# MOP 2017 Homework Problems

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June 2017

Congratulations on your excellent performance on the USA(J)MO, which has earned you an invitation to attend the Math Olympiad Summer Program! This program will be an intense and challenging opportunity for you to learn a tremendous amount of mathematics.

To celebrate your achievement, prepare you for MOP, and give you something to talk about on the way to Pittsburgh, you are asked to work on the following homework problems. The solutions will be discussed in the first week at MOP. The first few problems in each topic section are intended to be accessible to all, while experienced students (who typically score 21 or higher on USAMO/IMO) are encouraged to work instead on the last half of problems from each section. Of course, all students are welcome to attempt any and all problems.

This year's problems are adapted from the following sources:

- USA TSTST 2012
- USAMO 2007
- European Girl's Olympiad 2013
- Harvard-MIT Math Tournament 2015
- USA Team Selection Test for IMO 2017

When possible author information about the problems has been provided. Happy solving!

# 1 Algebra

**A1** (Victor Wang). Let $a, b \in \mathbb{Z}$ and $z = a + b\sqrt{-1}$. Let $p$ be an odd prime. Prove that the real part of $z^p - z$ is an integer divisible by $p$.

**A2** (Sam Vandervelde). Let $n$ be a positive integer. Define a sequence by setting $a_1 = n$ and, for each $k > 1$, letting $a_k$ be the unique integer in the range $0 \le a_k \le k-1$ for which $a_1 + a_2 + \cdots + a_k$ is divisible by $k$. (For instance, when $n = 9$ the obtained sequence is $9, 1, 2, 0, 3, 3, 3, \dots$.) Prove that for any $n$ the sequence $a_1$, $a_2$, ... eventually becomes constant.

**A3** (Palmer Mebane). Determine all infinite strings of letters with the following properties:

(a) Each letter is either $T$ or $S$,

(b) If position $i$ and $j$ both have the letter $T$, then position $i + j$ has the letter $S$,

(c) There are infinitely many integers $k$ such that position $2k - 1$ has the $k$th $T$.

**A4** (Pakawut Jiradilok). Let the sequences of real numbers $\{a_i\}_{i=1}^{\infty}$ and $\{b_i\}_{i=1}^{\infty}$ satisfy $a_{n+1} = (a_{n-1} - 1)(b_n + 1)$ and $b_{n+1} = a_n b_{n-1} - 1$ for $n \ge 2$, with $a_1 = a_2 = 2015$ and $b_1 = b_2 = 2013$. Evaluate the infinite sum

$$\sum_{n=1}^{\infty} b_n \left( \frac{1}{a_{n+1}} - \frac{1}{a_{n+3}} \right).$$

**A5** (Sung-yoon Kim). Positive real numbers $x, y, z$ satisfy $xyz + xy + yz + zx = x + y + z + 1$. Prove that

$$\frac{1}{3}\left( \sqrt{\frac{1+x^2}{1+x}} + \sqrt{\frac{1+y^2}{1+y}} + \sqrt{\frac{1+z^2}{1+z}} \right) \le \left( \frac{x+y+z}{3} \right)^{5/8}.$$

# 2 Combinatorics

**C1** (Kai Xiao). For positive integers $x$, let $g(x)$ be the number of blocks of consecutive 1's in the binary expansion of $x$. For example, $g(19) = 2$ because $19 = 10011_2$ has a block of one 1 at the beginning and a block of two 1's at the end, and $g(7) = 1$ because $7 = 111_2$ only has a single block of three 1's.

Evaluate $g(1) + g(2) + g(3) + \cdots + g(256)$.

**C2** (Carl Lian). \$indy has \$100 in pennies (worth \$0.01 each), nickels (worth \$0.05 each), dimes (worth \$0.10 each), and quarters (worth \$0.25 each). Prove that she can split her coins into two piles, each with total value exactly \$50.

**C3** (Reid Barton). An *animal* with $n$ *cells* is a connected figure consisting of $n$ equal-sized square cells (equivalently, a polyomino with $n$ cells). A *dinosaur* is an animal with at least 2007 cells. It is said to be *primitive* it its cells cannot be partitioned into two or more dinosaurs. Find with proof the maximum number of cells in a primitive dinosaur.

**C4** (Bulgaria). Snow White and the Seven Dwarves are living in their house in the forest. On each of 16 consecutive days, some of the dwarves worked in the diamond mine while the remaining dwarves collected berries in the forest. No dwarf performed both types of work on the same day. On any two different (not necessarily consecutive) days, at least three dwarves each performed both types of work. Further, on the first day, all seven dwarves worked in the diamond mine. Prove that, on one of these 16 days, all seven dwarves were collecting berries.

**C5** (Sam Vandervelde?)**.** Given a set $S$ of $n$ variables, a binary operation $\times$ on $S$ is called *simple* if it satisfies $(x \times y) \times z = x \times (y \times z)$ for all $x, y, z \in S$ and $x \times y \in \{x, y\}$ for all $x, y \in S$. Given a simple operation $\times$ on $S$, any string of elements in $S$ can be reduced to a single element, such as $xyz \to x \times (y \times z)$. A string of variables in $S$ is called *full* if it contains each variable in $S$ at least once, and two strings are *equivalent* if they evaluate to the same variable regardless of which simple $\times$ is chosen. For example $xxx$, $xx$, and $x$ are equivalent, but these are only full if $n = 1$. Suppose $T$ is a set of full strings such that any full string is equivalent to exactly one element of $T$. Determine the number of elements of $T$.

## 3 Geometry

**G1** (UK)**.** The side $BC$ of the triangle $ABC$ is extended beyond $C$ to $D$ so that $CD = BC$. The side $CA$ is extended beyond $A$ to $E$ so that $AE = 2CA$. Prove that if $AD = BE$ then the triangle $ABC$ is right-angled.

**G2** (Evan Chen)**.** Convex quadrilateral $ABCD$ with $BC = CD$ is inscribed in circle $\Omega$; the diagonals of $ABCD$ meet at $X$. Suppose $AD < AB$, the circumcircle of triangle $BCX$ intersects segment $AB$ at a point $Y \neq B$, and ray $\overrightarrow{CY}$ meets $\Omega$ again at a point $Z \neq C$. Prove that ray $\overrightarrow{DY}$ bisects angle $ZDB$.

**G3.** Triangle $ABC$ is inscribed in circle $\Omega$. The interior angle bisector of angle $A$ intersects side $BC$ and $\Omega$ at $D$ and $L$ (other than $A$), respectively. Let $M$ be the midpoint of side $BC$. The circumcircle of triangle $ADM$ intersects sides $AB$ and $AC$ again at $Q$ and $P$ (other than $A$), respectively. Let $N$ be the midpoint of segment $PQ$, and let $H$ be the foot of the perpendicular from $L$ to line $ND$. Prove that line $ML$ is tangent to the circumcircle of triangle $HMN$.

**G4.** Let $ABCD$ be a quadrilateral with $AC = BD$. Diagonals $AC$ and $BD$ meet at $P$. Let $\omega_1$ and $O_1$ denote the circumcircle and circumcenter of triangle $ABP$. Let $\omega_2$ and $O_2$ denote the circumcircle and circumcenter of triangle $CDP$. Segment $BC$ meets $\omega_1$ and $\omega_2$ again at $S$ and $T$ (other than $B$ and $C$), respectively. Let $M$ and $N$ be the midpoints of minor arcs $\widehat{SP}$ (not including $B$) and $\widehat{TP}$ (not including $C$). Prove that $\overline{MN} \parallel \overline{O_1O_2}$.

**G5** (Sung-yoon Kim)**.** Let $ABC$ be an acute triangle with $\omega$, $S$, and $R$ being its incircle, circumcircle, and circumradius, respectively. Circle $\omega_A$ is tangent internally to $S$ at $A$ and tangent externally to $\omega$. Circle $S_A$ is tangent internally to $S$ at $A$ and tangent internally to $\omega$.

Let $P_A$ and $Q_A$ denote the centers of $\omega_A$ and $S_A$, respectively. Define points $P_B$, $Q_B$, $P_C$, $Q_C$ analogously. Prove that

$$8P_AQ_A \cdot P_BQ_B \cdot P_CQ_C \le R^3$$

with equality if and only if triangle $ABC$ is equilateral.

## 4 Number Theory

**N1** (Slovenia)**.** Find all positive integers $a$ and $b$ for which there are three consecutive integers at which the polynomial $P(n) = \frac{1}{b}(n^5 + a)$ takes integer values.

**N2** (Titu Andreescu)**.** Prove that for every nonnegative integer $n$, the number $7^{7^n} + 1$ is the product of at least $2n + 3$ (not necessarily distinct) primes.

**N3.** A rational number $x$ is given. Prove that there exists a sequence $x_0, x_1, x_2, \dots$ of rational numbers with the following properties:

(a) $x_0 = x$;

(b) for every $n \ge 1$, either $x_n = 2x_{n-1}$ or $x_n = 2x_{n-1} + \frac{1}{n}$;

(c) $x_n$ is an integer for some $n$.

**N4.** Let $\mathbb{N}$ be the set of positive integers. Let $f \colon \mathbb{N} \to \mathbb{N}$ be a function satisfying the following two conditions:

(a) $f(m)$ and $f(n)$ are relatively prime whenever $m$ and $n$ are relatively prime.

(b) $n \le f(n) \le n + 2012$ for all $n$.

Prove that for any natural number $n$ and any prime $p$, if $p$ divides $f(n)$ then $p$ divides $n$.

**N5** (Romania)**.** Let $n$ be a positive integer.

(a) Prove that there exists a set $S$ of $6n$ positive integers such that the least common multiple of any two is at most $32n^2$.

(b) Show that every set $T$ of $6n$ positive integers contains two elements with least common multiple exceeding $9n^2$.

## 5 USA TST for IMO 2017

**TST1** (Linus Hamilton)**.** In a sports league, each team uses a set of at most $t$ signature colors. A set $S$ of teams is *color-identifiable* if one can assign each team in $S$ one of their signature colors, such that no team in $S$ is assigned *any* signature color of a different team in $S$. For all positive integers $n$ and $t$, determine the maximum integer $g(n, t)$ such that: In any sports league with exactly $n$ distinct colors present over all teams, one can always find a color-identifiable set of size at least $g(n, t)$.

**TST2** (Evan Chen)**.** Let $ABC$ be an acute scalene triangle with circumcenter $O$, and let $T$ be on line $BC$ such that $\angle TAO = 90^\circ$. The circle with diameter $\overline{AT}$ intersects the circumcircle of $\triangle BOC$ at two points $A_1$ and $A_2$, where $OA_1 < OA_2$. Points $B_1$, $B_2$, $C_1$, $C_2$ are defined analogously.

(a) Prove that $\overline{AA_1}$, $\overline{BB_1}$, $\overline{CC_1}$ are concurrent.

(b) Prove that $\overline{AA_2}$, $\overline{BB_2}$, $\overline{CC_2}$ are concurrent on the Euler line of triangle $ABC$.

**TST3** (Alison Miller)**.** Let $P, Q \in \mathbb{R}[x]$ be relatively prime nonconstant polynomials. Show that there can be at most three real numbers $\lambda$ such that $P + \lambda Q$ is the square of a polynomial.

**TST4** (Linus Hamilton)**.** You are cheating at a trivia contest. For each question, you can peek at each of the $n > 1$ other contestant's guesses before writing your own. For each question, after all guesses are submitted, the emcee announces the correct answer. A correct guess is worth 0 points. An incorrect guess is worth $-2$ points for other contestants, but only $-1$ point for you, because you hacked the scoring system. After announcing the correct answer, the emcee proceeds to read out the next question. Show that if you are leading by $2^{n-1}$ points at any time, then you can surely win first place.

(*Note* the change from $2^{n-1}$ to $2^{n-2}+1$.)

**TST5** (Danielle Wang and Evan Chen)**.** Let $ABC$ be a triangle with altitude $\overline{AE}$. The $A$-excircle touches $\overline{BC}$ at $D$, and intersects the circumcircle at two points $F$ and $G$. Prove that one can select points $V$ and $N$ on lines $DG$ and $DF$ such that quadrilateral $EVAN$ is a rhombus.

**TST6.** Prove that there are infinitely many triples $(a, b, p)$ of integers, with $p$ prime and $0 < a \le b < p$, for which $p^5$ divides $(a+b)^p - a^p - b^p$.

(*Note* the change from $p^3$ to $p^5$.)

## 6 If you finish early...

**X1** (Evan Chen)**.** 令不等邊三角形 $\triangle ABC$ 的內切圓圓心為 $I$， 且該內切圓分別為 $CA$, $AB$ 邊切於點 $E$, $F$。 $\triangle AEF$ 的外接圓在 $E$ 和 $F$ 的兩條切線的交點為 $S$。 直線 $EF$ 與 $BC$ 交於點 $T$。 試證：以 $ST$ 為直徑的圓垂直於 $\triangle BIC$ 的九點圓。

**X2.** What is the difference between a frog, an amphibian, and a mouse? (Hint: one of them likes fish.)

**X3.** A sloth is driving a miniature bulldozer.

(a) Draw a picture of you and the sloth.

(b) Show that the sloth cannot be swept away by any other bulldozer.