# 2016 MOP Homework

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Congratulations on your excellent performance on the USA(J)MO, which has earned you an invitation to attend the Math Olympiad Summer Program! This program will be an intense and challenging opportunity for you to learn a tremendous amount of mathematics.

To help prepare you for MOP, you should work on the following homework problems selected by the graders, taken from olympiads all around the world. The solutions will be discussed in the first few days of MOP. The first three problems in each section are intended to be accessible to all MOP students. Experienced students who typically score 21 or higher on USAMO/IMO are encouraged to work instead on the last three problems from each section. Of course, all students are welcome to attempt any and all problems.

Happy solving!

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## Algebra

**A1.** Let $n$ be an odd positive integer, and let $x_1, x_2, \ldots, x_n$ be nonnegative real numbers. Show that

$$\min(x_i^2 + x_{i+1}^2) \leq \max(2x_j x_{j+1})$$

where $1 \leq i, j \leq n$ and $x_{n+1} = x_1$.

**A2.** Prove that for any distinct integers $a_1,\ a_2,\ \ldots,\ a_n$ the polynomial $(x - a_1)(x - a_2)\ldots(x - a_n) - 1$ is irreducible over the integers.

**A3.** A finite sequence of integers $a_1, a_2, \ldots, a_n$ is called *regular* if there exists a real number $x$ satisfying

$$\lfloor kx \rfloor = a_k \quad \text{for } 1 \leq k \leq n.$$

Given a regular sequence $a_1, a_2, \ldots, a_n$, for $1 \leq k \leq n$ we say that the term $a_k$ is *forced* if the following condition is satisfied: the sequence

$$a_1,\ a_2,\ \ldots,\ a_{k-1},\ b$$

is regular if and only if $b = a_k$. Find the maximum possible number of forced terms in a regular sequence with 1000 terms.

**A4.** Prove that if $m$, $n$ are relatively prime positive integers, $x^m - y^n$ is irreducible in the complex numbers.

**A5.** Find all *smooth* functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f(x)^2 - f(y)^2 = f(x+y)f(x-y)$$

for all real numbers $x$ and $y$.

## Combinatorics

**C1.** Let $a_1, a_2, \ldots, a_9$ be nine real numbers, not necessarily distinct, with average $m$. Let $A$ denote the number of triples $1 \leq i < j < k \leq 9$ for which $a_i + a_j + a_k \geq 3m$. What is the minimum possible value of $A$?

**C2.** In a concert, 20 singers will perform. For each singer, there is a (possibly empty) set of other singers such that he wishes to perform later than all the singers from that set. Can it happen that there are exactly 2010 orders of the singers such that all their wishes are satisfied?

**C3.** Let $V = \{1, \ldots, 8\}$. How many permutations $\sigma : V \to V$ are automorphisms of some tree?

**C4.** There are $n > 2$ lamps arranged (evenly spaced) in a circle. Initially, one of them is turned on, and the rest are off. It is permitted to choose any regular polygon whose vertices are lamps and toggle all of their states simultaneously. For which positive integers $n$ is it possible to turn all the lamps off after a finite number of such operations?

**C5.** Let $T$ be a finite set of positive integers greater than 1. A subset $S$ of $T$ is called *good* if for every $t \in T$ there exists some $s \in S$ with $\gcd(s, t) > 1$. Prove that the number of good subsets of $T$ is odd.

## Geometry

**G1.** Let $A_0B_0C_0$ be a fixed triangle and $P$ a point inside it. For $n \geq 1$, let $A_n$ be the foot of $P$ to $B_{n-1}C_{n-1}$ and define $B_n$, $C_n$ similarly. Prove that triangles $A_3B_3C_3$ and $A_0B_0C_0$ are similar.

**G2.** Two circles $\omega_1$ and $\omega_2$, of equal radius intersect at different points $X_1$ and $X_2$. Consider a circle $\omega$ externally tangent to $\omega_1$ at $T_1$ and internally tangent to $\omega_2$ at point $T_2$. Prove that lines $X_1T_1$ and $X_2T_2$ intersect at a point lying on $\omega$.

**G3.** Let $ABC$ be an acute triangle with orthocenter $H$ and altitudes $BD$, $CE$. The circumcircle of $ADE$ cuts the circumcircle of $ABC$ at $F \neq A$. Prove that the angle bisectors of $\angle BFC$ and $\angle BHC$ concur at a point on $BC$.

**G4.** A circle $\omega$ is inscribed in a quadrilateral $ABCD$. Let $I$ be the center of $\omega$. Suppose that

$$(AI + DI)^2 + (BI + CI)^2 = (AB + CD)^2.$$

Prove that $ABCD$ is an isosceles trapezoid.

**G5.** Let $ABC$ be a triangle and let $I$ and $O$ denote its incentre and circumcentre respectively. Let $\omega_A$ be the circle through $B$ and $C$ which is tangent to the incircle of the triangle $ABC$; the circles $\omega_B$ and $\omega_C$ are defined similarly. The circles $\omega_B$ and $\omega_C$ meet at a point $A'$ distinct from $A$; the points $B'$ and $C'$ are defined similarly. Prove that the lines $AA', BB'$ and $CC'$ are concurrent at a point on the line $IO$.

## Number Theory

**N1.** Find all positive integers $k$ such that $3^k + 5^k$ is a power of an integer with exponent greater than 1.

**N2.** Solve $xy(x^2 + y^2) = 2z^4$ in positive integers.

**N3.** Positive integers $x_1$, $x_2$, $\ldots$, $x_n$ $(n \geq 4)$ are arranged in a circle such that each $x_i$ divides the sum of the neighbors; that is,

$$\frac{x_{i-1} + x_{i+1}}{x_i} = k_i$$

is an integer for each $i$, where $x_0 = x_n$, $x_{n+1} = x_1$. Prove that

$$2 \leq \frac{k_1 + \cdots + k_n}{n} < 3.$$

**N4.** Prove that for infinitely many positive integers $n$, the number $n^4 + 1$ has a prime divisor exceeding $2n$.

**N5.** Let $p$ be an odd prime number such that $p \equiv 2 \pmod 3$. Define a permutation $\pi$ of the residue classes modulo $p$ by $\pi(x) \equiv x^3 \pmod p$. Show that $\pi$ is an even permutation if and only if $p \equiv 3 \pmod 4$.

## If you finish early...

**X1.** On a square table of 2011 by 2011 cells we place a finite number of napkins that each cover a square of 52 by 52 cells. In each cell we write the number of napkins covering it, and we record the maximal number $k$ of cells that all contain the same nonzero number. Considering all possible napkin configurations, what is the largest value of $k$?

**X2.** Find a nontrivial solution to $a^3 + b^3 = 9$ in positive rational numbers, or prove that no such solutions exist. (Of course $\{a, b\} = \{1, 2\}$ is a trivial solution.)

**X3.** A sloth is sleeping near the centroid $G$ of $\triangle ABC$.

(a) Draw a picture of you and the sloth.

(b) A large tree is growing at the orthocenter $H$. What do you do?