◇ **OPAL Hunt 2**

# Conundrum

*Normally, puzzle hunts don't give instructions. This conundrum is much easier! It consists of nothing* but *instructions!*

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1. Write down five blanks on a sheet of paper.
2. For a function $f$, let $f^{-1}(y)$ denote the unique $x$ in its domain such that $f(x) = y$ (this notation will only be used when there is exactly one such $x$).
3. For any integer $n$ define the shorthand
$$f^n = \begin{cases} f \text{ applied } n \text{ times} & n > 0 \\ \mathrm{id} & n = 0 \\ f^{-1} \text{ applied } |n| \text{ times} & n < 0. \end{cases}$$
4. Let $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ denote the set of ordered pairs of nonnegative integers.
5. Define sequence 79, 0-indexed, as the sequence of powers of 2.
6. Define a function $\mathrm{Get}\colon \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ by mapping $(a, b)$ to the $b$th element of sequence $a$. (This is undefined if $b = 0$ and sequence $a$ is 1-indexed.)
7. Check your understanding by verifying that $\mathrm{Get}(79, 10) = 1024$.
8. Fill the first blank with the value of $\mathrm{Get}(79, 4) - \mathrm{Get}(79, 0)$.
9. For $i = 1, 2$, let $\pi_i\colon \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ denote the projection onto the $i$th component. That is, $\pi_1(a, b) = a$ and $\pi_2(a, b) = b$.
10. Define sequence 142, 0-indexed, such that $\mathrm{Get}(142, n)$ is the number of bijective functions from $\{1, \dots, n\}$ to itself for all $n$.
11. Define $\mathrm{Jump}\colon \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ such that there's a commutative diagram

$$\begin{tikzcd}
& \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \arrow[dl, "\pi_1"'] \arrow[d, "\mathrm{Jump}"] \arrow[dr, "\mathrm{Get}"] & \\
\mathbb{Z}_{\geq 0} & \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & \mathbb{Z}_{\geq 0}.
\end{tikzcd}$$

12. Check your understanding by verifying that
$$\pi_2(\mathrm{Jump}^2(142, 4)) = 620448401733239439360000.$$
13. Fill the third blank with the value of
$$\sum_{\varnothing \neq T \subseteq S} \mathrm{Get}\left(142, \sum_{t \in T} t\right)$$
where $S$ denotes the set of $z \in \mathbb{Z}_{\geq 0}$ for which $(142, z)$ is a fixed point of Jump.
14. We say a pair $(a, b)$ is *good* if $b$ only appears once in sequence $a$. When that occurs, let $\mathrm{Rev}(a, b)$ be the unique $c$ such that $\mathrm{Get}(a, c) = b$.
15. Define sequence 40, 1-indexed, by 2, 3, 5, 7, 11, …, and so on.
16. Define a function $\mathrm{Next}\colon \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ for which $\pi_1 \circ \mathrm{Next} = \pi_1$, and if $(a, b)$ is good then $\pi_2(\mathrm{Next}(a, b)) = \mathrm{Get}(a, \mathrm{Rev}(a, b) + 1)$.
17. Check your understanding by verifying that $\mathrm{Next}^{25}(40, 2) = (40, 101)$.
18. Fill the fourth blank with the value of $\pi_2(\mathrm{Next}^{-3}(40, 31))$.
19. Define sequence 108, 0-indexed, such that $\mathrm{Get}(108, 0) = \mathrm{Get}(108, 1) = 1$ and
$$\mathrm{Get}(108, n) = \sum_{i=0}^{n-1} \mathrm{Get}(108, i)\,\mathrm{Get}(108, n-1-i).$$
20. Fill the second blank with the value of $\pi_2(\mathrm{Jump}^{-2}(108, 3904442991190444395924 0))$.
21. Replace each of the four filled blanks with corresponding letters via A1Z26.
22. Check your understanding by verifying that $\mathrm{Get}(290, 45) = \mathrm{Get}(292, 22) + 1$.
23. Define a function $\mathrm{Down}\colon \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ such that for all good $(a, b)$,
$$\mathrm{Down}(a, b) = (a + 1, \mathrm{Get}(a + 1, \mathrm{Rev}(a, b)))\,.$$
24. Check your understanding by verifying that $\mathrm{Down}^2(40, 29) = (42, 1111111111)$.
25. Define a function $\mathrm{Seek}\colon \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ such that $\pi_2 \circ \mathrm{Seek} = \pi_2$ and
$$\pi_1(\mathrm{Seek}(a, b)) = \max\{c < a \mid \exists d : \mathrm{Get}(c, d) = b\}\,.$$
26. Check your understanding by verifying that $\mathrm{Seek}(43, 61) = (40, 61)$.
27. Fill the fifth blank with
$$\pi_2(\mathrm{Next}^{-1}(50128, \pi_2(\mathrm{Next}(\mathrm{Seek}(\mathrm{Next}^{-2}(\mathrm{Jump}(\mathrm{Down}^{-2}(\mathrm{Get}(10, 293), 10)))))))).$$
28. The blanks should have four letters followed by a number. Call this in!