MOP 2022 Homework Problems

Thug Raiders

May 2022

Instructions

> It is time to work now, not guess partials. You need 7s on TST, and we all need 7s in the IMO.
>
> Zuming Feng, 2008

Hello hello hey,

Congratulations on your excellent performance on the USA(J)MO, which has earned you an invitation to attend the Math Olympiad Summer Program! This program will be an intense and challenging opportunity for you to learn a tremendous amount of mathematics.

To celebrate your achievement, and give you something to talk about on the way to Pittsburgh, we invite you to work on the following homework problems. The solutions will be discussed in the first day of class at MOP.

You're welcome to share these problems with friends as there is nothing confidential in here.

Happy solving!

§1 Problems

The problems in this section should be approachable for all students.

1. A parabola and a circle intersect at exactly two points A and B. Suppose that the circle is tangent to the parabola at A. Does it follow that the circle is tangent to the parabola at B as well?

2. Triangle ABC has side lengths $AB = 20$, $BC = 15$, and $CA = 7$, and altitudes $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$. Find the distance between the orthocenter of $\triangle ABC$ and the incenter of $\triangle DEF$.

3. Exactly one side of a tetrahedron is greater than 1. Show that its volume is at most $\frac{1}{8}$.

4. Find all quadruples of odd integers (a, b, c, d) such that $0 < a < b < c < d$, $ad = bc$, and both $a + d$ and $b + c$ are powers of 2.

5. Let $f(x) = x^3 - 3x + 1$. Find all primes p such that exactly one of the numbers $f(0),\ f(1),\ \dots,\ f(p-1)$ is divisible by p.

6. Squares BCA_2A_1, CAB_2B_1, ABC_2C_1 are erected outside acute triangle ABC. Calculate $\angle A_1AA_2 + \angle B_1BB_2 + \angle C_1CC_2$.

7. Let $a_1,\ \dots,\ a_n,\ b_1,\ \dots,\ b_n$ be a permutation of $1,\ \dots,\ 2n$ satisfying $a_1 < \cdots < a_n$ and $b_1 > \cdots > b_n$. Prove that
$$|a_1 - b_1| + \cdots + |a_n - b_n| = n^2.$$

§2 Problems for returning students

The problems in this section are targeted at students who have previous MOP experience, but some of them should be accessible to new students too.

8. Prove there exists a constant $c > 0$, such that for any graph G with $n > 2$ vertices, we can split G into a forest and at most $cn \log n$ disjoint cycles.

9. Let S be the set of quadruples (a, b, c, d) of real numbers for which $[a, b]$ and $[c, d]$ are intervals with an intersection of positive length. Exhibit a smooth function $f \colon S \to \mathbb{R}$ for which $f(a, b, c, d) \in [a, b] \cap [c, d]$ for any $(a, b, c, d) \in S$.

10. Let $n, c > 1$ be integers. A diabolical combination lock has n dials $(d_1, \dots, d_n)$, with $d_i \in \{0, 1, \dots, c-1\}$ for each i. The d_i are always concealed, and the lock's initial state is also unknown. In a move, you may choose integers $x_1,\ \dots,\ x_n$ and replace d_i with $d_i + x_i \pmod c$. Then the lock secretly selects an integer k, and replaces each d_i with d_{i+k} (indices taken modulo n).

    The lock opens if all dials are set to 0. Determine for which (n, c) this can be accomplished regardless of the initial configuration and the choices of k.

11. Let `a` be a list of real numbers. Prove that the following Python code correctly sorts `a` in increasing order.

```
for i in range(n):
    for j in range(n):
        if a[i] < a[j]:
            a[i], a[j] = a[j], a[i]
```

12. There is a network of $n \geq 3$ cities and m roads connecting pairs of distinct towns. In city i, there is a sign with a positive integer a_i written on it. We say that city v is *directly reachable* from city u if we can start from city u and reach city v by repeating the following process a_u times:

    - Choose a road from the current city and move to the other town connected by the chosen road.

    We say that city v is *reachable* from city u if either v is *directly reachable* from u or there exist a city w such that v is *directly reachable* from w and w is *reachable* from u.

    The values of the signs have not been assigned. We want every town u to be *reachable* from every other town v. A labeling of the signs $a_1, a_2, ..., a_n$ is called *nice* if every town is *reachable* from every other town.

    The network of towns is called *beautiful* if **any** labeling $a_1, a_2, \cdots, a_n$ is *nice*.

    Determine, in terms of n, the minimum possible number of roads in a *beautiful* network of n cities.

§3 Advanced problems

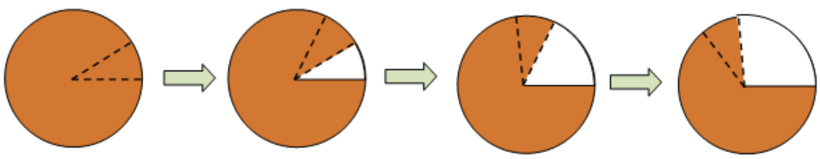
The problems in this section are more difficult or may require more background.

13. Is there a continuous surjective function from $[0, 1]$ to $(0, 1)$?

14. Let A and B be $n \times n$ complex matrices such that $A^2 + B^2 = 2AB$, and let 1 denote the $n \times n$ identity matrix. Prove that for any complex number z we have

$$\det(zA - 1) = \det(zB - 1).$$

15. Let $0° < x < 180°$ be a fixed angle. Consider a cake whose top is covered with icing (and whose bottom has no icing). Successively cut out slices of cake with angle x, flip them over, and insert them back into the cake, as shown below.



    Find all angles x such that after a finite number of moves, all the icing returns to the top of the cake.

16. Let S be a finite set of positive integers with $|S| \geq 2$. Suppose that when two distinct elements of S are selected uniformly randomly, the remainder when their sum is divided by 1024 is $0,\ 1,\ \ldots,\ 1023$ with equal probability. Find the smallest possible value of $|S|$.

17. Prove that there exists a positive integer n and a sequence $a_1,\ \ldots,\ a_n$ satisfying the following properties:

    - Each number a_i is equal to either -1, 0 or 1.
    - At least 99% of the numbers $a_1,\ a_2,\ \ldots,\ a_n$ are nonzero.
    - The sum $\frac{a_1}{1} + \frac{a_2}{2} + \cdots + \frac{a_n}{n}$ is 0.

18. Each of 100 baskets has some fruit (possibly none). Show that you can choose 51 of the baskets so that you obtain at least half of the apples, at least half of the bananas, and at least half of the cherries.

19. Consider a finite collection of complex numbers whose magnitudes sum to 1. Show that we can select some of these numbers whose sum has magnitude at least $\frac{1}{\pi}$.

20. Let $n > 1$ be a positive integer and $\mathcal{S}$ be the set of n^{th} roots of unity. Suppose P is an n-variable polynomial with complex coefficients such that for all $a_1, \dots, a_n \in \mathcal{S}$, $P(a_1, \dots, a_n) = 0$ if and only if $a_1, \dots, a_n$ are all different. What is the smallest possible degree of P?

§4 USA Remixed Math Olympiad

21. Find the largest integer N for which one may split $\{N, N+1, \dots, 2N\}$ into two sets, such that in each set, no two elements sum to a perfect square.

22. Find all pairs of primes (p, q) such that $p - q$ and $pq - q$ are perfect cubes.

23. Let p be a prime number and $k > 1$ be a positive integer. Prove that there is at most one pair of positive integers (a, b) such that $p = a^2 + kb^2$.

24. Find all positive integers a, b, c such that $ab + c$, $bc + a$, $ca + b$ are powers of 2. (The number 1 is considered a power of 2.)

25. Let n, k, ℓ be positive integers. Cathy is playing the following game. There are n cards, k piles, and ℓ cells; the cards are labelled 1 to n. Initially, all cards are placed inside one pile. Each turn, Cathy chooses a nonempty pile or cell and then moves the card with the smallest label, say i, to either any empty pile, any empty cell, or the pile containing card $i + 1$. (In particular, cells can never contain more than one card.)

    Cathy wins if at any point there is a pile containing only card n. Determine all triples of integers (n, k, ℓ) such that Cathy can win this game.