MOP 2018 Homework Problems

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Instructions

Hello hello,

Congratulations on your excellent performance on the USA(J)MO, which has earned you an invitation to attend the Math Olympiad Summer Program! This program will be an intense and challenging opportunity for you to learn a tremendous amount of mathematics.

To celebrate your achievement, and give you something to talk about on the way to Pittsburgh, you are asked to work on the following homework problems. The solutions will be discussed in the first day of class at MOP.

The first two problems in each topic section are intended to be accessible to all, while experienced students (who typically score 21 or higher on USAMO/IMO) are encouraged to work instead on the last half of problems from each section. Of course, all students are welcome to attempt any and all problems.

This year's problems are adapted from the following sources:

- European Girl's Math Olympiad 2018
- USA Team Selection Test for IMO 2018
- ELMO Shortlist 2013
- USA TST Selection Test 2011
- USA(J)MO 2018 adaptations
- Last year's MOP tests

When possible author information about the problems has been provided. Happy solving!

§1 Algebra

Problem A1. Consider the set

$$A = \left\{ 1 + \frac{1}{k} : k = 1, 2, 3, \dots \right\}.$$

For every integer $x \geq 2$, let $f(x)$ denote the minimum integer such that x can be written as the product of $f(x)$ elements of A (not necessarily distinct). Prove that there are infinitely many pairs of integers $x \geq 2$ and $y \geq 2$ for which

$$f(xy) < f(x) + f(y).$$

Problem A2. Fix a real number $0 < t < \frac{1}{2}$.

(a) Prove that there exists a positive integer n such that for every set S of n positive integers, the following holds: there exist distinct $x, y \in S$ and *nonnegative* integer $m \geq 0$ such that $|x - my| \leq ty$.

(b) Determine whether there exists an infinite set S of positive integers such that the following holds: for any distinct $x, y \in S$ and *positive* integer $m > 0$, we have $|x - my| > ty$.

Problem A3. Find the smallest constant $C > 0$ for which the following statement holds: among any five distinct positive real numbers, one can label four different ones as p, q, r, s such that

$$\left| \frac{p}{q} - \frac{r}{s} \right| \leq C.$$

Problem A4. Find all real-valued functions f defined on pairs of real numbers, having the following property: for all real numbers a, b, c, the median of $f(a, b), f(b, c), f(c, a)$ equals the median of a, b, c.

(The *median* of three real numbers, not necessarily distinct, is the number that is in the middle when the three numbers are arranged in nondecreasing order.)

Problem A5 (Evan Chen)**.** Let $\mathbb{N}$ denote the set of positive integers, and for a function f, let $f^k(n)$ denote the function f applied k times. Call a function $f : \mathbb{N} \to \mathbb{N}$ *saturated* if

$$f^{f^{f(n)}(n)}(n) = n$$

for every positive integer n. Find all positive integers m for which the following holds: every saturated function f satisfies $f^{2014}(m) = m$.

§2 Combinatorics

Problem C1. Let $n \geq 3$ be an integer. Several non-overlapping dominoes are placed on an $n \times n$ board. The *value* of a row or column is the number of dominoes that cover at least one cell of that row column. A domino configuration is called *balanced* if there exists some $k \geq 1$ such that every row and column has value k.

Prove that a balanced configuration exists for every $n \geq 3$ and find the minimum number of dominoes needed in such a configuration.

Problem C2 (Gregory Galperin)**.** For which integers $n \ge 3$ can one find a triangulation of regular n-gon consisting only of isosceles triangles?

Problem C3. The n contestants of EGMO are named C_1, C_2, ..., C_n. After the competition, they queue in front of the restaurant according to the following rules.

- The Jury chooses the initial order of the contestants in the queue.
- Every minute, the Jury chooses an integer i with $1 \le i \le n$.
  - If contestant C_i has at least i other contestants in front of her, she pays one euro to the Jury and moves forward in the queue by exactly i positions.
  - If contestant C_i has fewer than i other contestants in front of her, the restaurant opens and the process ends.

For every n, prove that this process must terminate and determine the maximum number of euros that the Jury can collect by cunningly choosing the initial order and the sequence of moves.

Problem C4. At a certain orphanage, every pair of orphans are either friends or enemies. For every three of an orphan's friends, an even number of pairs of them are enemies. Prove that it's possible to assign each orphan two parents such that every pair of friends shares exactly one parent, but no pair of enemies does, and no three parents are in a love triangle (where each pair of them has a child).

Problem C5 (Maria Monks, JMO6 variaton mentioned by Walter Stromquist)**.** Karl and his younger sister Kate play a game. Karl starts with n cards labeled $1, 2, \dots n$, and lines them up on his desk in an order of his choice. We call a pair (a, b) of cards *swapped* if $a > b$ and the card labeled a is to the left of the card labeled b.

The game then proceeds in n turns; on the Cth turn the following two operations happen:

- Karl picks up the card labeled C and inserts it back into the sequence in the opposite position (if the card labeled C had i cards to its left, then it now has i cards to its right).
- Kate then picks up the card labeled C and inserts it wherever she likes. If she moves it $i \ge 0$ positions to the left, she earns i points. If she moves it $i \ge 0$ positions to the right instead, she loses i points instead.

At the end of the game, Karl's score is the number of swapped pairs in the final lineup minus the number of swapped pairs in the starting lineup. The player with the larger score wins (or they tie). Which sibling has the winning strategy?

§3 Geometry

Problem G1. Let Γ be the circumcircle of triangle ABC. A circle Ω is tangent to the line segment AB and is tangent to Γ at a point lying on the same side of the line AB as C. The angle bisector of $\angle BCA$ intersects Ω at two different points P and Q. Prove that $\angle ABP = \angle QBC$.

Problem G2 (Michael Kural)**.** Let ABC be a scalene triangle with circumcircle Γ, and let D, E, F be the points where its incircle meets BC, AC, AB respectively. Let the circumcircles of $\triangle AEF$, $\triangle BFD$, and $\triangle CDE$ meet Γ a second time at X, Y, Z, respectively. Show that there exists a point P such that $\angle PAX = \angle PBY = \angle PCZ = 90°$.

Problem G3 (Allen Liu)**.** In non-right triangle ABC, a point D lies on line $\overline{BC}$. The circumcircle of ABD meets $\overline{AC}$ at F (other than A), and the circumcircle of ADC meets $\overline{AB}$ at E (other than A). Prove that as D varies, the circumcircle of AEF always passes through a fixed point other than A, and that this point lies on the median from A to $\overline{BC}$.

Problem G4. Let ABC be a triangle. Its excircles touch sides BC, CA, AB at D, E, F. Prove that the perimeter of triangle ABC is at most twice that of triangle DEF.

Problem G5 (Zuming Feng)**.** Let ABC be an acute, scalene triangle, and let M, N, and P be the midpoints of $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$, respectively. Let the perpendicular bisectors of $\overline{AB}$ and $\overline{AC}$ intersect ray AM in points D and E respectively, and let lines BD and CE intersect in point F, inside triangle ABC. Prove that points A, N, F, and P all lie on one circle.

§4 Number Theory

Problem N1 (Titu Andreescu)**.** Prove that for each positive integer n, there are pairwise relatively prime integers $k_0, \dots, k_n$, all strictly greater than 1, such that $k_0 k_1 \dots k_n - 1$ is the product of two consecutive integers.

Problem N2. Find the smallest prime number not of the form $|2^a - 3^b|$, where a and b are nonnegative integers.

Problem N3 (Ankan Bhattacharya; USAMO4 extension)**.** Find all odd positive integers n with the following property: if $a_1, \dots, a_n$ are integers, then there exists an integer k such that the numbers

$$a_1 + k, a_2 + 2k, \dots, a_n + nk$$

produce at least $\frac{1}{2}n$ distinct remainders upon division by n.

Problem N4. Let $x_0, x_1, \dots, x_{n_0 - 1}$ be integers, and let $d_1, d_2, \dots, d_k$ be positive integers with $n_0 = d_1 > d_2 > \dots > d_k$ and $\gcd(d_1, d_2, \dots, d_k) = 1$. For every integer $n \ge n_0$, define

$$x_n = \left\lfloor \frac{x_{n-d_1} + x_{n-d_2} + \dots + x_{n-d_k}}{k} \right\rfloor.$$

Show that the sequence (x_n) is eventually constant.

Problem N5 (Ivan Borsenco; USAMO3 converse)**.** Let $n \ge 2$ be an integer, and let $\{a_1, \dots, a_m\}$ denote the $m = \varphi(n)$ integers less than n and relatively prime to n. Suppose that m divides $a_1^k + \dots + a_m^k$ for every positive integer k. Prove that every prime divisor of m also divides n.

§5 USA TST for IMO 2018

Problem TST1 (Ashwin Sah)**.** Let $n \ge 2$ be a positive integer, and let $\sigma(n)$ denote the sum of the positive divisors of n. Prove that the n^{th} smallest positive integer relatively prime to n is at least $\sigma(n)$, and determine for which n equality holds.

Problem TST2 (Yang Liu and Michael Kural)**.** Find all functions $f \colon \mathbb{Z}^2 \to [0,1]$ such that for any integers x and y,

$$f(x,y) = \frac{f(x-1,y) + f(x,y-1)}{2}.$$

Problem TST3 (Evan Chen)**.** At a university dinner, there are 2017 mathematicians who each order two distinct entrées, with no two mathematicians ordering the same pair of entrées. The cost of each entrée is equal to the number of mathematicians who ordered it, and the university pays for each mathematician's less expensive entrée (ties broken arbitrarily). Over all possible sets of orders, what is the maximum total amount the university could have paid?

Problem TST4 (Josh Brakensiek)**.** Let n be a positive integer and let $S \subseteq \{0,1\}^n$ be a set of binary strings of length n. Given an odd number $x_1, \dots, x_{2k+1} \in S$ of binary strings (not necessarily distinct), their *majority* is defined as the binary string $y \in \{0,1\}^n$ for which the i^{th} bit of y is the most common bit among the i^{th} bits of $x_1, \dots, x_{2k+1}$. (For example, if $n = 4$ the majority of 0000, 0000, 1101, 1100, 0101 is 0100.)

Suppose that for some positive integer k, S has the property P_k that the majority of any $2k+1$ binary strings in S (possibly with repetition) is also in S. Prove that S has the same property P_k for all positive integers k.

Problem TST5 (Evan Chen)**.** Let $ABCD$ be a convex cyclic quadrilateral which is not a kite, but whose diagonals are perpendicular and meet at H. Denote by M and N the midpoints of $\overline{BC}$ and $\overline{CD}$. Rays MH and NH meet $\overline{AD}$ and $\overline{AB}$ at S and T, respectively. Prove there exists a point E, lying outside quadrilateral $ABCD$, such that

- ray EH bisects both angles $\angle BES$, $\angle TED$, and
- $\angle BEN = \angle MED$.

Problem TST6 (Mark Sellke)**.** Alice and Bob play a game. First, Alice secretly picks a finite set S of lattice points in the Cartesian plane. Then, for every line ℓ in the plane which is horizontal, vertical, or has slope $+1$ or -1, she tells Bob the number of points of S that lie on ℓ. Bob wins if he can then determine the set S.

Prove that if Alice picks S to be of the form

$$S = \left\{(x,y) \in \mathbb{Z}^2 \mid m \le x^2 + y^2 \le n\right\}$$

for some positive integers m and n, then Bob can win. (Bob does not know in advance that S is of this form.)

§6 If you finish early. . .

Problem X1 (Ray Li)**.** 令 $a_1, a_2, \dots, a_9$ 為九個實數，不必相異，具有平均 m。 令 A 為有序三元 $1 \le i < j < k \le 9$ 使得 $a_i + a_j + a_k \ge 3m$ 的數量。 試求A的最小值。

Problem X2. Solve for 5 in the equation

$$\sqrt{5-x} = 5 - x^2.$$

Problem X3. A sloth is located at the point $[0:1]$ in $\mathbb{CP}^1$. You are located at $[1+i:1]$.

(a) Compute the slow-homology groups of $\mathbb{CP}^1$.

(b) Pick an atlas on $\mathbb{CP}^1$. Draw a chart of this atlas containing you and the sloth.