2013 AIME I Problem 1

0.5 swim   30 bicycle   8 run

~~5r~~ r mph   10r mph   5r mph

$$4.25 = \frac{0.5}{r} + \frac{30}{10r} + \frac{8}{5r}$$

$$4.25r = 5.1 \Rightarrow r = \frac{51/10}{17/4} = \frac{\overset{3}{\cancel{51}} \cdot \overset{2}{\cancel{4}}}{\cancel{17} \cdot \underset{5}{\cancel{10}}} = \frac{6}{5}$$

$$\text{Time cycling} = \frac{30}{10 \cdot \frac{6}{5}} = \frac{30}{10 \cdot \frac{6}{5}}$$

$$= \frac{30}{12} = 2.5 \text{ hours}$$

$$2.5 \text{ hr} = \boxed{150 \text{ mins.}}$$

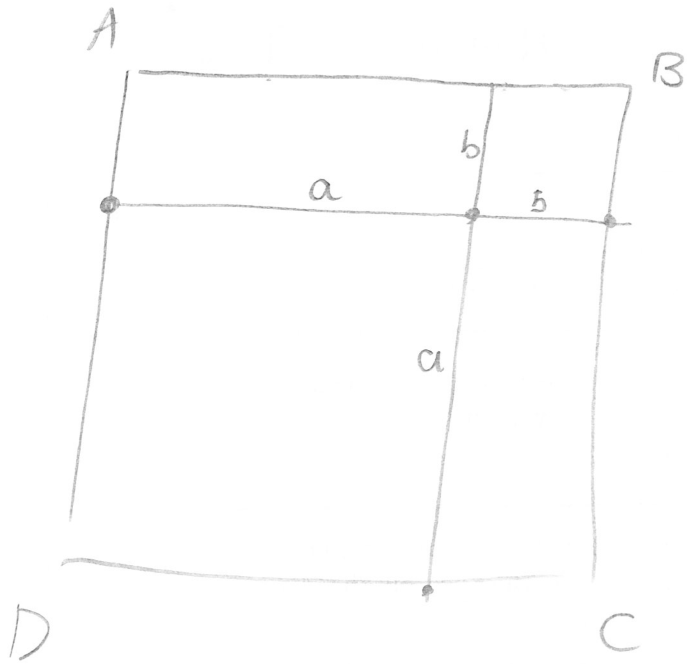
2013 AIME I Problem 2

$$\underline{5}\ \underline{x}\ \underline{y}\ \underline{z}\ \underline{5}$$

$$10 \times 10 \times 2$$

$$= \boxed{200}$$

2013 AIME I Problem 3



$$\frac{a^2+b^2}{(a+b)^2} = \frac{9}{10} \qquad \frac{a}{b}+\frac{b}{a} = \frac{a^2+b^2}{ab}$$

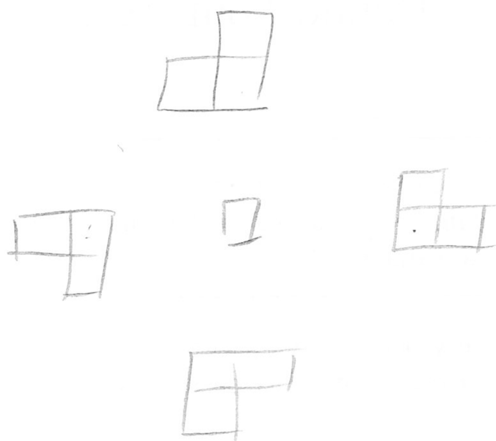
$$\Rightarrow \frac{(a+b)^2}{a^2+b^2} = \frac{10}{9}$$

$$\Rightarrow \frac{2ab}{a^2+b^2} = \frac{1}{9}$$

$$\Rightarrow \frac{a^2+b^2}{2ab} = 9$$

$$\Rightarrow \frac{a^2+b^2}{ab} = \frac{a}{b}+\frac{b}{a} = \boxed{018}$$

2013 AIME I Problem 4



$$\frac{3}{\binom{13}{5}} = \frac{\cancel{3}\cdot\cancel{120}}{13\cdot\cancel{12}\cdot 11\cdot\cancel{10}\cdot\cancel{9}\ 3}$$

$$= \frac{1}{13\cdot 11\cdot 3}$$

$$= \frac{1}{429} \Rightarrow n = \boxed{429}$$

2013 AIME I Problem 5

$$9x^3 = (x+1)^3$$

$$\Rightarrow \sqrt[3]{9}\, x = x+1$$

$$\Rightarrow (\sqrt[3]{9}-1)\, x = 1$$

Let $w = \sqrt[3]{9}$

$$\Rightarrow \cancel{x = \frac{1}{\sqrt[3]{9}-1}}$$

$$\cancel{= \frac{\sqrt[3]{9}^2 + \sqrt[3]{9} + 1}{\sqrt[3]{9}}}$$

$$x = \frac{1}{w-1}$$

$$= \frac{w^2+w+1}{(w-1)(w^2+w+1)} = \frac{w^2+w+1}{w^3-1}$$

$$= \frac{\sqrt[3]{81} + \sqrt[3]{9} + 1}{8}$$

$$81 + 9 + 8 = \boxed{098}$$

2013 AIME I Problem 6

$$\boxed{XYZ} \quad \boxed{4} \quad \boxed{5} \qquad \binom{9}{4}$$

$$\boxed{3} \quad \boxed{XYZ\ ?} \quad \boxed{5} \qquad \binom{9}{3,1,5}$$

$$\boxed{3} \quad \boxed{4} \quad \boxed{XYZ\ ??} \qquad \binom{9}{3,4,2}$$

$$p = \frac{\binom{9}{4} + \binom{9}{5,3,1} + \binom{9}{4,2,3}}{\binom{12}{3,4,5}}$$

$$= \frac{\frac{9\cdot 8\cdot 7 6}{24} + \frac{9\cdot 8\cdot 7 6}{24}\cdot 4 + \frac{9\cdot 8\cdot 7\cdot 6}{24}\cdot 10}{\frac{12\cdot 11\cdot 10}{6}\cdot\frac{9\cdot 8\cdot 7\cdot 6}{24}}$$

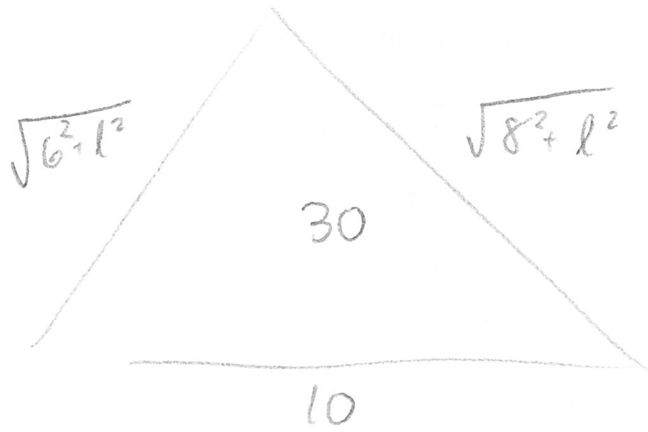
$$= \frac{1+4+10}{2\cdot 11\cdot 10}$$

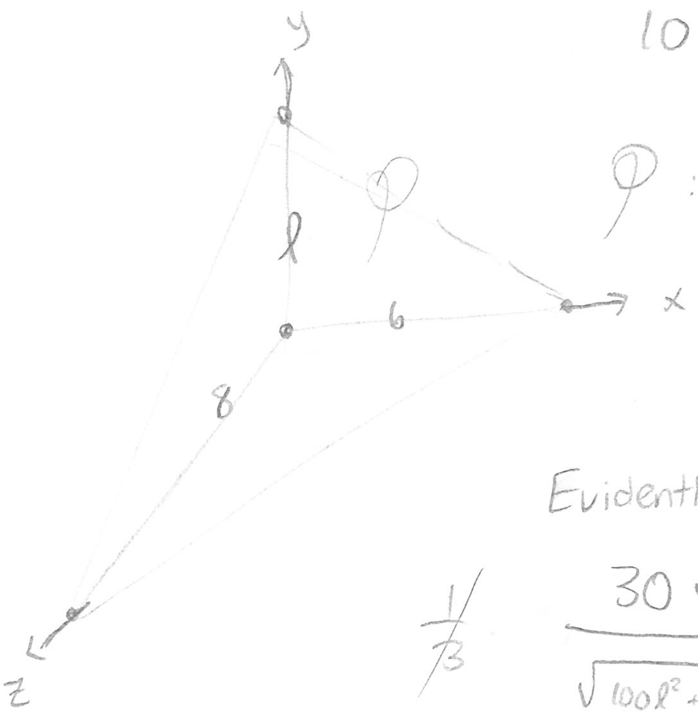
$$= \frac{15}{2\cdot 11\cdot 10} = \frac{3}{44}$$

$$3+44 = \boxed{047}$$

2013 AIME I Problem 7

Find 2ℓ





$\varphi : \quad 8\ell \cdot x + 48 \cdot y + 6\ell \cdot z = 48\ell$

$$d(O, \varphi) = \frac{48\ell}{\sqrt{100\ell^2 + 48^2 + \cancel{(48\ell)^2}}}$$

Evidently

$$\frac{1}{3} \cdot \frac{30 \cdot \cancel{48\ell}}{\sqrt{100\ell^2 + \cancel{(48\ell)^2} + 48^2}} = \frac{1}{2} \cdot \frac{1}{6} \cdot \cancel{8 \cdot 6 \cdot \ell}$$

$$\Rightarrow \quad 60 = \sqrt{100\ell^2 + \cancel{(48\ell)^2} + 48^2}$$

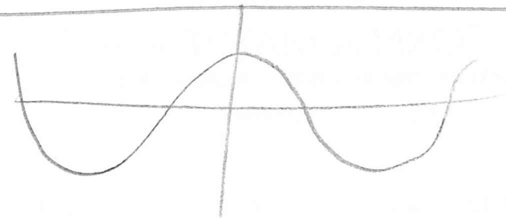
$$60^2 - 48^2 = (100 + \cancel{48^2})\,\ell^2$$

$$\Rightarrow \quad \underset{\parallel}{} 108 \cdot 12 = 100 \cdot \ell^2$$

$$\ell = \sqrt{\frac{1296}{100}} = \frac{36}{10} = \frac{18}{5}$$

$$2\ell = \frac{36}{\cancel{10}\,5} \rightarrow \boxed{041}$$

2013 AIME I Problem 8



$$-1 \le \log_m (nx) \le 1$$

$$\frac{1}{m} \le nx \le m$$

$$\frac{1}{mn} \le x \le \frac{m}{n}$$

$$\frac{m}{n} - \frac{1}{mn} = \frac{1}{2013}$$

$$2013(m^2 - 1) = mn$$

$$n = 2013\left(m - \frac{1}{m}\right)$$

$$m + n = 2014m - \frac{2013}{m}$$

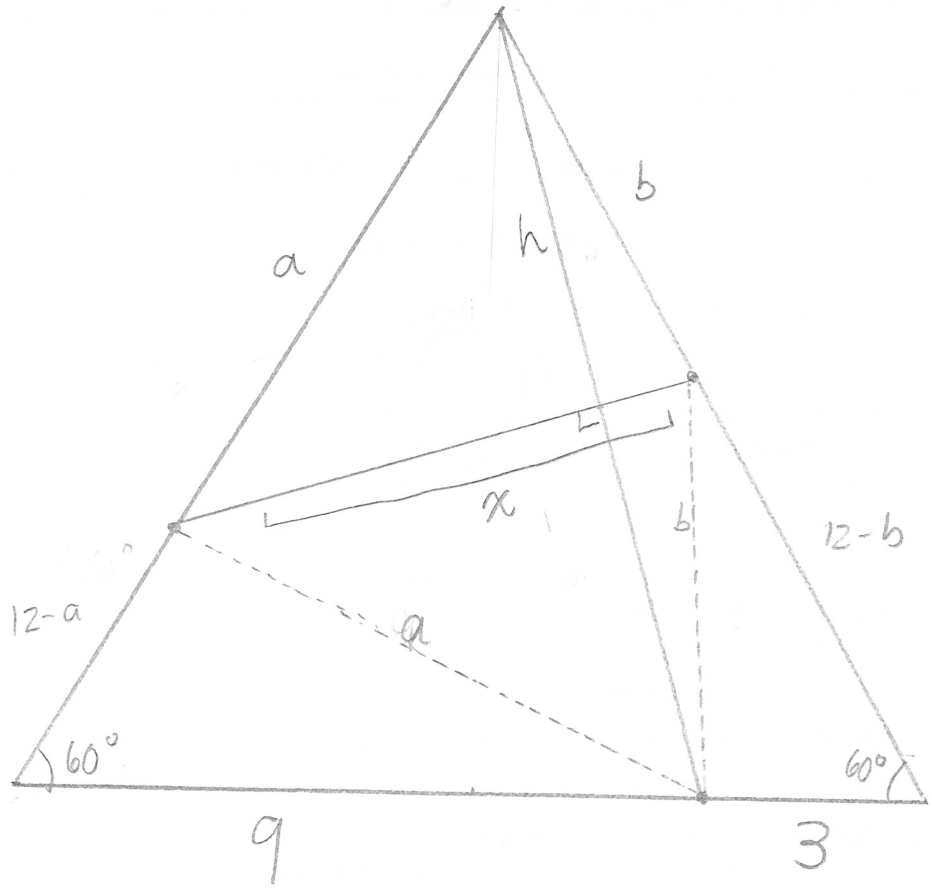
Minimized with $m = 3$; that is

$$2014 \cdot 3 - \frac{2013}{3}$$

$$\equiv 1042 - 671 \pmod{1000}$$

$$= \boxed{371}$$

2013 AIME I Problem 9 (1/2)



$\boxed{a}$:

$$(12-a)^2 + 9^2 - 9(12-a) = a^2$$

$$-24a + 144 + 81 - 108 + 9a = 0$$

$$\Rightarrow \quad 117 = 15a$$

$$\Rightarrow \quad a = \frac{39}{5}$$

$\boxed{b}$

$$(12-b)^2 + 3^2 - 3(12-b) = b^2$$

$$-24b + 144 + 9 - 36 + 3b = 0$$

$$117 = 21b$$

$$b = \frac{39}{7}.$$

2013 AIME I Problem 9 (2/2)

$$(2h)^2 = 3^2 + 12^2 - 3\cdot 12$$

$$= 117$$

$$\Rightarrow \quad h = \frac{1}{2}\sqrt{117}$$

$$[\triangle] = \cancel{\tfrac{1}{2}}ab\cdot\frac{\sqrt{3}}{2} = \cancel{\tfrac{1}{2}}x\cdot h$$

$$\Rightarrow \quad x = \frac{\frac{\sqrt{3}}{2}\cdot ab}{h}$$

$$= \frac{\frac{\sqrt{3}}{2}\cdot\frac{39}{5}\cdot\frac{39}{7}}{\frac{1}{2}\sqrt{117}}$$

$$= \frac{\cancel{2}\cdot 39^2}{35\cdot\sqrt{39}} = \frac{\cancel{2}\cdot 39\cdot\sqrt{39}}{35}$$

$$= \frac{39\sqrt{39}}{35} \approx \cancel{22.6}$$

$$39+39+35 = \boxed{113}$$

2013 AIME I Problem 10

$(r+si)(r-si) = r^2+s^2$ so $r,s \neq 0$.

$x^2 - 2r \cdot x + (r^2+s^2) \mid x^3 - ax^2 + bx - 65$

Suppose third root is u.

$$a = u + 2r \neq 0$$
$$b = r^2+s^2 + 2ru \neq 0$$
$$5 \cdot 13 = 65 = c = (r^2+s^2) \cdot u$$

clearly $\neq 0$ if $r,s \neq 0$

??

r^2+s^2	$\{r,s\}$	u	$\sum u + \cancel{2r}$	
5	$\pm 2, \pm 1$	13	$8 \cdot 13$	$= 104$
13	$\pm 3, \pm 2$	5	$8 \cdot 5$	$= 40$
65	$\pm 8, \pm 1$	1	$8 \cdot 1$	$= 8$
	$\pm 7, \pm 4$	1	$8 \cdot 1$	$= 8$

$$\boxed{160}$$

2013 AIME I Problem 11

$$\text{lcm}(16,15,14) = 2\cdot 8\cdot 15\cdot 7$$
$$= 1680 = 2^4\cdot 3\cdot 5\cdot 7$$

Let $N = 1680k$. Then

$$1680k - 3$$

has thee divisors $3 < x, y, z < 14$.

Possibilities: $9, 11, 13$ lol OK!

$$9\cdot 11\cdot 13 = 1287$$

$$N \equiv 3 \pmod{1287}$$
$$N \equiv 0 \pmod{1680}$$

$$1680k \equiv 3 \pmod{1287}$$

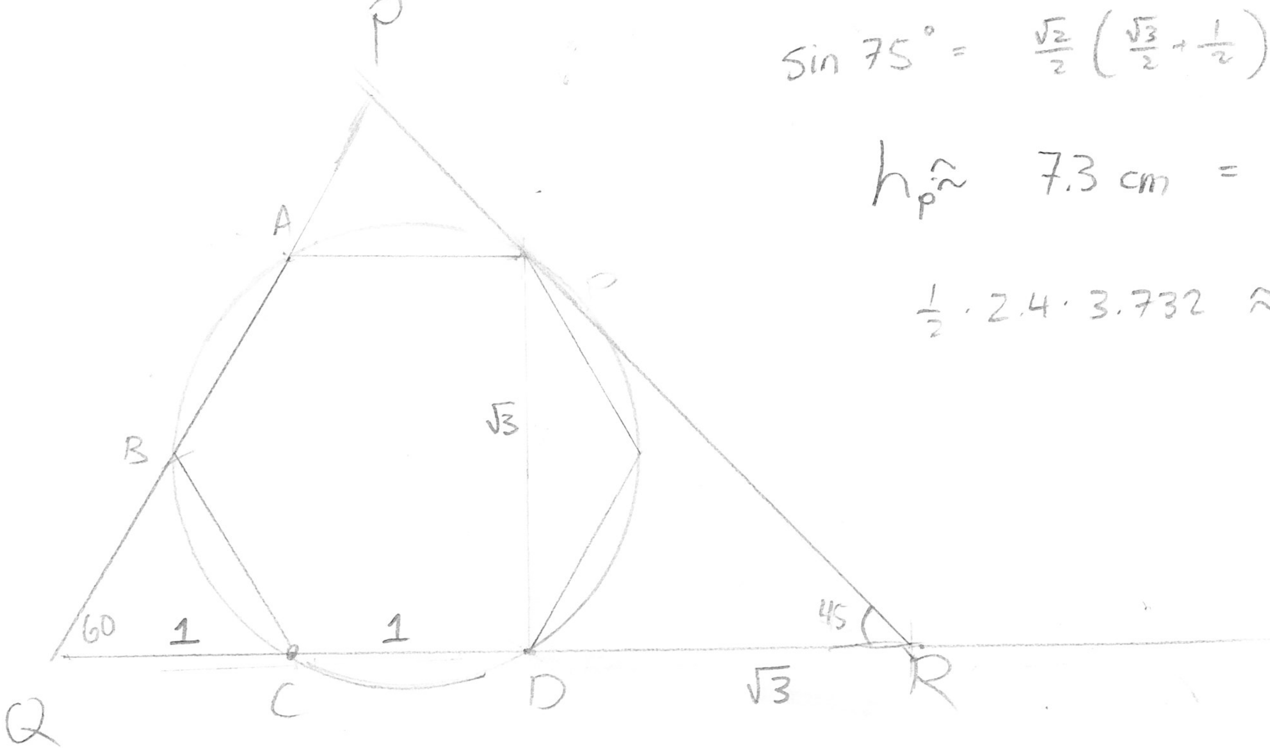
$$560k \equiv 131k \equiv 1 \pmod{429}$$

$$2k \equiv 1 \pmod{3}$$
$$\Rightarrow k \equiv -1 \pmod{3}$$

$$10k \equiv 1 \pmod{11}$$
$$\Rightarrow k \equiv -1 \pmod{11}$$

$$\Rightarrow \begin{rcases} k \equiv -1 \pmod{33} \\ k \equiv 1 \pmod{13} \end{rcases}$$

2013 AIME I Problem 12



$\sin 75^\circ = \frac{\sqrt{2}}{2}\left(\frac{\sqrt{3}}{2} + \frac{1}{2}\right)$

$h_p \approx 7.3 \text{ cm} = 2.4$

$\frac{1}{2} \cdot 2.4 \cdot 3.732 \approx 4.47$

$\frac{13.8 \cdot .8}{2.7}$

$$\frac{(2+\sqrt{3})^2 \cdot \sin 60^\circ \cdot \sin 45^\circ}{2 \cdot \sin 75^\circ} = \frac{(7+4\sqrt{3})\left(\frac{\sqrt{3}}{2}\right)}{\sqrt{3}+1}$$

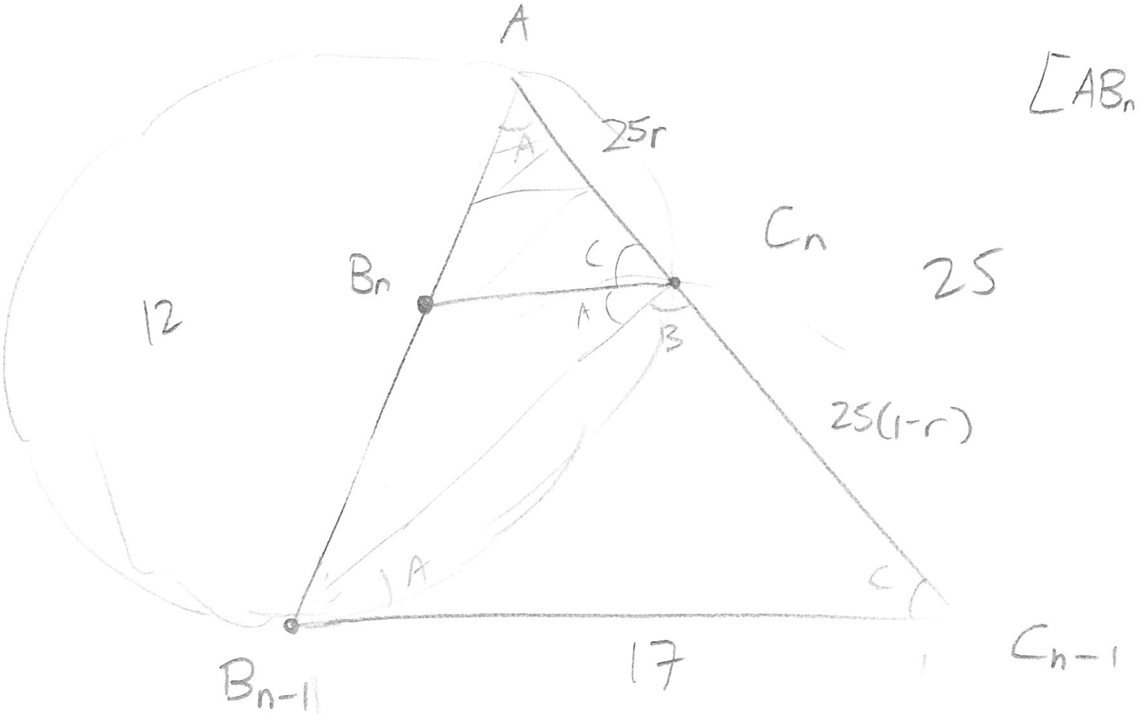
$$= \frac{7\sqrt{3}+12}{2\sqrt{3}+2}$$

$$= \frac{1}{2} \cdot \frac{7\sqrt{3}+12}{\sqrt{3}+1} \cdot \frac{\sqrt{3}-1}{\sqrt{3}-1}$$

$$= \frac{1}{4} \cdot \left(9 + 5\sqrt{3}\right)$$

$4 + 9 + 5 + 3 = \boxed{021}$

2013 AIME I Problem 13



$$[AB_{n-1}C_{n-1}] = \sqrt{\cancel{27}\cdot 10\cdot 15\cdot 2}$$
$$= 9\sqrt{100}$$
$$= 90$$

$$17^2 = 25^2(1-r)$$

$$\Rightarrow \quad r = 1 - \frac{17^2}{25^2}$$
$$= \frac{8\cdot 42}{25^2} = \frac{336}{625}$$

$$[B_{n-1}C_nB_n] = (1-r)\cdot[AB_{n-1}C_n]$$
$$= (1-r)\cdot r\cdot[AB_{n-1}C_{n-1}]$$

$$\text{Answer} = 90(1-r)\cdot r \cdot \frac{1}{1-r^2}$$
$$= 90\,\frac{r}{1+r}$$
$$= 90\left(1 - \frac{1}{\frac{961}{625}}\right) = \frac{90\cdot 336}{961}$$

$$\Rightarrow q = \boxed{961}$$

1 # 2013 AIME I Problem 14

$$P+Q = 1 + \frac{1}{2}(\cos\theta - \sin\theta) = \frac{1}{2}\cdot\frac{\sqrt{2}}{2}\cdot\cos(\theta+45^\circ)$$

$$- \frac{1}{4}(\sin 2\theta + \cos 2\theta) = -\frac{1}{4}\cdot\frac{\sqrt{2}}{2}\cdot\sin(2\theta+45^\circ)$$

$$- \frac{1}{8}(-\sin 3\theta + \cos 3\theta) = -\frac{1}{8}\cdot\frac{\sqrt{2}}{2}\cdot\cos(3\theta+45^\circ)$$

$$+ \frac{1}{16}(\sin 4\theta + \cos 4\theta) = \frac{1}{16}\frac{\sqrt{2}}{2}\cdot\sin(4\theta+45^\circ)$$

$$\sin\theta\cdot P - \cos\theta\cdot Q$$

$$= -\cos\theta + 0 + \frac{1}{4}\cos 3\theta + \frac{1}{16}$$

$$2P - \cos\theta\cdot Q = -\frac{1}{2}(\sin 2\theta - \sin\theta\cos\theta) = -\frac{1}{2}\sin\theta\cos\theta$$

$$- \frac{1}{4}(\cos 3\theta - \cos 2\theta\cos\theta) = +\frac{1}{4}\sin\theta\sin 2\theta$$

$$+ \frac{1}{8}(\sin 4\theta - \sin 3\theta\cos\theta) + \frac{1}{8}\sin\theta\cos 3\theta$$

$$+ \frac{1}{16}(\cos 5\theta - \cos 4\theta\cos\theta) - \frac{1}{16}\sin\theta\cos 4\theta$$

$$+ \cdots \qquad \vdots$$

$$= \sin\theta\cdot(-P)$$

2013 AIME I Problem 15

1	0	2	
1	3	5	1
1 4	6	11 8	2
1 4 7	9	17 14 11	3
	12		4
	15		5
	18		6
	21		7
	24		8
	27		9
	30		10
	33		11
	36		12
	39		13
	42		14
	45		15
1 46	48	95 50	16
1 4 49	51	101 98 53	16
~~1~~ ~~4~~ ~~7~~ 10 ⋮ 52	54	107 104 101 98 56	15 ⋮
94	96	98	1

$$2(1+\dots+16)$$

$$= 16\cdot 17$$

$$= \boxed{272}$$