "Damn. I've been waiting so long, and she won't even see my trick," Eve lamented.
    >
    > Eve shut the door. Suddenly, it was corn as far as one could see, with all traces of their old world now just a memory. Anne jerked up, her head spinning. "How!?" "Oh, it was easy, really," Eve replied. "After all, the field is now closed, so it's infinite."

30. There's a sloth sleeping in a tree. It's about to oversleep its morning class, but it looks really happy. Should you wake it up?