Cross Ratios

MOP 2016: Blue Group

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§1 Definitions

Definition 1.1. The **cross ratio** of four collinear points A, B, X, Y

$$(A, B; X, Y) = \frac{XA}{XB} \div \frac{YA}{YB}.$$

Definition 1.2. The **cross ratio** of four concurrent lines is the directed ratio

$$(a, b; x, y) = \pm\frac{\sin\angle(x, a)}{\sin\angle(x, b)} \div \frac{\sin\angle(y, a)}{\sin\angle(y, b)}.$$

Theorem 1.3 (Perspectivity)
Suppose four points A, B, X, Y lie on some line. Let P be not on this line. Then

$$(PA, PB; PX, PY) = (A, B; X, Y).$$

Definition 1.4. Let γ be a nondegenerate conic. Then four points A, B, X, Y on the conic, we define $(A, B; X, Y)_\gamma = (PA, PB; PX, PY)$ where P is any fifth point on the conic. This doesn't depend on the choice of P.

Exercise 1.5. Show that if γ is a circle, then in fact $(A, B; X, Y)_\gamma = \pm\frac{XA}{XB} \div \frac{YA}{YB}$.

§2 Harmonic Bundles

Exercise 2.1. Show that if A, B, X, Y are pairwise distinct then $(A, B; X, Y) = (A, B; Y, X) \iff (A, B; X, Y) = -1$.

Definition 2.2. If four points A, B, X, Y obey $(A, B; X, Y) = -1$ we say they are **harmonic bundle**. Often, we say Y is the **harmonic conjugate** of X with respect to $\overline{AB}$.

If four points on a nondegenerate conic γ have cross ratio -1, we say the resulting quadrilateral is a **harmonic quadrilateral** with respect to γ.

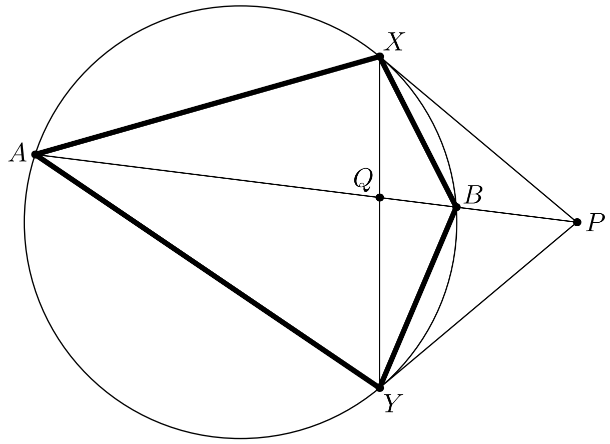
Lemma 2.3 (Midpoints and Parallel Lines)
Given points A and B, let M be the midpoint of $\overline{AB}$ and P_∞ the point at infinity of line AB. Then $(A, B; M, P_\infty)$ is a harmonic bundle.

Lemma 2.4 (Harmonic Quadrilaterals)
Let γ be a nondegenerate conic, and P a point with tangents PX, PY to γ. Consider another line through P meeting γ at A and B. Then

(a) $(A, B; P, \overline{AB} \cap \overline{XY}) = -1$ and

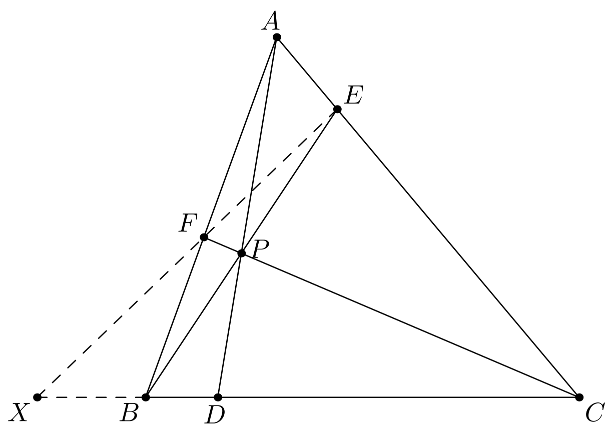
(b) $AXBY$ is harmonic.

Proof. $(A, B; P, Q) \stackrel{X}{=} (A, B; X, Y) \stackrel{Y}{=} (A, B; Q, P)$. □



Lemma 2.5 (Cevians Induce Harmonic Bundles)
Let ABC be a triangle with concurrent cevians $\overline{AD}$, $\overline{BE}$, $\overline{CF}$. Line EF meets BC at X. Then $(X, D; B, C)$ is a harmonic bundle.

Proof. $(B, C; X, D) \stackrel{A}{=} \left(F, E; X, \overline{AD} \cap \overline{EF}\right) \stackrel{P}{=} (C, B; X, D)$. □



Lemma 2.6 (Right Angles and Bisectors)

Let X, A, Y, B be collinear points in that order and let C be any point not on this line. Then any two of the following conditions implies the third condition.

(i) $(A, B; X, Y)$ is a harmonic bundle.

(ii) $\angle XCY = 90^\circ$.

(iii) $\overline{CY}$ bisects $\angle ACB$.

Proof. Amounts to the fact that if a, b meeting at C are two lines and x, y are their internal/external bisectors, then $(a, b; x, y) = -1$. ☐

§3 Poles and polars

Definition 3.1. Let γ be a nondegenerate conic and P a point. Consider lines through P intersecting γ at X, Y and let Q be the harmonic conjugate of P to $\overline{XY}$. Then the locus of Q lies on a line called the **polar** of P (with respect to γ). Point P is the **pole** of Q.

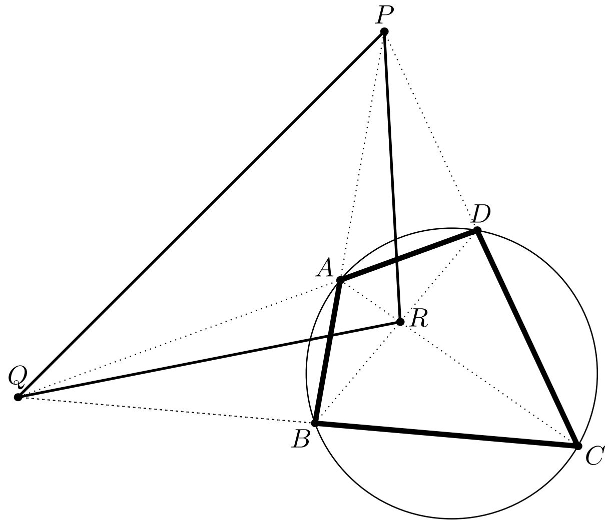
> **Remark 3.2.** If PX, PY are tangents from P to γ, then the polar of P coincides with line XY.

Lemma 3.3 (La Hire)

Let γ be a nondegenerate conic. Point P lies on the polar of Q if and only if Q lies on the polar of P.

Theorem 3.4 (Brokard)

Let $ABCD$ be a quadrilateral inscribed in nondegenerate conic γ. Set $P = \overline{AB} \cap \overline{CD}$, $Q = \overline{BC} \cap \overline{DA}$, and $R = \overline{AC} \cap \overline{BD}$. Then P, Q, R are the poles of QR, RP, PQ, respectively. (In particular, if γ is a circle, then the orthocenter of PQR is the center of the circle.)



Proof. Follows from Lemma 2.5. (Try it!) □

§4 Projective Transformations

Definition 4.1. A **collineation** of the projective plane is a bijection which preserves collinearity.

A **projective transformation** is a bijection of the projective plane which can be expressed in homogeneous coordinates as multiplication by a matrix M. (Don't worry about this).

> **Remark 4.2.** The **fundamental theorem of projective geometry** asserts that these two concepts coincide, *a la* functional equation style. Note that if you replace the base field $\mathbb{R}$ by $\mathbb{C}$, the theorem actually becomes false!

A projective transformation also sends conics to conics, and preserves the cross ratio.

Theorem 4.3 (Projective Transformations)

Each of the following is achievable with a unique projective transformation.

(a) Taking four points A, B, C, D (no three collinear) to any other four points W, X, Y, Z (no three collinear).

(b) Map any nondegenerate conic to another nondegenerate conic while moving any inscribed triangle into another inscribed triangle.

All cross ratios preserved.

Corollary 4.4

Let $ABCD$ be a convex cyclic quadrilateral. One can take a projective transformation sending $ABCD$ to a rectangle and preserving its circumcircle.

Proof. Let $ABCD$ be the original inscribed quadrilateral. Assume it has cross ratio $k < 0$. Let $WXYZ$ be a rectangle with cross ratio k in the same circle. Take a projective transformation fixing the circumcircle, sending ABC to WXY. Since cross ratios are preserved, D maps to Z as well. □

Exercise 4.5. Give "one-line" proofs of results in §2 and §3 by taking a suitable projective transformation.

§5 Examples

Example 5.1 (Brazilian Olympiad 2011/5)

Let ABC be an acute triangle with orthocenter H and altitudes $\overline{BD}$, $\overline{CE}$. The circumcircle of ADE cuts the circumcircle of ABC at $F \neq A$. Prove that the angle bisectors of $\angle BFC$ and $\angle BHC$ concur at a point on $\overline{BC}$.

Example 5.2 (APMO 2008/3)

Let Γ be the circumcircle of a triangle ABC. A circle passing through points A and C meets the sides $\overline{BC}$ and $\overline{BA}$ at D and E, respectively. The lines AD and CE meet Γ again at G and H, respectively. The tangent lines to Γ at A and C meet the line DE at L and M, respectively.

Prove that the lines LH and MG meet at Γ.

§6 Problems

Students who have never seen projective geometry before are encouraged to try the "beginner" problems, in order to get used to the tools at hand. On the other hand, those of you who have significant experience should try the "advanced" problems.

§6.1 Beginner problems

Problem 6.1 (JMO 2011/5)**.** Points A, B, C, D, E lie on a circle ω and point P lies outside the circle. The given points are such that (i) lines PB and PD are tangent to ω, (ii) P, A, C are collinear, and (iii) $\overline{DE} \parallel \overline{AC}$.

Prove that $\overline{BE}$ bisects $\overline{AC}$.

Problem 6.2 (Symmedians)**.** Let ABC be a triangle, and let the tangents to B and C meet at D. Prove that line AD is isogonal to the A-median.

Problem 6.3 (Canada 1994/5)**.** Let ABC be an acute triangle. Let $\overline{AD}$ be the altitude on $\overline{BC}$, and let H be any interior point on $\overline{AD}$. Lines BH and CH, when extended, intersect $\overline{AC}$, $\overline{AB}$ at E and F respectively. Prove that $\angle EDH = \angle FDH$.

Problem 6.4 (TSTST 2014)**.** Consider a convex pentagon circumscribed about a circle. We name the lines that connect vertices of the pentagon with the opposite points of tangency with the circle *gergonnians*.

(a) Prove that if four gergonnians are concurrent, then all five of them are.

(b) Prove that if there is a triple of gergonnians that are concurrent, then there is another triple of gergonnias that are concurrent.

§6.2 Intermediate problems

Problem 6.5 (Butterfly theorem)**.** Chord XY of a circle has midpoint M. We draw chords AB and CD concurrent at M. Prove that AC and BD meet XY at points equidistant from M.

Problem 6.6 (APMO 2016/3)**.** Let AB and AC be two distinct rays not lying on the same line, and let ω be a circle with center O that is tangent to ray AC at E and ray AB at F. Let R be a point on segment EF. The line through O parallel to EF intersects line AB at P. Let N be the intersection of lines PR and AC, and let M be the intersection of line AB and the line through R parallel to AC. Prove that line MN is tangent to ω.

Problem 6.7 (APMO 2013/5)**.** Let $ABCD$ be a quadrilateral inscribed in a circle ω, and let P be a point on the extension of $\overline{AC}$ such that $\overline{PB}$ and $\overline{PD}$ are tangent to ω. The tangent at C intersects $\overline{PD}$ at Q and the line AD at R. Let E be the second point of intersection between $\overline{AQ}$ and ω. Prove that B, E, R are collinear.

Problem 6.8 (Taiwan TST 2014)**.** In $\triangle ABC$ with incenter I, the incircle is tangent to $\overline{CA}$, $\overline{AB}$ at E, F. The reflections of E, F across I are G, H. Let Q be the intersection of $\overline{GH}$ and $\overline{BC}$, and let M be the midpoint of $\overline{BC}$. Prove that $\overline{IQ}$ and $\overline{IM}$ are perpendicular.

Problem 6.9 (Iran 2002)**.** Let ABC be a triangle. The incircle of triangle ABC touches the side BC at A', and the line AA' meets the incircle again at a point P. Let the lines CP and BP meet the incircle of triangle ABC again at N and M, respectively. Prove that the lines AA', BN and CM are concurrent.

§6.3 Advanced problems

Problem 6.10. Let $\mathscr{H}$ be a rectangular hyperbola with center Z, and let PQ be a chord of $\mathscr{H}$ with midpoint M. Then the asymptotes of $\mathscr{H}$ are the angle bisectors of line ZM and the line through Z parallel to PQ.

Problem 6.11 (Shortlist 2004/G8)**.** Given a cyclic quadrilateral $ABCD$, let M be the midpoint of the side CD, and let N be a point on the circumcircle of triangle ABM. Assume that the point N is different from the point M and satisfies

$$\frac{AN}{BN} = \frac{AM}{BM}.$$

Prove that the points E, F, N are collinear, where $E = \overline{AC} \cap \overline{BD}$ and $F = \overline{BC} \cap \overline{DA}$.

Problem 6.12 (Brazil 2013/6)**.** Let ABC be a triangle whose incircle is tangent to BC, CA, AB at D, E, F. Lines BE and CF intersect at G. Denote the reflection of G across EF, FD, DE by X, Y, Z, respectively. Prove that lines AX, BY, CZ are concurrent at a point on line IO, where I and O are the incenter and circumcenter of $\triangle ABC$.

Problem 6.13 (Taiwan TST 2015)**.** In scalene triangle ABC with incenter I, the incircle is tangent to sides CA and AB at points E and F. The tangents to the circumcircle of $\triangle AEF$ at E and F meet at S. Lines EF and BC intersect at T. Prove that the circle with diameter ST is orthogonal to the nine-point circle of triangle BIC.