# MOP 2020 Homework Problems

Thug Raiders

July 2020

## Instructions

> It is time to work now, not guess partials. You need 7s on TST, and we all need 7s in the IMO.
>
> Zuming Feng, 2008

Hello hello hey,

Congratulations on your excellent performance on the USA(J)MOO, which has earned you an invitation to attend the Math Olympiad Summer Program! This program will be an intense and challenging opportunity for you to learn a tremendous amount of mathematics.

In normal years, we provide a set of 20 or so homework problems to the newly invited MOPpers to give them some math problems to talk about on the way to Pittsburgh; the solutions are discussed during the first day of classes. Well, this year the timeline is too tight for this to work, but apparently some people said they wanted some homework problems anyways, so we compiled this list of about 15 problems of various difficulties so you have something to think about before the camp actually starts. However, **the problems are optional this year**. Nonetheless, we encourage you to discuss the problems with each other.

The first five problems are intended to be accessible to all, while more experienced students are encouraged to work instead on the last half of problems. Of course, all students are welcome to attempt any and all problems.*

This year's homework also contains variants and generalizations of the problems from this year's USA(J)MO. This is because we want you to think about how the problems and solutions you've already done might be stretched, beyond just what gets 7 points. Also, they're sort of funny. Fair warning: we don't know solutions to all of these.

You're welcome to share the MOP homework problems with your friends as well; unlike most MOP materials, the homework is not considered "internal use only".

Happy solving!

*Those of you on the IMO team, or close to it, may recognize a large number of these problems; in that case, don't let the lack of homework distract you from your usual practice routine.

## §1 Problems

1. A convex quadrilateral $ABCD$ of unit area is inscribed in a unit circle. Diagonals $AC$ and $BD$ intersect at $P$. What is the minimum possible area of the quadrilateral formed by the circumcenters of $PAB$, $PBC$, $PCD$, and $PDA$?

2. (Based on discussions with Alex Zhai and Walter Stromquist) An empty $99 \times 99 \times 99$ cube is given, and a $99 \times 99$ grid of square unit cells is drawn on each of its six faces. A *beam* is a $1 \times 1 \times 99$ rectangular prism. We wish to place some beams in the cube such that:

   - The two $1 \times 1$ faces of each beam coincide with unit cells lying on opposite faces of the cube. (Hence, there are $3 \cdot 99^2$ possible positions for a beam.)
   - No two beams have intersecting interiors.
   - The interiors of each of the four $1 \times 99$ faces of each beam touch either a face of the cube or the interior of the face of another beam.

   Find the minimum possible nonzero number of beams that can be placed, and calculate the number of ways this minimum could be achieved.

3. (EGMO 2020) The positive integers $a_0, a_1, a_2, \ldots, a_{3030}$ satisfy

   $$2a_{n+2} = a_{n+1} + 4a_n \text{ for } n = 0, 1, 2, \ldots, 3028.$$

   Prove that at least one of the numbers $a_0, a_1, a_2, \ldots, a_{3030}$ is divisible by $2^{2020}$.

4. (Canada 2020) A set $S$ contains $n \geq 3$ distinct positive real numbers. Show that there are at most $n - 2$ distinct integer powers of three that can be written as the sum of three distinct elements from $S$.

5. (2016 ELMO Longlist C5, by Michael Ren) Carl is writing subsets of $\{1, 2, 3, ..., n\}$ on a blackboard. He initially writes $N$ distinct two-element subsets on the board. For $k \geq 3$, he writes a $k$-element subset of $\{1, 2, 3, ..., n\}$ on the blackboard if at least $k - 1$ of its subsets with $k - 1$ elements are already written on the board and he continues with this process until he can't write any more subsets. Determine the least positive integer $N$ such that for any choice of the initial $N$ two-element subsets, Carl will write $\{1, 2, 3, \ldots, n\}$ on the board.

6. Draw a segment between $(a, b)$ and $(c, d)$ whenever $a$, $b$, $c$, $d$ are nonnegative integers satisfying $|ad - bc| = 1$. Prove that no two drawn segments intersect (except at endpoints).

7. (Based on discussions with Evan Chen and Ashwin Sah) Let $n \geq 2$ be an integer. Suppose $P(x_1, \ldots, x_n)$ is an $n$-variable polynomial with real coefficients which vanishes whenever two of its arguments are equal.

   (a) Find all such $P$ which have exactly $n!$ terms.

   (b) If $P$ has strictly more than $n!$ terms, what is the smallest number of terms it could have?

8. (Dorlir Ahmeti and Alex Gunning) Show that for any cyclic hexagon $ABCDEF$ we have:

   $$\sqrt[3]{AD \cdot BE \cdot CF} \geq \sqrt[3]{AB \cdot CD \cdot EF} + \sqrt[3]{BC \cdot DE \cdot FA}$$

   with equality if and only if the lines $AB, CF, DE$ are concurrent or parallel; $BC, AD, EF$ are concurrent or parallel and $CD, BE, FA$ are concurrent or parallel.

9. (Bulgaria) Let $n$ be a positive integer. Find the number of square-free positive integers $a$ such that $\left\lfloor \frac{n}{\sqrt{a}} \right\rfloor$ is odd.

10. A collection of countably many regular pentagons is given with total area 2020. Show that they can be arranged as to completely cover a regular pentagon of area 1.

11. Suppose that $(a_1, b_1)$, $(a_2, b_2)$, $\dots$, $(a_{100}, b_{100})$ are distinct ordered pairs of non-negative **reals**. Let $N$ denote the number of pairs of integers $(i, j)$ satisfying $1 \le i < j \le 100$ and $|a_i b_j - a_j b_i| = 1$. Determine the largest possible value of $N$ over all possible choices of the 100 ordered pairs.

12. Call a function from $f : \mathbb{N} \to \mathbb{N}$ *beautiful* if

$$S(\gcd(f(x), f(y))) = S(\gcd(x, y)) \quad \text{and} \quad S(\operatorname{lcm}(f(x), f(y))) = S(\operatorname{lcm}(x, y))$$

for $x, y \in \mathbb{N}$. Find all beautiful functions. (Here $\mathbb{N}$ is the set of positive integers and $S(x)$ is the sum of digits of $x$.)

13. (Bluher 2017) Let $p > 3$ be a prime. An integer $x$ is called a quadratic non-residue if $p$ does not divide $x - t^2$ for any integer $t$.

    Denote by $A$ the set of all integers $a$ such that $1 \le a < p$, and both $1 - a$ and $3 + a$ are quadratic non-residues. Calculate the remainder when the product of the elements of $A$ is divided by $p$.

14. Let $n$ be a fixed positive integer. Show that for all sufficiently large primes $p \equiv 1 \pmod 3$, there exist $n$ consecutive nonzero cubic residues modulo $p$.

    (In other words: there exists $N$, dependent on $n$, such that for all primes $p > N$ satisfying $p \equiv 1 \pmod 3$, there exists $a$, dependent on $p$, such that $a+1, a+2, a+3, \dots, a+n$ are each cubes of nonzero residues modulo $p$.)

15. (Taiwan TST 2019, by Evan Chen, almost accidentally used as USAMO 2020/6) Triangle $ABC$ is inscribed in an ellipse with foci $H$ and $K$. Prove that if $H$ is the orthocenter of $\triangle ABC$ then the incenter of $\triangle ABC$ lies on line $HK$.