MOP 2021 Homework Problems

Thug Raiders

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Instructions

> It is time to work now, not guess partials. You need 7s on TST, and we all need 7s in the IMO.
>
> Zuming Feng, 2008

Hello hello hey,

Congratulations on your excellent performance on the USA(J)MO, which has earned you an invitation to attend the Math Olympiad Summer Program! This program will be an intense and challenging opportunity for you to learn a tremendous amount of mathematics.

In normal years, we provide a set of 20 or so homework problems to the newly invited MOPpers to give them some math problems to talk about on the way to Pittsburgh; the solutions are discussed during the first day of classes. We are upholding this tradition, except rather than talking about them on the way to Pittsburgh, you're encouraged to discuss them in Facebook or Discord. Just something to kick off this virtual program. These problems are optional and you don't *have* to work on them, but if we picked the problems well, you'll probably want to anyways.

Within each section, the problems aren't sorted in any particular order; some are much easier than others. Work on whatever looks interesting to you, and ignore the ones that aren't.*

This year's homework also contains variants and generalizations of the problems from this year's USA(J)MO. This is because we want you to think about how the problems and solutions you've already done might be stretched, beyond just what gets 7 points. Also, they're sort of funny. Fair warning: we don't know solutions to all of these.

You're welcome to share the MOP homework problems with your friends as well; unlike most MOP materials, the homework is not considered "internal use only".

Happy solving!

*Those of you on the IMO team, or close to it, may recognize a large number of these problems; in that case, don't let the lack of homework distract you from your usual practice routine.

§1 Problems

The problems in this section should be approachable for all students.

1. (Ukraine TST 2021) Initially, let n be an integer greater than 1. Every second one chooses a prime p dividing n, and replaces n with $n + \frac{n}{p}$. Prove that number 3 was chosen as p infinitely many times.

2. Can you find six different points on the plane such that no three are collinear and the distances between them are all integers?

3. Show that if a tournament[†] has a directed cycle of length k, then it has a directed cycle of length $(k-1)$.

4. You are given two noncongruent parallel line segments. Using straightedge alone, divide one of the segments into nine equal parts.

5. (EGMO 2021) The number 2021 is fantabulous. For any positive integer m, if any element of the set $\{m, 2m+1, 3m\}$ is fantabulous, then all the elements are fantabulous. Does it follow that the number 2021^{2021} is fantabulous?

6. An equilateral triangle is covered by 1000 disks of radius 1. Show that the triangle may be covered by 4000 disks of radius $\frac{1}{2}$.

7. (EGMO 2021) Does there exist a nonnegative integer a for which the equation
$$\left\lfloor \frac{m}{1} \right\rfloor + \left\lfloor \frac{m}{2} \right\rfloor + \left\lfloor \frac{m}{3} \right\rfloor + \cdots + \left\lfloor \frac{m}{m} \right\rfloor = n^2 + a$$
has more than one million different solutions (m, n) where m and n are positive integers?

8. (EGMO 2021) Let ABC be a triangle with incenter I and let D be an arbitrary point on the side BC. Let the line through D perpendicular to BI intersect CI at E. Let the line through D perpendicular to CI intersect BI at F. Prove that the reflection of A across the line EF lies on the line BC.

9. Let $N = 10^{2021}$. There are 2021 concentric circles centered at O, and N equally-spaced rays are emitted from point O. Among the $2021N$ intersections of the circles and the rays, some are painted red while the others remain unpainted.

   It is known that, no matter how one intersection point from each circle is chosen, there is an angle θ such that after a rotation of θ with respect to O, all chosen points are moved to red points. Given this information, find the smallest possible number of red points.

10. Let ABC be a triangle with incircle ω and circumcircle Γ. Define I to be the center of ω and D, E, F to be the points where ω touches BC, CA, AB respectively. Let line AI meet BC at J and Γ again at M, and let N be the point diametrically opposite M on Γ. Suppose that K is a point on ID such that JK is parallel to MD. Let the perpendicular to AI passing through K intersect IN at Q. Prove that $DQ \perp EF$.

[†] A tournament on n vertices is a directed graph such that for any pair of distinct vertices a, b, exactly one of $a \to b$ and $b \to a$ is an edge.

§2 Problems for returning students

The problems in this section are targeted at students who have previous MOP experience, but some of them should be accessible to new students too.

11. Three graders A, B, and C are called, in some order, True, False, and Random. True always speaks truly, False always speaks falsely, but whether Random speaks truly or falsely is a completely random matter. Your task is to determine the identities of A, B, and C by asking three yes-no questions; each question must be put to exactly one grader. The graders understand English, but will answer all questions in their own language, in which the words for "yes" and "no" are "combo" and "geo," in some order. You do not know which word means which, and you also cannot ask questions for which no answer can be given (e.g. "is this sentence false?").

12.
    - Fix $k \geq 2$. Are there infinitely many solutions (n, a, b) to $a^k + b^k = n!$ where $\gcd(a, b) = 1$, $n \geq 3$, and $a, b \in \mathbb{Z}$?
    - What about solutions (k, n, a, b) where $k \geq 2$ is allowed to vary?

13. Let S be a set of size n, and let $f_1, f_2, \dots, f_k$ be functions from $S \to S$. A *synchronizing sequence of length* ℓ is a sequence of integers $1 \leq i_1, i_2, \dots, i_\ell \leq k$ such that $f_{i_1} \circ f_{i_2} \circ \cdots \circ f_{i_\ell}(s)$ is the same for all $s \in S$, i.e. is a constant function.

    Show that if a synchronizing sequence exists, then there exists one of length at most n^3.

14. (China TST) Let $f(x)$ and $g(x)$ be polynomials with integer coefficients. Suppose that for infinitely many primes p, there exists an integer m_p such that
$$f(a) \equiv g(a + m_p) \pmod{p}$$
for every integer a. Prove there exists $r \in \mathbb{Q}$ such that $f(x) \equiv g(x + r)$.

15. Let ABC be a triangle and P be a point on (ABC). Line ℓ meets PA, PB, PC at A', B', C', respectively. Points D, E, F lie on (ABC) such that the lines AD, BE, CF and ℓ are concurrent. Prove that circumcircle of the triangle formed by the lines passing through A', B', C' and parallel to EF, FD, DE, respectively, is tangent to (ABC).

16. Exhibit a set S of 100-bit binary strings such that $|S| = 2^{99} + 1$ and, for each $s \in S$, there are at most 10 strings $t \in S$ for which s and t differ in exactly one character.

17. Let $p_1, \dots, p_n$ and $q_1, \dots, q_n$ be positive real numbers such that $p_1 + \cdots + p_n = q_1 + \cdots + q_n = 1$. Prove that
$$\left(\sum_i |p_i - q_i| \right)^2 \leq 2 \sum_i p_i \log \frac{p_i}{q_i}.$$

18. (almost HMIC 2021) In an $n \times n$ square grid, n squares are marked so that every rectangle composed of exactly n grid squares contains at most one marked square. Determine all possible values of n.

§3 Advanced problems

The problems in this section are more difficult and targeted at students who can solve about half the problems on TSTST 2020.

19. An $n \times m$ matrix is *nice* if it contains every integer from 1 to mn exactly once and 1 is the only entry which is the smallest both in its row and in its column. Prove that the number of $n \times m$ nice matrices is $(nm)!n!m!/(n+m-1)!$.

20. Let a, b, c, d, e, f be fixed positive integers. For $r, s \geq 0$, define
$$T_{rs} = X_1^r Y_1^s + X_2^r Y_2^s + X_3^r Y_3^s.$$
Show that the polynomial
$$\begin{aligned} X_1^a X_2^b X_3^c Y_1^d Y_2^e Y_3^f + X_1^a X_2^c X_3^b Y_1^d Y_2^f Y_3^e + X_1^b X_2^a X_3^c Y_1^e Y_2^d Y_3^f \\ + X_1^b X_2^c X_3^a Y_1^e Y_2^f Y_3^d + X_1^c X_2^a X_3^b Y_1^f Y_2^d Y_3^e + X_1^c X_2^b X_3^a Y_1^f Y_2^e Y_3^d \end{aligned}$$
can be expressed as a polynomial in $T_{00}, T_{01}, T_{02}, T_{03}, T_{10}, T_{11}, T_{12}, T_{20}, T_{21}, T_{30}$ with complex coefficients. (T_{12} is "tee-one-two", not "tee-twelve".)

21. Find all functions $f\colon \mathbb{Z} \to \mathbb{R}$ satisfying
$$f(a) + f(b) = f(c) + f(d)$$
whenever a, b, c, d are integers such that $a^2 + b^2 = c^2 + d^2$.

22. Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ satisfying
$$f(f(x)y - f(y)x) = f(f(x))y - f(x)f(y)$$
for all real x and y.

23. Let A, B, C, D, E, F be points on a conic which is not a rectangular hyperbola such that ABC and DEF share a common orthocenter H. Show that H lies on the radical axis of (ABC) and (DEF).

24. Let $n \geq 3$ be a positive integer. Consider an infinite grid of square unit cells. Find, in terms of n, the greatest positive integer C which satisfies the following condition: For every colouring of the cells of the grid in n colours, there is some polyomino[‡] within the grid which contains at most $n-1$ colours and whose area is at least C.

[‡]A polyomino is a figure which consists of unit squares joined together by their sides. (A polyomino may contain holes.)

§4 USA Remixed Math Olympiad

The problems in this section vary in difficulty. Not all of them have known solutions.

25. Find all functions $f\colon \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ satisfying $f(a^2+b^2) = f(a)f(b)$ for all positive integers a and b.

26. Rectangles BCC_1B_2, CAA_1C_2, and ABB_1A_2 are erected outside an acute triangle ABC. Suppose that lines B_1C_2, C_1A_2, and A_1B_2 are concurrent. Prove that the areas of triangles AA_1A_2, BB_1B_2, CC_1C_2 sum to the area of triangle ABC.

27. The Planar National Park is a undirected 3-regular planar graph (i.e. all vertices have degree 3) on n vertices. A visitor walks through the park as follows: she begins at a vertex and starts walking along an edge. When she reaches the other endpoint, she turns left. On the next vertex she turns right, and so on, alternating left and right turns at each vertex. She does this until she gets back to the vertex where she started.

    Prove that there exists a park layout with the property that, during the visitor's journey, she visits 99% of the vertices three or more times.

28. Let $n \geq 2$ be an integer. An $n \times n$ board is initially empty. Each minute, you may perform one of three moves:

    - If there is an L-shaped tromino region of three cells without stones on the board (rotations *are* allowed), you may place a stone in each of those cells.
    - If all cells in a column have a stone, you may remove all stones from that column.
    - If all cells in a row have a stone, you may remove all stones from that row.

    For which n is it possible that, after some non-zero number of moves, the board has no stones?

29. Classify all finite sets S of integers such that for each $s \in S$, if we pick $x \in S$ randomly, then $\gcd(x, s)$ is equally likely to equal each divisor of s.

30. Let n be a fixed positive integer. Carina has four pins, labeled A, B, C, D, respectively, located at the origin of the coordinate plane. In a *move*, Carina may move a pin to an adjacent lattice point at distance 1 away. What is the least number of moves that Carina can make in order for the convex hull of A, B, C, D to have area n?

31. Classify all infinite sequences of positive real numbers $\dots, a_{-1}, a_0, a_1, a_2, \dots$ such that for every integer i,

    $$a_i = \begin{cases} a_{i-1} + a_{i+1} & \text{if } i \text{ is even} \\ \frac{1}{a_{i-1}} + \frac{1}{a_{i+1}} & \text{if } i \text{ is odd.} \end{cases}$$

32. Let $ABCDEF$ be a convex hexagon satisfying $\overline{AB} \parallel \overline{DE}$, $\overline{BC} \parallel \overline{EF}$, and $\overline{CD} \parallel \overline{FA}$. Let X, Y, and Z be the midpoints of $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$.

    Suppose that the orthocenter of $\triangle XYZ$ is the midpoint of the circumcenter of $\triangle ACE$ and the circumcenter of $\triangle BDF$. Decide whether it follows that

    $$AB \cdot DE = BC \cdot EF = CD \cdot FA.$$