MOP 2019 Homework Problems

Evan Chen

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Instructions

> "A lot of progress" usually mean a *wonderful* 2 at IMO. Man, let's work hard and solve those problems completely. Do not expect to get 3 or 4's on the IMO. Instead, get 7's!!! You can do it, man!
>
> Zuming Feng, 2008

Hello hello,

Congratulations on your excellent performance on the USA(J)MO, which has earned you an invitation to attend the Math Olympiad Summer Program! This program will be an intense and challenging opportunity for you to learn a tremendous amount of mathematics.

To celebrate your achievement, and give you something to talk about on the way to Pittsburgh, we invite you to work on the following homework problems. The solutions will be discussed in the first day of class at MOP.

The first half problems in most sections are intended to be accessible to all, while more experienced students are encouraged to work instead on the last half of problems from each section. Of course, all students are welcome to attempt any and all problems.*

This year's homework also contains variants and generalizations of the problems from this year's USA(J)MO. This is because we want you to think about how the problems and solutions you've already done might be stretched, beyond just what gets 7 points. Also, they're sort of funny. Fair warning: we don't know solutions to all of these.

One change from past years: in the past MOP homework was supposed to be for internal use, for no really good reason. But then I decided I was having too much fun writing these, so starting from this year you're welcome to share the MOP homework problems with your friends as well.

Happy solving!

*Those of you on the IMO team, or close to it, may recognize a large number of these problems; in that case, don't let the lack of homework distract you from your usual practice routine.

§1 Algebra

Problem A1 (JMO2'). For any pair of integers (a, b), describe all pairs of functions $f\colon \mathbb{Z} \to \mathbb{Z}$ and $g\colon \mathbb{Z} \to \mathbb{Z}$ which obey

$$f(g(x)) = x + a \quad \text{and} \quad g(f(x)) = x + b$$

for all integers x.

Problem A2 (USAMO1'). Find all functions $f\colon \mathbb{N} \to \mathbb{N}$ satisfying

$$\underbrace{f(f(\dots f}_{n \text{ times}}(n)\dots)) = \frac{n^2}{f(f(n))}$$

for all positive integers n.

Problem A3 (USMCA 2019/4). Solve over $\mathbb{R}$ the functional equation

$$[f(f(x) + y)]^2 = (x - y)\,(f(x) - f(y)) + 4f(x)f(y).$$

Problem A4 (Putnam 2018 B4). Given a real number a, we define a sequence by $x_0 = 1$, $x_1 = x_2 = a$, and $x_{n+1} = 2x_n x_{n-1} - x_{n-2}$ for $n \ge 2$. Prove that if $x_n = 0$ for some n, then the sequence is periodic.

Problem A5 (Old version of USMCA 2019/6). Let n be a positive integer. Prove that if f is a real polynomial satisfying $x^{2n} f(1/x) = f(x)$, and the x^n coefficient of f is zero, then f has a complex root z with $|z| = 1$.

Problem A6 (USMCA 2019/6). A *mirrored polynomial* is a polynomial f of degree 100 with real coefficients such that the x^{50} coefficient of f is 1, and $f(x) = x^{100} f(1/x)$ holds for all real nonzero x. Find the smallest real constant C such that any mirrored polynomial f satisfying $f(1) \ge C$ has a complex root z obeying $|z| = 1$.

Problem A7 (TSTST 2013/2). A finite sequence of integers $a_1, a_2, \dots, a_n$ is called *regular* if there exists a real number x satisfying

$$\lfloor kx \rfloor = a_k \quad \text{for } 1 \le k \le n.$$

Given a regular sequence $a_1, a_2, \dots, a_n$, for $1 \le k \le n$ we say that the term a_k is *forced* if the following condition is satisfied: the sequence

$$a_1,\ a_2,\ \dots,\ a_{k-1},\ b$$

is regular if and only if $b = a_k$.

Find the maximum possible number of forced terms in a regular sequence with 1000 terms.

§2 Combinatorics

Problem C1 (EGMO 2019/5). Let $n \ge 2$ be an integer, and let $a_1, a_2, \dots, a_n$ be positive integers. Show that there exist positive integers $b_1, b_2, \dots, b_n$ satisfying the following three conditions:

(a) $a_i \le b_i$ for $i = 1, 2, \dots, n$;

(b) the remainders of b_1, b_2, $\dots$, b_n on division by n are pairwise different,

(c) $b_1 + \dots + b_n \le n\left(\frac{n-1}{2} + \left\lfloor \frac{a_1+\dots+a_n}{n} \right\rfloor\right)$.

Problem C2 (EGMO 2019/2)**.** Let n be a positive integer. Dominoes are placed on a $2n \times 2n$ board in such a way that every cell of the board is (orthogonally) adjacent to exactly one cell covered by a domino. For each n, determine the largest number of dominoes that can be placed in this way.

Problem C3 (IMO 1999/3)**.** Let n be an even positive integer. Find the minimal number of cells on the $n \times n$ board that must be marked so that any cell (marked or not marked) has a marked neighboring cell.

Problem C4 (Putnam 2018 B6)**.** Prove that the number of length 2018-tuples whose entries are in $\{1, 2, 3, 4, 5, 6, 10\}$ and sum to 3860, is at most

$$2^{3860} \cdot \left(\frac{2018}{2048}\right)^{2018}.$$

Problem C5 (USMCA 2019/5)**.** The number 2019 is written on a blackboard. Every minute, if the number a is written on the board, Evan erases it and replaces it with a number chosen from the set

$$\{0, 1, 2, \dots, \lceil 2.01a \rceil\}$$

uniformly at random (here $\lceil \bullet \rceil$ is the ceiling function). Is there an integer N such that the board reads 0 after N steps with at least 99% probability?

Problem C6 (USAMO4')**.** Let n be a positive integer. Determine the number of ways to choose sets $S_{ijk} \subseteq \{1, 2, \dots, 3n\}$, for all $0 \le i, j, k \le n$, such that

- $|S_{ijk}| = i + j + k$, and
- $S_{ijk} \subseteq S_{pqr}$ if $0 \le i \le p \le n$, $0 \le j \le q \le n$, and $0 \le k \le r \le n$.

Problem C7 (Misread USAMO4')**.** What if the sets in USAMO4' must be distinct?

§3 Geometry

Problem G1 (JMO4')**.** Let ABC be a triangle with $\angle B > 90^\circ$ and let E and F be the feet of the altitudes from B and C. Can line EF be tangent to the incircle?

Problem G2 (USMCA 2019/3)**.** Let ABC be a scalene triangle. The incircle of ABC touches $\overline{BC}$ at D. Let P be a point on $\overline{BC}$ satisfying $\angle BAP = \angle CAP$, and M be the midpoint of $\overline{BC}$. Define Q to be on $\overline{AM}$ such that $\overline{PQ} \perp \overline{AM}$. Prove that the circumcircle of $\triangle AQD$ is tangent to $\overline{BC}$.

Problem G3 (JMO4' x TST6)**.** Let ABC be a triangle and let E and F be the feet of the altitudes from B and C. Assume line EF is tangent to the incircle of ABC.

Let the excircle of triangle ABC opposite the vertex A be tangent to $\overline{BC}$ at point A_1. Define points B_1 on $\overline{CA}$ and C_1 on $\overline{AB}$ analogously, using the excircles opposite B and C, respectively. Prove that points A, A_1, B_1, C_1 are concyclic.

Problem G4 (USAMO2')**.** Let $ABCD$ be a cyclic (convex) quadrilateral. The diagonals of $ABCD$ intersect at E. Suppose there is a point P on side $\overline{AB}$ satisfying $\angle APD = \angle BPC$, and PE bisects $\overline{CD}$. Prove that either $AD = BC$, or $AD^2 + BC^2 = AB^2$.

§4 Number theory

Problem N1 (USAMO5'). Two rational numbers a and b are written on a blackboard. At any point, Evan may take two of the numbers x and y written on the board and write either their arithmetic mean $\frac{1}{2}(x+y)$ or their harmonic mean $\frac{2xy}{x+y}$. For which (a,b) can Evan write the geometric mean $\sqrt{ab}$ on the board in finitely many steps?

Problem N2 (USMCA 2019/1). Kelvin the Frog and Alex the Kat are playing a game on an initially empty blackboard. Kelvin begins by writing a digit. Then, the players alternate inserting a digit anywhere into the number currently on the blackboard, including possibly a leading zero (e.g. 12 can become 123, 142, 512, 012, etc.). Alex wins if the blackboard shows a perfect square at any time, and Kelvin's goal is prevent Alex from winning. Does Alex have a winning strategy?

Problem N3 (Putnam 2015 A2). Define $a_0 = 1$, $a_1 = 2$ and $a_n = 4a_{n-1} - a_{n-2}$. Find an odd prime dividing a_{2015}.

Problem N4 (TSTST 2013/5). Let p be a prime. Prove that in a complete graph with $1000p$ vertices whose edges are labelled with integers, one can find a cycle whose sum of labels is divisible by p.

Problem N5 (USAMO3'). Let L be the set of positive integers containing the decimal digit 7. Determine all polynomials $f(x)$ with nonnegative coefficients such that $f(x) \in L$ for all $x \in L$.

§5 USA TST for IMO 2019

Problem TST1 (M. Staps). Let ABC be a triangle and let M and N denote the midpoints of $\overline{AB}$ and $\overline{AC}$, respectively. Let X be a point such that $\overline{AX}$ is tangent to the circumcircle of triangle ABC. Denote by ω_B the circle through M and B tangent to $\overline{MX}$, and by ω_C the circle through N and C tangent to $\overline{NX}$. Show that ω_B and ω_C intersect on line BC.

Problem TST2 (A. Sah and Y. Liu). Let $\mathbb{Z}/n\mathbb{Z}$ denote the set of integers considered modulo n (hence $\mathbb{Z}/n\mathbb{Z}$ has n elements). Find all positive integers n for which there exists a bijective function $g \colon \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$, such that the 101 functions

$$g(x), \quad g(x)+x, \quad g(x)+2x, \quad \dots, \quad g(x)+100x$$

are all bijections on $\mathbb{Z}/n\mathbb{Z}$.

Problem TST3 (N. Beluhov). A *snake of length k* is an animal which occupies an ordered k-tuple $(s_1, \dots, s_k)$ of cells in an $n \times n$ grid of square unit cells. These cells must be pairwise distinct, and s_i and s_{i+1} must share a side for $i = 1, \dots, k-1$. If the snake is currently occupying $(s_1, \dots, s_k)$ and s is an unoccupied cell sharing a side with s_1, the snake can *move* to occupy $(s, s_1, \dots, s_{k-1})$ instead. The snake has *turned around* if it occupied $(s_1, s_2, \dots, s_k)$ at the beginning, but after a finite number of moves occupies $(s_k, s_{k-1}, \dots, s_1)$ instead.

Determine whether there exists an integer $n > 1$ such that one can place some snake of length at least $0.9n^2$ in an $n \times n$ grid which can turn around.

Problem TST4 (A. Bhattacharya). We say a function $f \colon \mathbb{Z}_{\ge 0} \times \mathbb{Z}_{\ge 0} \to \mathbb{Z}$ is *great* if for any nonnegative integers m and n,

$$f(m+1,n+1)f(m,n) - f(m+1,n)f(m,n+1) = 1.$$

If $A = (a_0, a_1, \dots)$ and $B = (b_0, b_1, \dots)$ are two sequences of integers, we write $A \sim B$ if there exists a great function f satisfying $f(n, 0) = a_n$ and $f(0, n) = b_n$ for every nonnegative integer n (in particular, $a_0 = b_0$).

Prove that if A, B, C, and D are four sequences of integers satisfying $A \sim B$, $B \sim C$, and $C \sim D$, then $D \sim A$.

Problem TST5 (Y. Yao)**.** Let n be a positive integer. Tasty and Stacy are given a circular necklace with $3n$ sapphire beads and $3n$ turquoise beads, such that no three consecutive beads have the same color. They play a cooperative game where they alternate turns removing three consecutive beads, subject to the following conditions:

- Tasty must remove three consecutive beads which are turquoise, sapphire, and turquoise, in that order, on each of his turns.
- Stacy must remove three consecutive beads which are sapphire, turquoise, and sapphire, in that order, on each of her turns.

They win if all the beads are removed in $2n$ turns. Prove that if they can win with Tasty going first, they can also win with Stacy going first.

Problem TST6 (A. Bhattacharya)**.** Let ABC be a triangle with incenter I, and let D be a point on line BC satisfying $\angle AID = 90^\circ$. Let the excircle of triangle ABC opposite the vertex A be tangent to $\overline{BC}$ at point A_1. Define points B_1 on $\overline{CA}$ and C_1 on $\overline{AB}$ analogously, using the excircles opposite B and C, respectively.

Prove that if quadrilateral $AB_1A_1C_1$ is cyclic, then $\overline{AD}$ is tangent to the circumcircle of $\triangle DB_1C_1$.

§6 Extra

Problem X1. Let a_1, a_2, … be a sequence of nonnegative integers which satisfies the recurrence

$$a_{n+2} = a_n a_{n+1} + 1$$

for $n = 1, 2, \dots$. Prove that $a_{2018} - 22$ is not a prime.

Problem X2 (EGMO 2019/6)**.** On a circle, Alina draws 2019 chords, the endpoints of which are all different. A point is considered marked if it is either

(i) one of the 4038 endpoints of a chord; or

(ii) an intersection point of at least two chords.

Of the 4038 points meeting criterion (i), Alina labels 2019 points with a 0 and the other 2019 points with a 1. She labels each point meeting criterion (ii) with an arbitrary integer (not necessarily positive).

Along each chord, Alina considers the segments connecting two consecutive marked points. (A chord with k marked points has $k-1$ such segments.) She labels each such segment in yellow with the sum of the labels of its two endpoints and in blue with the absolute value of their difference. Alina finds that the $N+1$ yellow labels take each value $0, 1, \dots, N$ exactly once. Show that at least one blue label is a multiple of 3.

Problem X3. Amber and Canmoo encounter a sloth. What is the sloth's name?