18.218 Lecture Notes

Taught by Alex Postnikov

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This is MIT's graduate 18.218, instructed by Alex Postnikov. The formal name for this class is "Topics in Combinatorics". All errors are my responsibility.

The permanent URL for this document is http://web.evanchen.cc/coursework.html, along with all my other course notes.

Contents

§1 February 8, 2017

Examples of chip-firing games.

§1.1 Bert Kostant's game

Actually "find the highest root".

Let $G = (V, E)$ be a simple graph, and set $V = [n]$. For $i \in V$ let $N(i)$ denote the neighbors of i.

For $i \in V$ we have $c_i \geq 0$ chips; the vector $(c_i)_{1 \leq i \leq n}$ is called a *configuration*. We say a vertex i is:

- Happy if $c_i = \frac{1}{2} \sum_{j \in N(i)} c_j$.
- Unhappy if $c_i < \frac{1}{2} \sum_{j \in N(i)} c_j$.
- Excited if $c_i > \frac{1}{2} \sum_{j \in N(i)} c_j$.

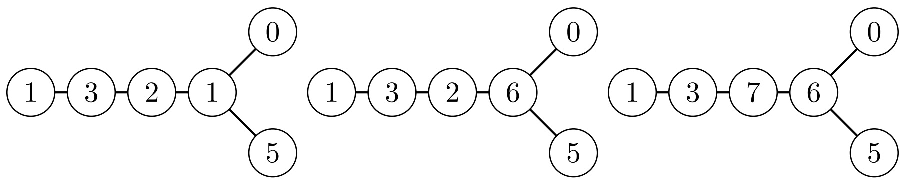
Goal: make everyone happy or excited.

The game is played as follows. Initially no chips are present (hence $c_i = 0$ for all i, and all vertices are happy). Then, we place a chip at vertex $v_{i_0} = 1$, so i_0 is excited but neighbors of i_0 are unhappy. Subsequently, do the following "reflection":

Pick any unhappy vertex i, and replace c_i by

$$c_i \mapsto -c_i + \sum_{j \in N(i)} c_j.$$

Here's an example of a couple steps.



More examples.

Example 1.1 (Kostant's game on P_n and C_n)

Check that:

- Let $G = P_n$, the game ends with all vertices having exactly one chip.
- Let $G = C_n$, the game never ends.

We now say:

Definition 1.2. The graph G is of **finite type** if the game ends.

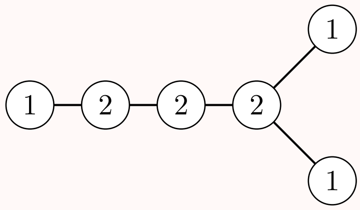
Of course, in order for this definition to make sense, we have to prove the following claim.

Proposition 1.3

If there is a way to play so that the game ends, then any sequence of moves eventually leads to a terminating state. Moreover, the final configuration vector does not depend on the choice of moves, nor on the initial vertex we added a chip on.

Example 1.4

- If G is a path on n vertices, then the terminating state is all 1.
- If G is the graph at the beginning, the terminating state is:



§1.2 Sponsor game

(This name is not standard, and is idiosyncratic. Postnikov says that if anyone sponsors his next teaching of the class with $\$10^6$, he will henceforth name the game after them.)

Everything is the same as previous game except the reflection step, which is replaced by:

Pick any unhappy vertex i, then replace c_i by

$$c_i \mapsto c_i + 1.$$

So instead of the reflection process, the sponsor gives them a chip.

Example 1.5 (Sponsor game on P_n and C_n)

If $G = P_n$ or $G = C_n$, both games terminate after n steps, with all vertices having exactly one chip.

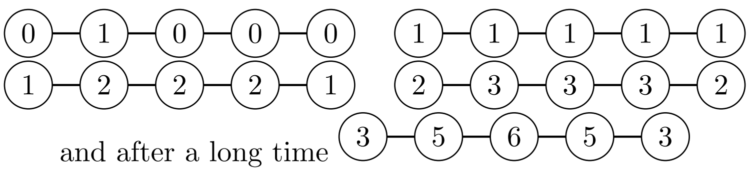
§1.3 Excited sponsor game

Everything is the same as previous except the reflection step, which is replaced by:

Pick any unhappy *or happy* vertex i, then replace c_i by

$$c_i \mapsto c_i + 1.$$

In other words, the sponsor wants everyone to be excited, not just happy.



On the other hand the excited sponsor game never terminates on a cycle, because it is impossible for the inequality $c_i > \frac{1}{2}(c_{i-1} + c_{i+1})$ to hold for all i (by noting that the minimal vertex is always un-excited).

So the excited sponsor game feels more like Kostant's game, but the terminal state is different in the path case (35653 rather than 11111).

§1.4 Chip-firing game

Also called "abelian sandpile model".

Retain the notation $G = (V, E)$, $V = [n]$, and c_i.

Definition 1.6. A vertex i is **stable** if $c_i < \deg_G(i)$, and **unstable** if $c_i \geq \deg_G(i)$.

In a **firing** move, we pick an unstable i, we move a chip from i to each neighbor of i.

Obviously the game goes on forever if the number of chips is sufficiently large (since the number of chips is invariant). To fix this, we add a *sink*: a vertex which eats all chips fired at it.

It turns out:

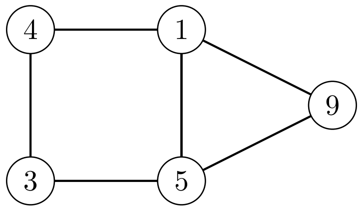
Proposition 1.7

With a sink, chip-firing games always terminate.

§2 February 10, 2017

§2.1 Chip-firing with games

Chip-firing game (with sink) continued.



Chip-firing with sink has the following property

Lemma 2.1

In chip-firing with sink:

- (Finiteness) After finitely many steps the game stops.
- (Uniqueness) The result is unique.

In contrast to other games: not all games we consider who have finiteness, but when they do we will have uniqueness results.

In fact finiteness is easy to see (olympiad-style monovariant). The correct invariant is

$$\sum_{d \geq 0} (\text{num chips distance } d) \cdot \varepsilon^d$$

with ε being positive but smaller than any particular real number (i.e. lex sort by coefficients).

Someone considers asking multiple sinks. One way to subsume it is to contract the sinks together, allowing multiple edges (which doesn't change anything).

§2.2 Cartan firing

Our fourth game: same setup as chip-firing, but without a sink. Instead of using $\deg_G i$, we use the number two. In other words, we say

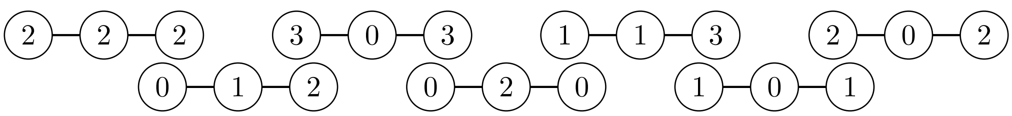
- vertex i is **stable** if $c_i \in \{0, 1\}$ and
- vertex i is **unstable** if $c_i \geq 2$.

As usual we fire unstable vertices. If i is unstable,

- $c_i \mapsto c_i - 2$ (i.e. it loses two chips)
- $c_j \mapsto c_j + 1$ for all neighbors j (meaning each neighbor gains a chip).

So that means the number of chips total is not invariant.

Here is an example.



Actually one can get.

Claim 2.2. For any path, the game terminates.

In contrast:

Claim 2.3. In a cycle, if weights of 2 are placed at each vertex, then the game goes on forever.

Actually, for cycles (and essentially only cycles) chip-firing and Cartan firing coincide.

§2.3 Matrix firing

We now formulate a more general game.

Let $A = (a_{ij})$ be a $n \times n$ matrix, symmetric for now, satisfying the following condition: all diagonal entries are positive, and all other entries are nonpositive. (In symbols, $a_{ii} \ge 0$ and $a_{ij} < 0$ for $i \ne j$.)

Definition 2.4. We let A_i denote the ith row of the matrix.

A configuration is then a vector $c = (c_1, \dots, c_n)$ where $c_i \ge 0$. Then a **firing** move for a vertex i consists of the following: if $c_i \ge a_{ii}$, then we do the map

$$c \mapsto c - A_i.$$

Example 2.5 (Special cases of matrix firing)

Let G be a graph.

(a) Chip-firing *without a sink* corresponds to setting A to the **Laplacian matrix** L_G.

(b) Standard chip-firing corresponds to A being the **truncated Laplacian matrix** in which one row and column of L'_G are deleted (corresponding to the sink).

(c) The **Cartan matrix** for G, denoted A_G, is the same as L_G except with 2's on diagonal. (The matrix is actwually called the **generalized simply-laced Cartan matrix**.)

Remark 2.6. The truncated Laplacian matrix already has combinatorial interest; for example $\det L'_G$ is equal to the number of spanning trees of G, by Kirchoff's matrix theorem.

The general terminating condition is as follows.

Proposition 2.7 (Finiteness of A-firing)

Let A be as above ($a_{ii} \ge 0$, $a_{ij} < 0$ for $i \ne j$). Then the following are equivalent.

(1) A-firing is finite for any initial configuration.

(2) There exists $h = (h_1, \dots, h_n) > 0$ such that $A \cdot h > 0$.

(3) A is positive definite (for example, all principal minors are positive).

The notation $h > 0$ means $h_i > 0$ for all i. Since the proof (2) $\iff$ (3) is linear algebra, we will prove (2) $\implies$ (1).

Proof that (2) $\implies$ (1). $A \cdot h > 0$ implies the dot product $\langle h, A_i \rangle$ is positive for each i. Thus over configurations c, the dot product $\langle h, c \rangle$ is decreasing over time, as it decreases by $\langle h, A_i \rangle$ when i is fired. On the other hand $h > 0$ and $c \ge 0$ so done. $\square$

So it remains to show uniqueness in the strongest sense possible: for fixed A, games either terminate always in exactly the same way, else they never terminate. The proof of this recalls on so-called diamond lemma.

§2.4 Diamond lemma

We state the diamond lemma in the context of A-firing.

Lemma 2.8 (Diamond lemma)

If there are two ways to fire, say $c \xrightarrow{i} c_1$ and $c \xrightarrow{j} c_2$, where $i \neq j$, then we can complete the diagram to get

$$\begin{tikzcd} & c \arrow[dl, "i"'] \arrow[dr, "j"] & \\ c_1 \arrow[dr, "j"'] & & c_2 \arrow[dl, "i"] \\ & c_3 & \end{tikzcd}$$

Proof. Just $c_3 = c - A_i - A_j$. $\square$

Remark 2.9. This "commutativity property" expressed by the diamond lemma is why we physicists call this game "*abelian* sand piles".

Proof of uniqueness from diamond lemma. Consider the following diagram:

$$\begin{tikzcd} c \arrow[r] \arrow[d] & c_1' \arrow[r] & c_2' \arrow[r] & \dots \\ c_1 \arrow[d] & & & \\ c_2 \arrow[d] & & & \\ \vdots \arrow[d] & & & \\ c_\ell & & & \end{tikzcd}$$

Assume for contradiction c_ℓ is final. Then, assume ℓ is minimal. Then diamond lemma repeatedly gives a downward path from c_1', until we find an index k such that $c_{k+1} = c_k''$, for example

$$\begin{tikzcd}
c \arrow[r] \arrow[d] & c_1' \arrow[r] \arrow[d] & c_2' \arrow[r] & \dots \\
c_1 \arrow[r] \arrow[d] & c_1'' \arrow[d] \\
c_2 \arrow[r] \arrow[d] & c_2'' \arrow[dl, equal] \\
c_3 \arrow[d] \\
\vdots \arrow[d] \\
c_\ell
\end{tikzcd}$$

This contradicts minimality of ℓ then. $\square$

In fact, this diamond lemma applies for every game (sponsor game, A-firing, etc.) except the second game.

Definition 2.10. G is **Cartan finite** if Cartan firing is finite for any initial configuration.

Lemma 2.11

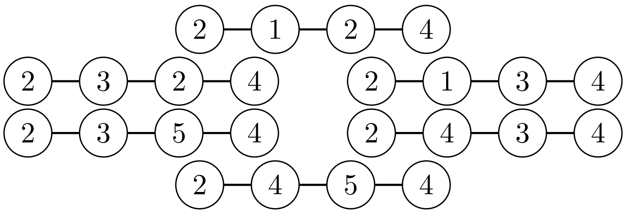
If a graph is Cartan finite, then any subgraph is Cartan finite.

§3 February 15, 2017

§3.1 Diamond/Hex Lemma

We now consider Kostant's game, since it is not subsumed by the previous theory. (This is the one firing on $c_i \mapsto \sum_{j \in N(i)} c_j - c_i$.)

Here is a fake example (which is "fake" in the sense that the initial configuration should have just 1 chip on a single vertex), in the shape of hexagon (see the lemma below).



These fit in a so-called "diamond / hexagon" lemma.

Lemma 3.1 (Diamond or hexagon lemma)

For Kostant game, suppose $i \neq j$ are two moves for which $c \xrightarrow{i} c'$ and $c \xrightarrow{j} c''$. Then

- If i and j are not adjacent, we get the same statement as in the diamond lemma.
- Otherwise, we get a hexagon

$$\begin{tikzcd}
& c \arrow[dl, "i"'] \arrow[dr, "j"] & \\
c' \arrow[d, "j"] & & c'' \arrow[d, "i"] \\
\bullet \arrow[dr, "i"] & & \bullet \arrow[dl, "j"'] \\
& c_{\text{last}} &
\end{tikzcd}$$

§3.2 Roman lemma

More generally, *suppose* that we have a diamond/hexagon lemma. The following theorem tells us something good about this.

Theorem 3.2 (Roman lemma)

Let C be a connected directed graph without self-loops (possibly infinite!). Suppose that: for every vertex c with at least 2 outgoing edges, we can find 2 converging paths of the same length (as in the diamond/hexagon lemma) to some other vertex. Then one of the following is true:

- C has no end-points (vertex of C with outdegree zero).
- C has exactly one end-point c_{end} and all directed paths eventually reach c_{end}, and have the same length. In other words C should be a graded poset with a unique minimum.

Here the graph C should be interpreted as the set of possible configurations. One interpretation of the name is "all roads lead to Rome". (We can joke that this has religious connotations in the sense that: either you wander forever, or we always end up in the same place no matter what we try to do.)

Proof. Assume C has an endpoint. ... $\square$

§4 February 17, 2017

§4.1 Review

- *Cartan's firing game*: Let G be a simple graph, and define the Cartan matrix

$$A = A_G = 2I - \text{adj matrix of } A.$$

  Then a configuration $c = (c_1, \dots, c_n) \in \mathbb{Z}^n_{\ge 0}$. Now let $e_1, \dots, e_n$ be the standard basis of $\mathbb{R}^n$.

  Then firing f_i as usual corresponds to

$$f_i : c \mapsto c - Ae_i.$$

- *Kostant's game* (a reflection game):

$$s_i : c \mapsto c - (Ac, e_i)\, e_i$$

  if $(Ac, e_i) < 0$. (We are using $(\bullet, \bullet)$ for the dot product.)

We have written this in terms of an arbitrary matrix A, since we will use this generality later. In what follows, all graphs G are connected.

§4.2 Vinberg's additive function

We replace the notion with "happy" now.

Definition 4.1. A configuration $h \in \mathbb{Z}^n_{\ge 0}$ is called a(n)

- (Vinberg) **additive function** if $Ah = \mathbf{0}$.
- **subadditive function** if $Ah \ge 0$ (happy or excited)
- **strictly subadditive function** if $Ah > 0$ (excited).

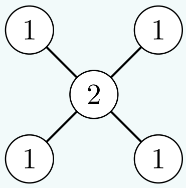
We think of h as a function from vertices to $\mathbb{Z}_{\ge 0}$, hence the name.

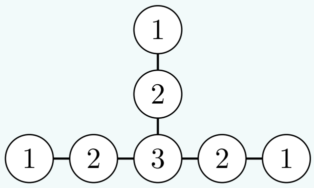
We now give a complete classification of all additive functions.

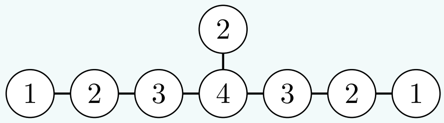
Theorem 4.2 (Vinberg's theorem)

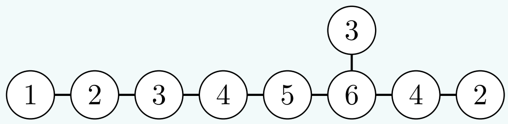
All additive functions h (up to scaling) are given by the following list.

- If G is a cycle, then $h(v) = 1$ for each vertex v. We call this the graph $\widetilde{A}_n$ if there are $n+1$ vertices.
- G is a tree as follows: a path on k vertices, say $v_1, v_2, \dots, v_k$ is given, and then adds two leaves on each of v_1 and v_k for a total of $k+4$ vertices. Then we let $h(v_i) = 2$ for each $i = 1, \dots, k$ and $h(w) = 1$ for each of the four leaves w. We call this $\widetilde{D}_n$ where $n = k+3$.
- The following graphs:

$\widetilde{D}_4 =$ 

$\widetilde{E}_6 =$ 

$\widetilde{E}_7 =$ 

$\widetilde{E}_8 =$ 

We won't prove this yet, but we'll show the existence of such an additive function implies:

Lemma 4.3

For all graphs in the list above, both Kostant's game and the Cartan game are infinite.

Proof. Assume $(h_1, \dots, h_n)$ be an additive function. Indeed, take any initial configuration such that $c = (c_1, \dots, c_n)$ where $c_i \ge 2h_i$ for each i. Then we contend that if we fire each vertex i exactly h_i times, then we can check that we return to the same configuration c, as

$$c \mapsto c - Ah = c.$$

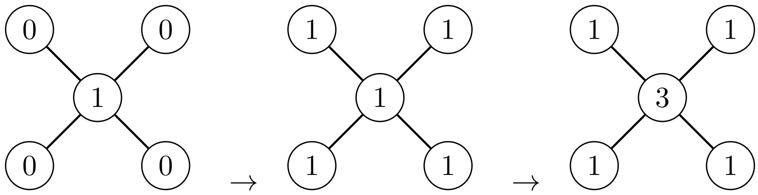
□

§4.3 Infinitude of Kostant games

Proposition 4.4

For all graphs G in the above list let h be the (nonzero) additive function on G above, and let e_i be the configuration where one chip is dropped on a vertex i. Then there exists a way to play Kostant's game such that we reach the configuration $e_i + h$, unless G is a cycle in which case we can reach $e_i + 2h$ instead.

One just checks this manually for each of the graphs. Example for the five-vertex graph above:



Now we contend that Kostant's game is invariant under addition by additive functions.

Proposition 4.5

For Kostant's game, if there exists a sequence of moves taking c to c', then the same sequence of moves takes $c + h$ to $c' + h$.

Corollary 4.6

Kostant's game is infinite on any of the graphs above.

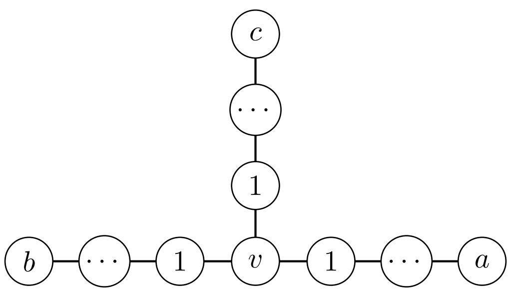
Proof. By the two preceding propositions, we have a sequence $e_i \to e_i + h \to e_i + 2h \to \dots$ for every i. So there is one infinite way to play the game, hence all ways of playing the game never terminate. □

§4.4 Simply-laced Dynkin diagrams

If G is a graph and has one of the above graphs as a subgraph, then we see the game is infinite on it too. Consequently, if G is a graph on which Kostant's game is finite, then

- G has no cycles, hence is a tree
- G has no vertex of degree ≥ 4 (because of the 4-star)
- G has at most two trivalent vertices (since then there's a path between them).

Hence, G is either a chain, or it is a "three-legged graph" consisting of a single trivalent vertex v with paths of length a, b, c leaving them.



On the other hand, the three exceptional graphs in Vinberg's theorem imply that $\min(a, b, c) < 2$, hence WLOG $a = 1$. Then we must have $\min(b, c) < 3$, hence $b \le 2$, and also when $b = 2$ it follows that $c \le 4$.

Thus we conclude that

Theorem 4.7 (Simply-laced Dynkin diagrams)
The only graphs on which Kostant game is finite are the **simply-laced Dynkin diagrams**:

- A_n, the path on n vertices.
- D_n, which has n vertices, one trivalent vertex with two leaves and a path on it.
- E_6, E_7, E_8.

The earlier graphs are then called the **extended Dynkin diagrams** since they are achieved from the Dynkin diagrams by adding one vertex (hence the tilde notation).

As for the Cartan game:

Proposition 4.8 (Finiteness of Cartan game)
For each of the A-D-E graphs, there exists a strictly sub-additive function $h > 0$ (meaning $Ah > 0$). Thus the Cartan game is finite exactly on these graphs.

§4.5 Recap

Putting everything together for today and previous lectures:

Theorem 4.9 (Master finiteness theorem)
The following are equivalent for a connected graph G.

(1) Kostant's game is finite.

(2) Cartan's firing game is finite for any initial configuration.

(3) The Cartan matrix A is positive definite (all principal minors are positive).

(4) All principle minors of A are nonzero.

(5) There exists a strictly sub-additive function $h > 0$ on G.

(6) G has no subgraphs isomorphic to any of the extended Dynkin diagrams $\widetilde{A}_n$, $\widetilde{D}_n$, $\widetilde{E}_6$, $\widetilde{E}_7$, $\widetilde{E}_8$.

(7) G is isomorphic to one of A_n,D_n, E_6, E_7, E_8.

This is the so-called "ADE classification". However, we mentioned that the main study of object in the class is the *root system*, which are classified in this way: so we have managed to classify the root system before defining them! We'll see the definition of this next time.

§5 February 21, 2017

We will now consider so-called *cluster algebra games*, in which we not only change configurations but also alter the graph.

§5.1 Reflection game (generalize Kostant game)

Definition 5.1. A **generalized Cartan matrix** is a matrix $A = (a_{ij})$ with integer entries such that:

(1) $a_{ii} = 2$ for all i

(2) $a_{ij} \le 0$ for $i \ne j$.

(3) if $a_{ij} < 0$ then $a_{ji} < 0$.

As usual we can obtain a graph G on vertices $\{1, \dots, n\}$ by letting (i, j) be an edge exactly when $a_{i,j} < 0$. As usual we may assume G is connected (since otherwise we may sub-divide the matrix).

We then have the **reflection game** (generalizing Kostant's game) as follows.

Definition 5.2. Let $e_1, \dots, e_n$ be a standard basis of $\mathbb{R}^n$. Then the reflection game consists of moves

$$s_i : \mathbb{R}^n \to \mathbb{R}^n$$

by

$$c \mapsto c - \left(A^\top c, e_i\right) e_i.$$

To be explicit, the map is

$$(c_1, \dots, c_n) \mapsto (c_1, \dots, c_i', \dots, c_n)$$

where

$$c_i' = -c_i - \sum_{j \ne i} a_{ji} c_j.$$

As usual $s_i^2 = \mathrm{id}$.

So this is somewhat symmetric, but not quite as symmetric as before. This time we (for now) place no constraints on the c_i in order to "reflect" by s_i.

§5.2 Weyl group

We now give an unorthodox definition of root systems. This is not the "usual" definition, but it is equivalent to them.

Definition 5.3. A **Weyl group** W and real **root system** Φ is defined as follows.

- $W \subseteq \mathrm{GL}(n)$ is the subgroup generated by s_i.
- $\Phi = W\{e_1, \dots, e_n\} \subseteq \mathbb{Z}^n$ is the image of basis elements under the reflections in W. The elements of Φ can be called **roots**. (Remark for experts: they are currently written in some particular choice of coordinates.)

We are interested in when W and Φ are finite.

Example 5.4 ($n = 2$ case)
Let $n = 2$. Then we can write

$$A = \begin{bmatrix} 2 & -a \\ -b & 2 \end{bmatrix}.$$

In that case, the following are equivalent:

(1) W is finite

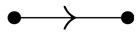
(2) Φ is finite

(3) $ab < 4$. That is, A must be one of the matrices

$$A_2 = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \qquad B_2 = \begin{bmatrix} 2 & -2 \\ -1 & 2 \end{bmatrix} \qquad G_2 = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$$

or one of the transposes.

So we can imagine the graphs as follows:

- A_2 corresponds to two vertices joined by a single edge. 
- B_2 corresponds to two arrows right and an arrow left: 
- G_2 corresponds to three arrows left and one arrow right: 

By convention a picture

with k lines means that we have k arrows in the direction of the arrow head, and just 1 arrow in the reverse direction.

§5.3 Classification of finite Weyl groups

Theorem 5.5
The following are equivalent.

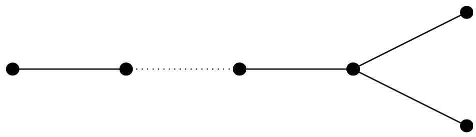
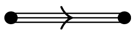
(1) Φ is finite.

(2) W is finite.

(3) A is corresponds to one of the following Dynkin diagrams: A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2.

Here are pictures of each of them:

- A_n:
- B_n: 
- C_n: 

- D_n: 
- E_6, E_7, E_8 as before.
- F_4: 
- G_2: 

The types A_n, B_n, D_n are thus called **simply-laced** because they don't have double edges.

(1) $\iff$ (2). Obviously if W is finite then Φ is finite ($|\Phi| \le n\,|W|$). To see the other direction, assume Φ is finite; there is then a canonical map

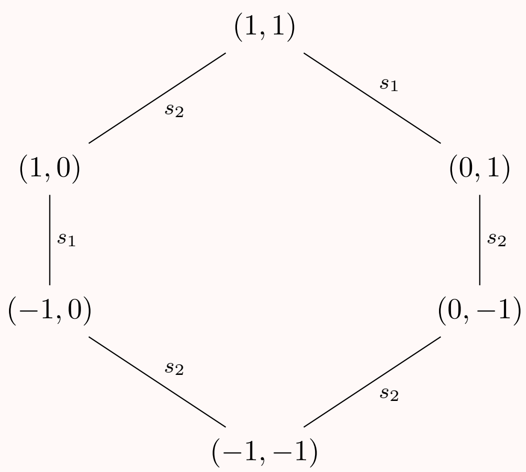
$$W \to S_\Phi$$

onto permutations of Φ. As Φ contains the basis elements $e_1 \dots, e_n$ the map is injective; hence $|W| \le |S_\Phi| < \infty$. $\square$

Now let's give an example.

Example 5.6 (Root system for A_2)

Take A_2, so $W = \{s_1, s_2\}$ and Φ can be interpreted in the following diagram.

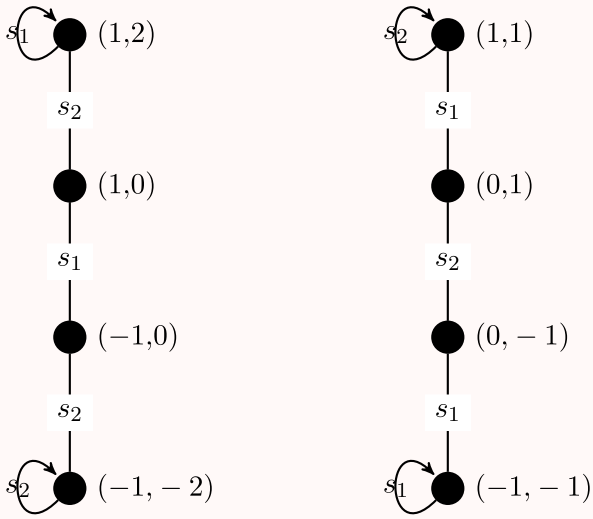


The roots $(1,0)$ and $(0,1)$ are called **simple roots**. They are divided into **positive roots** and **negative roots** in the obvious manner.

Example 5.7 (Root system for B_2)

Let's consider B_2, so the matrix is

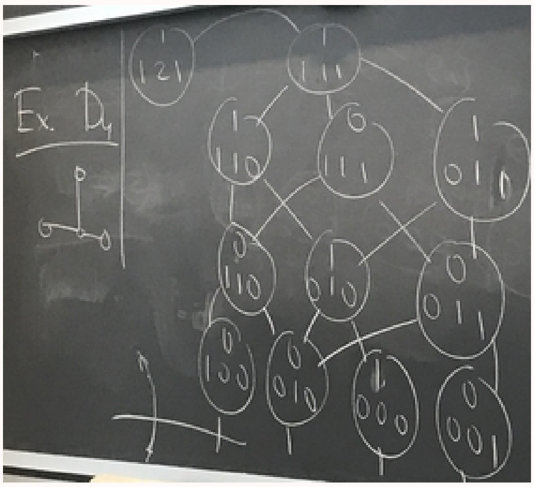
$$\begin{bmatrix} 2 & -2 \\ -1 & 2 \end{bmatrix}.$$



Note that this is not connected. This is the difference between simply laced and non simply laced diagrams.

Example 5.8 (D_4 poset)

Here is the picture of the roots of D_4.



In fact, we get a poset which is graded by the sum of all the components.

§5.4 Proof of uniqueness

From now on we will orient the s_i edges of Φ we drew earlier in one direction, namely upwards in the poset. Thus the undirected graphs earlier now have directions on them, similar to our situation before, and we would like for the game to terminate.

We now prove the lemma from before. The main idea remains intact from before.

The analog of "diamond/hexagon lemma" from before is:

Lemma 5.9 (Diamond / hexagon / octagon / dodecagon lemma)
Suppose $s_i : c \mapsto c'$ and $s_j : c \mapsto c''$. Then there are four cases:

- If $a_{ij} = a_{ji} = 0$, then we have a diamond lemma as before.
- If $a_{ij} = a_{ji} = -1$, we have a hexagon lemma as before.
- If $a_{ij}a_{ji} = 2$, then there are two paths of length 4 converging to a given end configuration (octagon).
- If $a_{ij}a_{ji} = 3$, then there are two paths of length 6 converging to a given end configuration (dodecagon).

Consequently, the Roman Lemma implies the uniqueness result: if we climb a (finite) root system following the edges s_i, we will always end at a unique place.

(Note that this is per connected component; for example the root system of B_2 has two connected components.)

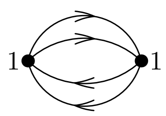
§5.5 Proof of finiteness

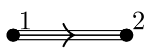
All that remains to do is show finiteness. Again:

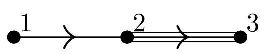
- A function $h \in \mathbb{R}^n$ is called an **additive function** if $h > 0$ and $A^\top h = \vec{0}$.

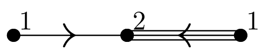
As before we just need to exhibit a bunch of additive functions in order to get a list of forbidden subgraphs. We now give a list of all additive functions.

- $\widetilde{A}_n$, $\widetilde{D}_n$, $\widetilde{E}_{6,7,8}$ are as before.
- If $A = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$, we have the additive function



- If $A = \begin{bmatrix} 4 & -2 \\ -1 & 2 \end{bmatrix}$, the additive function is 

- 

- 

- . . . to be finished next lecture.

§6 February 22, 2017

We are now going to generalize from graphs to general matrices with real entries.

§6.1 Linear algebra results

We first state the following linear algebra result.

Lemma 6.1 (Farkas lemma)
Let A be an $m \times n$ matrix, $b \in \mathbb{R}^m$. Exactly one of the following is true:

(1) There exists $u \in \mathbb{R}^n$ such that $u \geq 0$ and $Au = b$.

(2) There exists $v \in \mathbb{R}^m$ such that $A^\top v\ le0$ and $(v, b) > 0$.

Proof. Let $a_1, \dots, a_n$ denote the columns of A. Consider the hypercone spanned by the a_i. Then

- If b lies in the cone, then some nonnegative combination of a_i equals b, hence u exists as desired.
- Else there is a separating hyperplane: there exists a linear function $f(x)$ such that $f(b) < 0$ while $f(a_i) \leq 0$. Then the coefficients of f give the vector v. □

Here is a variant of this lemma.

Lemma 6.2
Let A be an $m \times n$ matrix. Exactly one of the following is true:

(1) There exists $u \in \mathbb{R}^n$ such that $u \geq 0$, u is not the zero vector, and $Au \geq 0$.

(2) There exists $v \in \mathbb{R}^m$ such that $v > 0$ and $A^\top v < 0$.

The proof is left as a homework question.

§6.2 Axioms for square matrices

We know consider a square matrix A on which we will again play A-firing games.
Consider the following set of matrices

(M0) All a_{ij} are integers.

(M1) $a_{ij} \leq 0$ for all $i \neq j$.

(M2) $a_{ij} \neq 0$ if and only if $a_{ji} \neq 0$.

(M3) $a_{ii} = 2$ for all i.

Then we have the following situations we've seen.

1. All real matrices: will examine this class.
2. Matrices satisfying M1: this implies the diamond lemma, so then the Roman lemma then gives us a *uniqueness theorem* right away.

3. Matrices satisfying M1 and M2. This already implies a lot about the matrix.
4. Matrices satisfying all four: these are the *generalized Cartan matrices* we discussed last class. Kostant's game makes sense in this context.
5. Simply-laced generalized Cartan matrices: those matrices coming from simple graphs (meaning $a_{ij} \in \{0, -1\}$ for $i \neq j$). For these we have the ADE classification, and the very strong uniqueness result that the result of the Kostant game doesn't depend on the starting position of the initial chip.

§6.3 Finite, affine, indefinite type

Here is the theorem:

Theorem 6.3 (Vinberg, see [Kat, Theorem 4.3])
Let A be an indecomposable $n \times n$ matrix which satisfies conditions (M1), (M2). Then exactly one of the following is true:

- (Finite type) There exists a vector $u > 0$ such that $Au > 0$.
- (Affine type) There exists a vector $u > 0$ such that $Au = \vec{0}$.
- (Indefinite type) There exists a vector $u > 0$ such that $Au < 0$.

Moreover, A and $A^\top$ are of the same type.

Note (M0) and (M3) are explicitly not required.

Remark 6.4 (Contiuining the religious connotations)**.** In our old terminology:

- Finite type is "heaven" because we can make everyone excited.
- Indefinite type is "hell" because we can make everyone unhappy.
- Affine type is "purgatory" because we can make everyone happy but not excited.

In fact, affine type and finite type turn out to be much more closely related to each other (while indefinite type is much "worse" than both), in the same way that people go to heaven after purgatory.

Remark 6.5. Note that if any diagonal is negative (note that there's no axioms on diagonal entries) then A is automatically of indefinite type.

In fact, there's more.

Theorem 6.6 (Vinberg, continued)
Retain the setting of the previous theorem. Then

- Suppose A is of **finite type**. Whenever $Av \geq 0$, either $v > 0$ or $v = \vec{0}$. Moreover $\det A \neq 0$.
- Suppose A i s of **affine type**. If $Av \geq 0$, then $Av = \vec{0}$. Moreover, the column rank of A is exactly 1.
- Suppose A is of **indefinite type**. If $Av \geq 0$ and $v \geq 0$, then $v = \vec{0}$.

In other words:

- Whenever A is of finite type, there are no additive functions but there exists a sub-additive function.
- Whenever A is of affine type, then every sub-additive function is additive.
- Whenever A is of indefinite type, then we cannot find any sub-additive functions at all.

We now give a characterization in terms of the firing game.

Corollary 6.7 (via Firing Game)
For A satisfying (M1) and (M2):

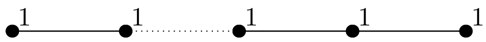
(1) A is of finite type if and only if the A-firing game is finite for every initial configuration.

(2) Assume also A has integer entries. Then A is of affine type if and only if there exists a cycle in the A-firing game.

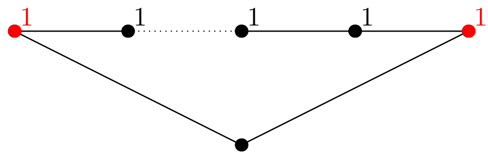
§6.4 Generalized Cartan matrices

From now on assume all of (M0)-(M3).

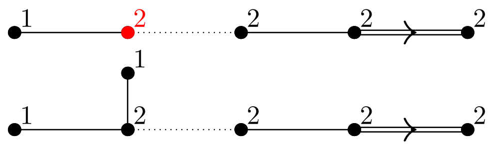
Recall that we have the list of finite type diagrams, with their labels marked, and with excited vertices marked in red.

- A_n: 

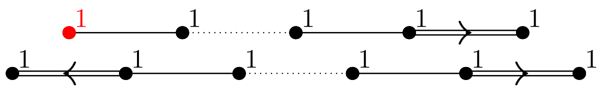
  We can make this affine by adding a single vertex to get $\widetilde{A}_n$.

- B_n: depending on where we drop the initial chip, we can get the following two diagrams. First, if we drop the initial chip at the second vertex, we get:

  On the other hand, if we drop the initial chip at the far right, we instead get the picture

- Type C_n will be more interesting: there will be three extensions, not just two.

§7 February 24, 2017

§7.1 Outline of proof of Vinberg theorem

Definition 7.1. Let A be an *indecomposable* real $n \times n$ matrix (so the resulting directed multi-graph is connected). We say $c \in \mathbb{R}^n_{\ge 0}$ is

- **additive** if $A^\top c = \vec{0}$,
- **coadditive** if $Ac = \vec{0}$,
- **subadditive** if $A^\top c \ge 0$, and
- **co-subadditive** if $Ac \ge 0$.

Here is the key observation for the Vinberg trichotomy theorem we mentioned last time, which seems almost trivial at first.

Lemma 7.2
For any (co)subadditive c, either $c = \vec{0}$ or $c > 0$.

Proof. Assume $c \neq \vec{0}$. Consider neighboring vertices i and j, where $c_j \neq 0$. (meaning $a_{ij} < 0$). Since $A^\top c \ge 0$, we require

$$a_{ii}c_i \ge \sum_{j \neq i} -a_{ij}c_j.$$

The right-hand side is strictly positive now, so $c_i > 0$.
Hence connected-ness now implies $c_k > 0$ for every k. □

Now, consider the cone

$$K_A = \{u \in \mathbb{R}^n \mid Au \ge 0\}$$

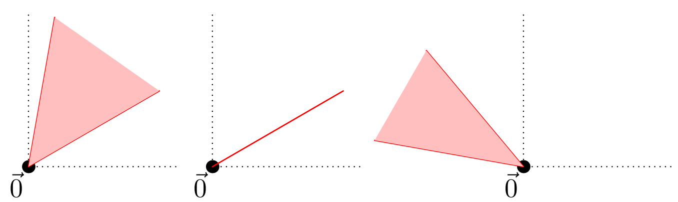
and the positive orthant

$$O = \{u \in \mathbb{R}^n \mid u \ge 0\}$$

and observe that $K_A \cap O$ consists exactly of the co-subadditive functions. The previous lemma then implies K_A intersects the boundary of O *only* at the origin. This geometric surprise then implies that one of three situations happens:

- K_A is completely contained inside O — the finite case.
- K_A is a line through the origin — the affine case.
- K_A is completely disjoint from O — the indefinite case.

(Image below, with K_A drawn in red.)



§7.2 Matrix firing for A via Vinberg theorem

We consider firing as before, with real entries. Explicitly, if the ith position has more than a_{ii} counters, then we may fire by subtracting off the ith column.

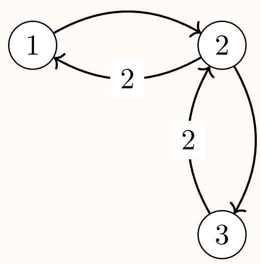
We can imagine the configuration space as points in $\mathbb{R}^n$. Then the three cases we mentioned are as follows:

- Finite: any firing process is finite.
- Affine: any firing process is bounded (stays inside some simplex).
- Indefinite: any firing process is unbounded given sufficiently large starting configurations.

Here's the key example: let G be an undirected graph or network[1]. Then we let L_G denote the Laplacian matrix, meaning the ith diagonal entry contains the outdegree of i and the (i, j)th entry is the negative of the weight from i to j.

Example 7.3 (Laplacian matrix)

Consider the graph



The Laplacian matrix is then

$$L_G = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -1 \\ 0 & -2 & 2 \end{bmatrix}$$

This is of affine type because it has an eigenvector $[1, 1, 1]^\top$. This corresponds to chip-firing without a sink.

Example 7.4 (Reduced Laplacian matrix)

Instead consider L'_G the reduced Laplacian (corresponding to chip-firing with a sink), which is L_G with a row or column deleted. Then L'_G has finite type.

Thus in fact

- Finite case corresponds exactly to reduced Laplacian matrix.
- Affine case corresponds exactly to Laplacian matrix.
- Indefinite case is weird.

[1] Networks are graphs where we allow multiplicity of edges to be nonnegative real numbers rather than integers

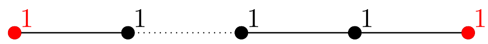
§7.3 Cartan matrices via Vinberg theorem

Let A be a generalized Cartan matrix now, meaning $a_{ij} \in \mathbb{Z}$, $a_{ii} = 2$ in addition to previous assumptions. These two conditions "make everything very rigid", meaning that there are only a few finite and affine cases.

Remark 7.5 (Classification philosophy)**.** The hardest part is to write down this list. Once it's done, one can "by examination" verify that it's correct.

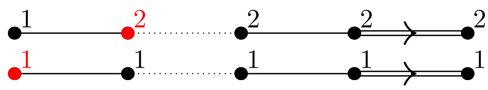
§7.4 Finite list

- For $n > 1$, on A_n there are two excited vertices, shown in red.

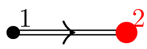


When $n = 1$, there's only a single vertex which is somehow "doubly excited".

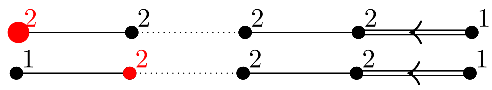
- B_n: there are two ways to play the Kostant game, depending on whether we place the initial chip on the left and right, respectively.



Like with A_n there is an exceptional case B_2:

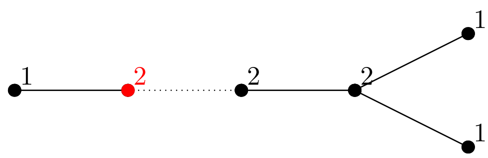


- C_n: there are two ways to play the Kostant game, depending on whether we place the initial chip on the left and right, respectively. Again we have "doubly excited points".

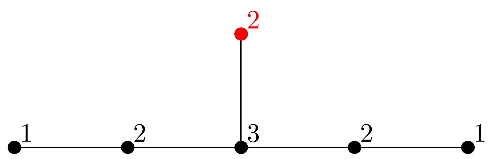


Note $C_2 = B_2$.

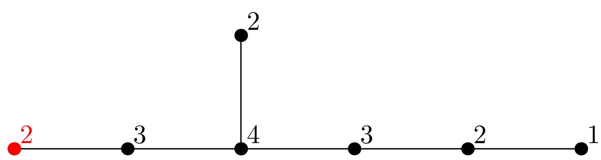
- D_n:



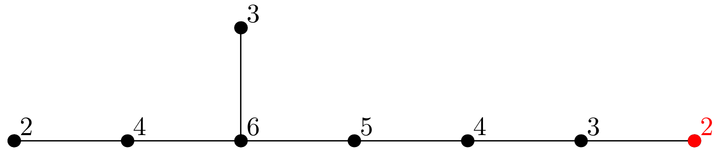
- E_6:



- E_7:



- E_8:



- F_4:

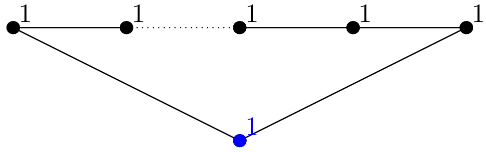
- G_2:

Remark 7.6. Postnikov mentions that combinatorialists like A_n and lie theorists like E_8.

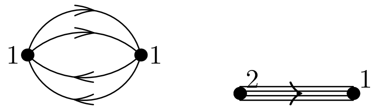
§7.5 Affine list

We now complete the finite diagrams by adding one vertex.

For $n > 1$ we extend A_n to $\widetilde{A}_n$ by adding one vertex (marked blue):



As for $\widetilde{A}_1$, there are two extensions, one with two arrows going each direction and one with a quadruple arrow (!).



§8 February 27, 2017

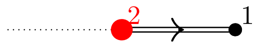
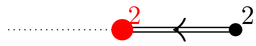
§8.1 Constructing affine diagrams

We obtain the affine diagrams from the finite ones by the following procedure. Consider an affine diagram with some excited ones. Take a new node and add it adjacent to these excited nodes.

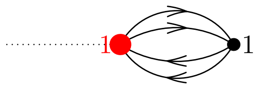
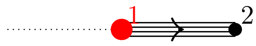
- Given an excited node with 2 chips, we add one edge and a chip with 1.

- If there is just 1 chip, do the same but add 2 edges to the new one and 1 back.

- If there is a doubly excited node with 2 chips, there are two possible extensions.

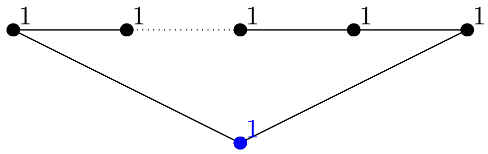
- If there is a doubly excited node with 1 chip again there are two possible extensions as mentioned last time (the last involving a quadruple arrow).

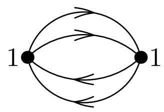
Finally, in the case A_n when there are two excited vertices we simply add a single vertex adjacent to both. This gives us a recipe to construct the affine diagram from the finite one.

§8.2 Draw all affine diagrams

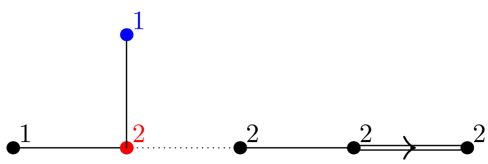
For $n > 1$ we extend A_n to $\widetilde{A}_n$ by adding one vertex (marked blue):



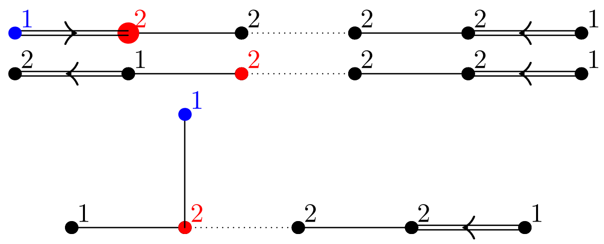
As for $\widetilde{A}_1$, there are two extensions, one with two arrows going each direction and one with a quadruple arrow (!).

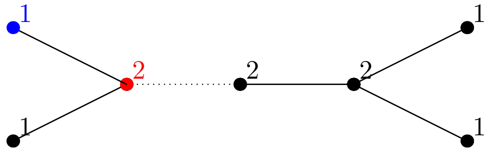
Here is $\widetilde{B}_n$:



Here is $\widetilde{C}_n$:

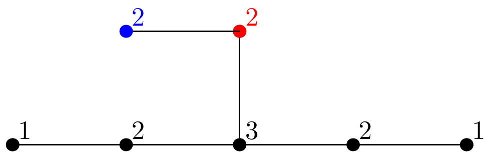


Here is $\widetilde{D}_n$:

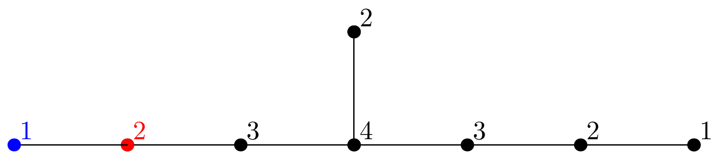


Here are $\widetilde{E}_6$, $\widetilde{E}_7$, $\widetilde{E}_8$:

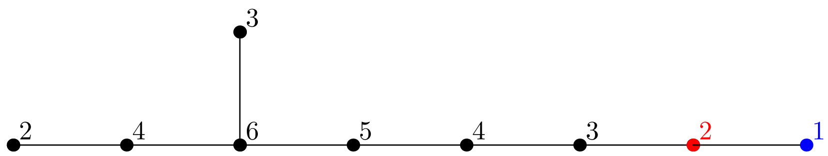
- $\widetilde{E}_6$:



- $\widetilde{E}_7$:



- $\widetilde{E}_8$:



Here are the two versions $\widetilde{F}_4$:

Here are the extensions of $\widetilde{G}_2$:

§8.3 Observations

Recall that:

- Finite type iff there exists a nonzero subadditive function which is not additive.
- Affine type iff there exists a nonzero additive function.

Lemma 8.1

Any proper induced subgraph of a graph of finite or affine type is of affine type.

Proof. Simply restrict the additive function to the subgraph. $\square$

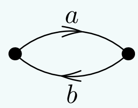
Lemma 8.2

The following Dynkin diagrams are of indefinite type:

(1) The two-vertex graph corresponding to

$$\begin{bmatrix} 2 & -a \\ -b & 2 \end{bmatrix}$$

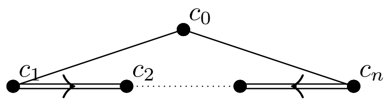
Pictorially this is:



(2) Any cycle which has at least one nonsimple edge.

(3) A triple edge adjacent to any non-simple edge.

Proof. We check (2) only. Indeed, we have some inequalities $2c_i \ge c_{i-1} + c_{i+1} + \text{stuff}$, where the stuff is some more nonnegative coefficients corresponding to the non-simple edges.



Then adding all the inequalities gives

$$2\sum c_i \ge 2\sum c_i + \text{stuff}$$

where the stuff has at least one more term; hence it follows that $c = \vec{0}$. $\square$

The main observation that

Theorem 8.3

Any graph G has no subgraph from the affine list or from the previous lemma belongs to the finite list we provided.

So the converse implies that our finite list is complete.

Proof. Indeed, suppose G avoids both obstructions (not in affine list or in the previous lemma). Then:

- G has no cycles, since A_n is contained in the affine list and the previous lemma excludes cycles which at least one non-simple edge.
- G can't have more than four arrows on any edge (because of $\widetilde{A}_1$).

- If G contains a triple edge, then $G = G_2$ (because of the extensions $\widetilde{G}_2$ and the cycle condition).
- There is at most one double edge, because of $\widetilde{B}_n$, $\widetilde{C}_n$.
- There exists at most one trivalent vertex (because of $\widetilde{D}_n$) and moreover in such a graph we have no double edges (because of the last $\widetilde{C}_n$).
- ...

Thus we have shown the lemma together with the forbidden subgraphs $\widetilde{A}_n$, $\widetilde{B}_n$, $\widetilde{C}_n$, $\widetilde{D}_n$, $\widetilde{E}_6$, $\widetilde{E}_7$, $\widetilde{E}_8$, $\widetilde{F}_4$, $\widetilde{G}_2$ give us the conclusion. □

This proof is weird because it seems almost circular. The algorithm is:

- Write down the finite list, and claim it's complete.
- Generate the affine list by augmenting the finite list appropriately.
- Use the lemma along with the affine list as forbidden subgraphs, and check that these forbidden conditions restrict us back to the finite list we started with.

Thus we have

Theorem 8.4
This is a complete classification of finite and affine generalized Cartan matrices.

Corollary 8.5
Kostant's game is finite if and only if the generalized Cartan matrix is finite.

Proof. If the game stops, then the ending point is a subadditive function which is not additive, since in a final configuration has at least one excited vertex. □

Exercise 8.6. Prove that if A is of finite type, then Kostant's group is finite, without using classification.

§8.4 Root system

We now define a root system (at last!).

Definition 8.7. Suppose V is a Euclidean space ($\mathbb{R}^n$ with a dot product) and let α be a nonzero vector. By H_α we denote the hyperplane orthogonal to α and by s_α the reflection about H_α.

Definition 8.8. A **root system** is a finite subset

$$\Phi \subset V \setminus \{0\}$$

such that

(1) If $\alpha \in \Phi$, then $s_\alpha \Phi = \Phi$. In other words reflecting one root with another root gives another root.

(2) Φ spans V.

(3) If α and β are linearly dependent, then either $\alpha = \beta$ or $\alpha = -\beta$.

(We alluded to a "crystallographic" condition that can be added. But we give the full definition next lecture.)

Condition (1) is the main condition. For (2), we can restrict any non-spanning Φ to its span anyways. Condition (3) is cosmetic, and some authors omit it.

In the next lecture we will see that this root system picture corresponds exactly to Cartan matrices of finite type.

§9 March 1 2017

Today we'll be going through standard notations and definitions of root systems, and then in the near future talk about some "numerology" of these root systems (some special numbers associated to them). Afterwards the direction of the course may vary depending on interest and demand.

§9.1 Notations

From now on V is a Euclidean space of dimension r with inner product $(-,-)$ (we have switched to the letter r, which is standard in this area of mathematics). As in last lecture, we let H_α be the hyperplane perpendicular to a nonzero vector $0 \neq \alpha \in V$ and we let s_α be the reflection across H_α by

$$s_\alpha : \lambda \mapsto \lambda - 2\frac{(\lambda,\alpha)}{(\alpha,\alpha)}\alpha.$$

(We can check this works since $s_\alpha(\alpha) = -\alpha$ and that it fixes the hyperplane.) We introduce the following notation:

Definition 9.1. For each $0 \neq \alpha \in V$ denote

$$\alpha^\vee \overset{\text{def}}{=} \frac{2}{(\alpha,\alpha)}\alpha.$$

This lets us simplify the formula s_α to

$$s_\alpha(\lambda) = \lambda - (\lambda, \alpha^\vee)\alpha.$$

§9.2 Definition of root systems

We recall the definition of the root system from the previous lecture. We add in the following terminology:

- The **rank** of a root system Φ is the dimension of the ambient vector space.
- The elements of Φ are called **roots**.

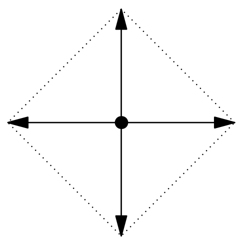
Finally, we add a new condition:

Definition 9.2. A root system Φ is **crystallographic** if for any $\alpha, \beta \in \Phi$, we have $(\alpha^\vee, \beta) \in \mathbb{Z}$. Thus $s_\alpha(\beta)$ will be an integer linear combination of α and β.

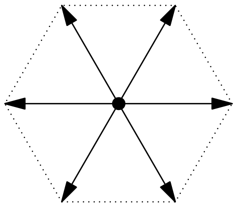
§9.3 Examples of root systems

§9.3.1 Crystallographic examples of rank two

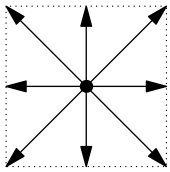
The following root system is $A_1 \times A_1$, a square.



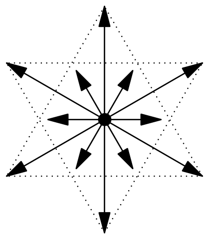
A hexagon also works, called A_2.



Next is B_2, inscribed in a square.



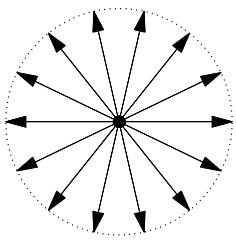
Finally, there's a root system G_2.



These are all such systems up to isomorphism.

§9.3.2 Non-crystallographic examples of rank two

Once the crystallographic condition is dropped, one can take any rotation or scaling. In addition, we obtain a new family of examples $I_2(n)$, the vertices of a $2n$-gon.



I think I should add the construction for A_n here

§9.4 Structure of root systems

Definition 9.3. Let $\Phi_1 \subset V_1$, $\Phi_2 \subset V_2$ be root systems. Then we can define the root system

$$\Phi_1 \cup \Phi_2 \subseteq V_1 \oplus V_2$$

in the obvious way. (Really this is $(\Phi_1, 0) \cup (0, \Phi_2)$.)

Definition 9.4. We say Φ is **irreducible** if it cannot be written as the union of root systems of lesser rank.

Definition 9.5. The **Weyl group** $W = W_\Phi \subseteq \mathrm{GL}(V)$ is the group generated by s_α by all $\alpha \in \Phi$. Since W acts by permutations on Φ, it is finite.

Definition 9.6. Pick a generic linear form $f(x) = (\lambda, x)$ on V, not vanishing on any element of Φ. We define

- The **positive roots** $\alpha \in \Phi_\lambda^+$ such that $f(\alpha) > 0$.
- The **negative roots** $\beta \in \Phi_\lambda^-$ such that $f(\beta) < 0$.

Thus vectors of Φ are split into Φ^+ and Φ^-.

There are multiple choices of λ but in fact they are all equivalent.

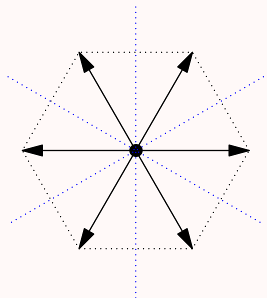
Lemma 9.7 (Positive roots are unique)
Let Φ_λ^+ and $\Phi_{\lambda'}^+$ be two choices of positive roots. Then there exists $w \in W$ such that

$$w(\Phi_\lambda^+) = \Phi_{\lambda'}^+.$$

In fact we will later see that the choice of this w is unique. For now, to prove this lemma we use the notion of a Weyl chamber.

Definition 9.8. The **Coxeter arrangement** is the collection of hyperplanes orthogonal to any some root of Φ. The resulting regions are called **Weyl chambers**.

Example 9.9 (Coxeter arrangement for A_2)
We draw in blue the three hyperplanes in the Coxeter arrangement for A_2. This divides the two-dimensional space into six Weyl chambers.



Proof of lemma. First note that the choice of roots depends only on the Weyl chamber that λ lies in.

If λ and λ' live in Weyl chambers separated by H_α, we then see that $w = s_\alpha$ works.

Otherwise, all we have to do is take a path between λ and λ' which is generic in the sense that it doesn't pass through any intersection points of multiple hyperplanes. This path intersects hyperplanes $H_{\alpha_1}, \dots, H_{\alpha_k}$, say. Then the reflection $w = s_{\alpha_k} \dots s_{\alpha_1}$ works. $\square$

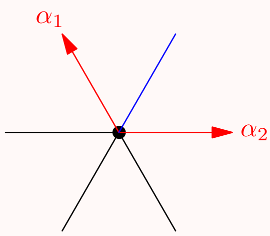
Thus, since the choice of Weyl chamber doesn't matter, we will fix a **fundamental chamber**, thus fixing a choice of Φ_+ of positive roots. We then consider the **cone** generated by $\alpha \in \Phi_+$,

$$\left\{ \sum_{\alpha \in \Phi_+} c_\alpha \alpha \mid c_\alpha \geq 0 \right\}.$$

Thus we can take $\alpha_1, \dots, \alpha_m \in \Phi_+$ a minimal set of **generators** for this cone.

Example 9.10 (Cone in A_2)

Here is an example of a cone for A_2, with α_1 and α_2 being positive roots generating the cone, and the third root in the cone being marked in blue.



The key lemma for these roots is:

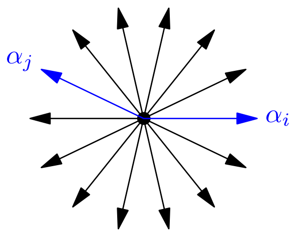
Lemma 9.11 (All angles between generators are non-acute)

Retain the notation above. For any $i \neq j$, we have $(\alpha_i, \alpha_j) \le 0$.

Proof. Take the two-dimensional plane spanned by α_i, α_j, and look at the subset of Φ inside it. We have a root system of rank two inside Φ, some of which are positive.

Assume for contradiction that $(\alpha_i, \alpha_j) > 0$, meaning the angle is acute. Then consider $\pm s_{\alpha_i}(\alpha_j)$. Exactly one of these is positive and neither lie inside it. □

Remark 9.12. Actually it isn't hard to see that in a two-dimenisonal root system, all angles around the origin due to closure under reflections. So α_i and α_j must end up being "nearly opposite" vectors.



These α_i are span V, but in fact.

Lemma 9.13 (Generators of positive cone form a basis)

Retain the notation $\alpha_1, \dots, \alpha_m$ as before. Then α_i form a basis for V. In particular, $m = r = \dim V$.

Proof. It's clear they are spanning, so we want to check linear dependence. This will be the same as the proof of Farkas lemma. Suppose that we have a linear dependence

$$c_1\alpha_1 + \dots + c_m\alpha_m = 0.$$

First if $c_i \ge 0$ for all i, we claim all c_i are zero. This is obvious geometrically; formally, apply a dot product α_i on both sides of the dependence. Similarly if all c_i are negative, we have another contradiction.

On the other hand if some c_i are negative and some are positive, we move all the coefficients to the same side,

$$v = d_1\alpha_1 + \dots + d_s\alpha_s = d_{s+1}\alpha_{s+1} + \dots + d_m\alpha_m$$

where each d_i is nonnegative. We have $v \neq 0$ by the preceding paragraph. But now

$$0 < (v, v) = (d_1\alpha_1 + \dots + d_s\alpha_s, d_{s+1}\alpha_{s+1} + \dots + d_m\alpha_m) .$$

The right hand side is non-positive by expanding the dot product. Contradiction. $\square$

Thus in summary, we have that:

- A choice of Weyl chamber gives a set of positive roots.
- This gives us a choice of basis, which we call **simple roots**.

§10 March 3, 2017

I was not feeling well this fine Friday afternoon, and thus did not attend lecture. Thanks to Tom Roby for sending me his handwritten notes.

§10.1 Simple reflections

Let Φ be a root system for the Weyl group W, and let $\Phi^+ \subset \Phi$ be the positive roots. We denote by $\alpha_1, \dots, \alpha_r$ the simple roots. We have the following properties:

- $\alpha_1, \dots, \alpha_r$ form a basis of V.
- $(\alpha_i, \alpha_j) \le 0$ for $i \ne j$.
- Any $\alpha \in \Phi^+$ is a $\mathbb{N}$-linear combination of α_i's.
- For any other choice of simple roots $\alpha_1', \dots, \alpha_r'$, there exists $w \in W$ such that $w\{\alpha_1, \dots, \alpha_r\} = \{\alpha_1', \dots, \alpha_r'\}$.
- For all $\alpha \in \Phi$, there exists α_i and $w \in W$ such that $w(\alpha_i) = \alpha$.

Definition 10.1. A **simple reflection** is one of the form

$$s_i \overset{\text{def}}{=} s_{\alpha_i}$$

for some $1 \le i \le r$ (as noted α_i is a simple root).

Lemma 10.2 (Simple reflections generate W)
W is generated by simple reflections $s_1, \dots, s_r$.

Proof. Here is a geometric proof. Note that s_i correspond to reflections around walls of the fundamental chamber C_0. So if C' is an adjacent chamber, then $C' = s_i(C_0)$.

Then reflections with respect to walls of C' are of teh form

$$s_j' = s_{s_i(\alpha_j)} = s_i s_j s_i$$

for some j, with i fixed ($C' = s_i(C_0)$). And so on. Later we'll see more details of this construction. $\square$

§10.2 Cartan matrices

Recall that $\alpha^\vee = \frac{2\alpha}{(\alpha,\alpha)}$. Now given a root system Φ with simple roots $\{\alpha_1, \dots, \alpha_r\}$, we can construct the matrix

$$A = A_\Phi = (a_{ij}) \qquad \text{where} \qquad a_{ij} = \left(\alpha_i^\vee, \alpha_j\right).$$

We now have:

Proposition 10.3 (It's a generalized Cartan matrix)
Let $A = A_\Phi = (a_{ij})$ as above.

(i) $a_{ii} = 2$.

(ii) $a_{ij} \le 0$ for $i \neq j$.

(iii) $a_{ij} \neq 0 \iff a_{ji} \neq 0$.

(iv) $a_{ij} \in \mathbb{Z}$ if Φ is crystallographic.

Hence A is a generalized Cartan matrix.

This lets us relate configurations to the Kostant game in the following way: a configuration maps to a vector via

$$\vec{c} = (c_1, \dots, c_r) \mapsto \lambda = c_1\alpha_1 + \dots + c_r\alpha_r.$$

We now compute

$$\begin{aligned} s_i(\lambda) &= \lambda - \left(\alpha_i^\vee, \lambda\right)\alpha_i \\ &= \lambda - \sum_j a_{ij}c_j\alpha_i \\ &= c_1\alpha_1 + \dots + c_i'\alpha_i + \dots + c_r\alpha_r. \end{aligned}$$

with c_i' corresponding to Kostant game.

Then we observe that

Theorem 10.4 (Root system $\leftrightarrow$ Kostant game)
Thus vectors lying in the cone of the root system Φ correspond exactly to configurations in Kostant's game with the matrix $A = A_\Phi$, with simple reflections corresponding to firings.

Accordingly,

Theorem 10.5 (Irreducible $\Phi \leftarrow$ finite type Cartan)
Crystallographic irreducible root systems correspond to generalized indecomposable Cartan matrices of finite type, up to re-ordering the rows and columns of the matrix (equivalently, relabelling the nodes of the Dynkin diagram).

§11 March 6, 2017

Let W be a Weyl group with simple reflections $s_1, \dots, s_r$ as usual.

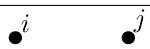
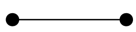
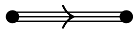
§11.1 Presentation of the Weyl group

Theorem 11.1 (Presentation of the Weyl group)
The group W is generated by $s_1, \dots, s_r$ with the following **Coxeter relations**:

(1) $s_i^2 = 1$ for all i, and

(2)
$$\underbrace{s_i s_j s_i \dots} = \underbrace{s_j s_i s_j \dots} \qquad \text{for all } i \neq j$$
where $2m_{ij} = \#\left\{\pm\alpha_i, \pm\alpha_j, \pm s_i(\alpha_j), \pm s_j(\alpha_i), \pm s_i s_j(\alpha_i), \dots\right\}$.

Remark 11.2. In the crystallographic case, we have only four cases for m_{ij}.

m_{ij}	Graph Picture	Matrix
$m_{ij} = 2$		$a_{ij} = a_{ji} = 0$
$m_{ij} = 3$		$a_{ij} = a_{ji} = -1$
$m_{ij} = 4$		$a_{ij} = -2$, $a_{ji} = -1$
$m_{ij} = 6$		$a_{ij} = -3$, $a_{ji} = -1$.

§11.2 Proof of Coxeter relations

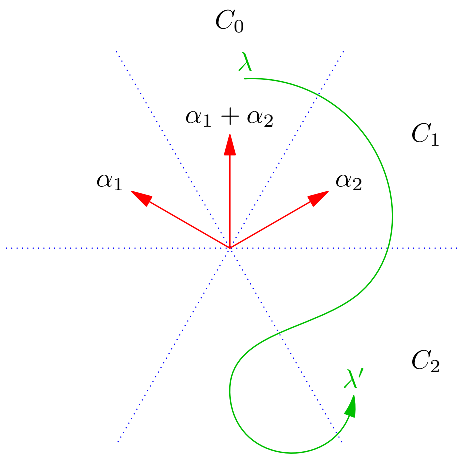
It's not hard to see the Coxeter relations are true; we want to show they are necessary.

Proposition 11.3
Let C_0 be the fundamental Weyl chamber and let C any other Weyl chamber. If
$$s_{i_1} \dots s_{i_k}(C_0) = s_{j_1} \dots s_{j_{k'}}(C_0) = C$$
then $s_{i_1} \dots s_{i_k}$ and $s_{j_1} \dots s_{j_{k'}}$ are related by Coxeter moves.

Proof. Suppose $w = s_{i_1} \dots s_{i_k} \in W$. We are going to write
$$w = \dots \left(s_{i_1} s_{i_2} s_{i_3} s_{i_2}^{-1} s_{i_1}^{-1}\right)\left(s_{i_1} s_{i_2} s_{i_1}^{-1}\right) s_{i_1} = s_{\beta_k} \dots s_{\beta_1}$$
where $s_{\beta_1} = \alpha_{i_1}^{-1}$, $\beta_2 = s_{i_1}\alpha_{i_2}$, $\beta_3 = s_{i_1} s_{i_2}(\alpha_{i_3})$, and so on. The trick is that:

> **The s_β will tell us which hyperplanes we cross when we go from C_0 to C.**

For example, consider the following figure:



Then we have $\beta_1 = \alpha_1$, $\beta_2 = \alpha_1 + \alpha_2$ $\beta_3 = \alpha_2$ and $\beta_4 = -\alpha_2$, corresponding to the four walls that we pass.

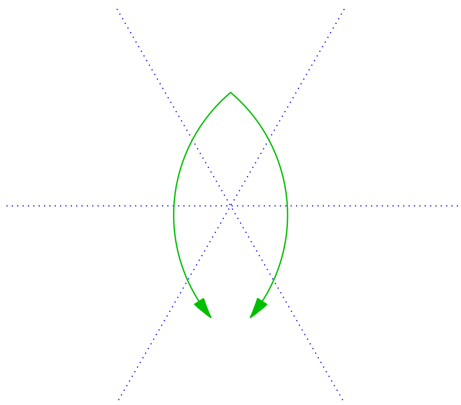
Thus we have a way to translate a path into a sequence of reflections in W. We would like to go the other way, and we can go do so as long we have some "generic" condition.

Definition 11.4. Let p be any smooth continuous path from $\lambda \in C_0$ to $\lambda' \in C$ such that all interescitons of p wyth hyperplanes H_β are **transversal**, meaning,

- p is not tangent to any H_β and
- p does not pass any intersection of H_β's which is codimension of ≥ 2.

Now what we would like is to "deform" one path p to another path $\widetilde{p}$. We can do this, but we will occasionally (in the process of deforming) run into non-transversal paths. So we need two **local moves** in addition to continuous deformation:

- If a path crosses H_β and then crosses H_β again, then we can pretend it never crossed H_β to begin with. Geometrically this corresponds to tangency; algebraically this gives us the relation $As_is_iB \to AB$.
- If a path crosses a codimension two intersection, we can push it through this intersection:



Algebraically, this is the same as changing $s_is_js_i \dots$ to $s_js_is_j \dots$.

We will avoid codimension three or higher intersections (geometrically it's clear this is possible). Thus these two local moves complete the proof. $\square$

Thus we arrive at the result:

Corollary 11.5 (W bijects to Weyl chambers)
There are exactly $|W|$ Weyl chambers, corresponding to $w(C_0)$ for $w \in W$.

§11.3 Dual description

Here is a dual description of the result we just proved. Let W be a Weyl group with fundamental chamber C_0.

Definition 11.6. For λ any point strictly inside C_0, we define the ***W*-permutohedron** $\Pi(\lambda)$ by taking the convex hull $w(\lambda)$ for $w \in W$.

Then we have the following results.

Theorem 11.7 (W-permutohedron properties)
Take $\Pi(\lambda)$ as above.

(1) $\Pi(\lambda)$ has $|W|$ vertices.

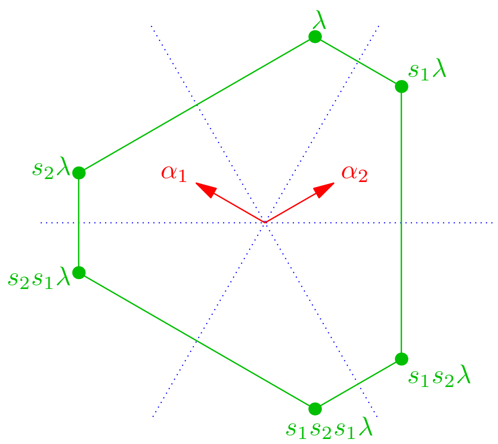
(2) Two vertices $u(\lambda)$ and $w(\lambda)$ are connected if and only if the corresponding Weyl chambers are adjacent, if and only if they differ by a simple reflection (meaning $w = u \cdot s_i$).

Proof. (1) It is obvious each $w(\lambda)$ are distinct, since they are in different Weyl chambers. To see they are all vertices, use the fact that W acts on the points in a symmetric transitive way, meaning either all points are vertices of $\Pi(\lambda)$ or none of them are.

(2) Again by symmetry.

□

As an example,



§11.4 Weak Bruhat

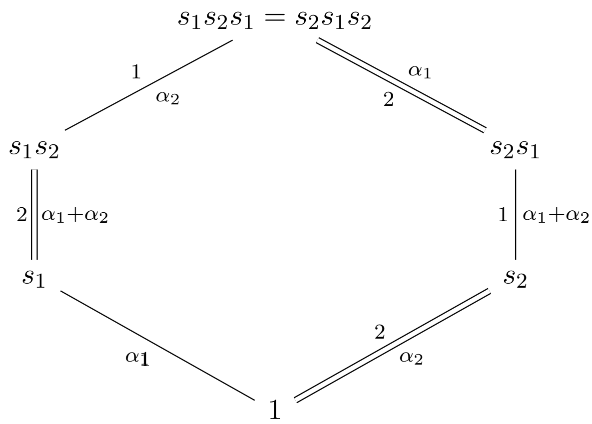
We can now construct the following graph out of this.

Definition 11.8. The **weak Bruhat graph** is defined by setting the vertices to be $w \in W$ and the edges (u, w) where $w = us_i$.

There are two ways to label the edge $[u, w]$:

- The right label of $[u, us_i]$ is i
- The left label of $[u, w]$ is $w = s_\beta u$.

It will be convenient to have 1 on the bottom, so our example now reads:



A decomposition of $w = s_{i_1} \dots s_{i_k} = s_{\beta_k} \dots s_{\beta_1}$ corresponds then to a walk (path with repeated vertices) from 1 to w, where i_1, i_k are right labels of edges and $\beta_1, \dots, \beta_k$ are left labels.

Actually, the graph we drew came be made by a poset as follows.

Definition 11.9. For a $w \in W$, we let $\ell(w)$ be the length of the shorted decomposition of $w = s_{i_1} \dots s_{i_\ell}$. Such a decomposition is called a **reduced decomposition**.

Thus the Weyl group becomes a graded poset by ℓ, called the **weak Bruhat order**.

§12 March 8, 2017

As usual W is a Weyl group with simple reflections $s_1, \dots, s_r$ satisfying the Coxeter relations.

We retain the notation $\ell(w)$ from last lecture. Notice that $\ell(w) = \ell(w^{-1})$ just because if $w = s_{i_1} \dots s_{i_\ell}$ hen $w^{-1} = s_{i_\ell} \dots s_{i_1}$.

§12.1 Inversions

Definition 12.1. An **inversion** of w is a root $\alpha \in \Phi^+$ such that $w(\alpha) \in \Phi^-$. We let $\operatorname{Inv}(w)$ denote the set of such inversions.

Lemma 12.2 (Length is number of inversions)
For any w we have $\ell(w) = \ell(w^{-1}) = \# \operatorname{Inv}(w)$.

Proof. Let λ be **strictly dominant**, meaning λ is in the interior of the fundamental chamber C_0. Then we've seen already that $\ell(w)$ is the minimal number of hyperplanes H_α we need to cross to get from λ to $w(\lambda)$.

Now take $\alpha \in \Phi^+$, so $(\alpha, \lambda) > 0$. Then we have that H_α separates λ and $w(\lambda)$ if and only if

$$(w(\lambda), \alpha) < 0 \iff (\lambda, w^{-1}(\alpha)) < 0 \iff w^{-1}(\alpha) \in \Phi^{-1}.$$

So the number of separating planes is exactly $\operatorname{Inv}(w^{-1})$. $\square$

Corollary 12.3 (β_i are inversion set)
Let $w = s_{i_1} \dots s_{i_\ell}$ be a reduced decomposition and assume $w = s_{\beta_\ell} \dots s_{\beta_1}$ as last lecture. Then $\operatorname{Inv}(w^{-1}) = \{\beta_1, \dots, \beta_\ell\}$.

Corollary 12.4 (Weak Bruhat order via inversions)
We have $u \le w$ in the weak Bruhat order exactly if $\operatorname{Inv}(u^{-1}) \subseteq \operatorname{Inv}(w^{-1})$.

Remark. $u \le w \not\Longleftrightarrow u^{-1} \le w^{-1}$.

§12.2 Construction of root system of type A_{n-1}

We'll adopt the convention $r = n - 1$. Let V be the vector space

$$V = \left\{ (x_1, \dots, x_n) \in \mathbb{R}^n \mid \sum x_i = 0 \right\} \subseteq \mathbb{R}^n$$

be a hyperplane of codimension one. If we equip $\mathbb{R}^n$ with the usual basis $e_1, \dots, e_n$. Then, we let

$$\Phi = \left\{ \alpha_{ij} \overset{\text{def}}{=} e_i - e_j \mid i \neq j \right\}$$

and let $H_{\alpha_{ij}} = \{x \in V \mid x_i = x_j\}$. This forms the so-called **braid arrangement**. Finally, the reflection $s_{\alpha_{ij}}$ turns out to be

$$s_{\alpha_{ij}} : (x_1, \dots, x_i, \dots, x_j, \dots, x_n) \mapsto (x_1, \dots, x_j, \dots, x_i, \dots, x_n).$$

We pick simple roots $\alpha_i = \alpha_{i,i+1}$ whence s_i are adjacent transpositions. Then, $W \cong S_n$ is the symmetry group.

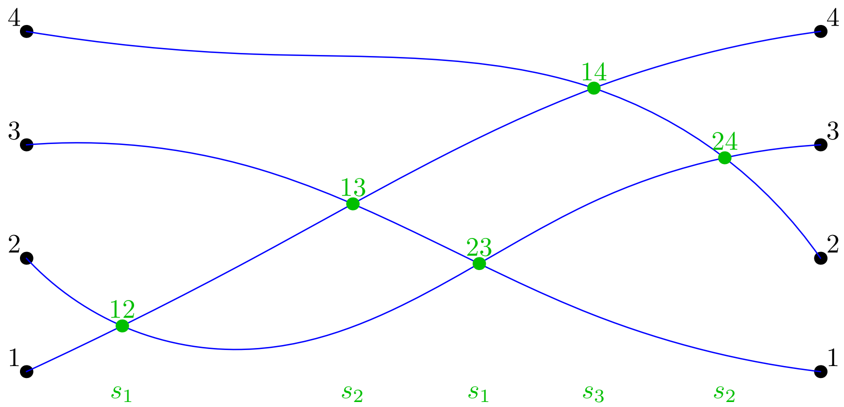
Why is this related to A_{n-1}? To see this, we simply construct the Cartan matrix, and find that it coincides with that of a chain on n vertices.

§12.3 Wiring diagrams

Here is just an example when $n = 4$ (hence $r = 3$) Consider the following permutation

$$w = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 1 & 2 \end{pmatrix}.$$

We can imagine this as a series of "wires", which have some intersections from left to right; this gives us a **wiring diagram**. We label each intersection of wires with s_h where $1 \le h \le r$ is the "height" of the intersection (which is the number of wires below that intersection point, plus one).



The intersections are height 1, 2, 1, 3, 2, and writing these from right to left we obtain

$$\boxed{w = s_2 s_3 s_1 s_2 s_1}.$$

Moreover, it happens that the intersection points correspond exactly to the inversions of the permutation (with the "reduced" condition corresponding to no two paths intersecting twice).

§12.4 Generalizing permutation statistics to Weyl groups

In combinatorics, people are quite interested in the *numerology* of permutations. Roughly, most falls into one of there categories:

- **Mahonian** statistics, based on the number of inversions in the permutation.
- **Eulerian** statistics, related to the number of descents (meaning $w(i) > w(i+1)$). These are based on the Eulerian numbers $A(n,k)$ which is the number of $w \in S_n$ with $k+1$ descents (or k descents, depending on convention).
- Stirling-ian statistics, based on the number of cycles in w.

We're going to make the following point:

All of these make sense for general Weyl groups.

Specifically, we've said $i \in \{1, \dots, r\}$ is an inversion if $w \in W$ when $i \in \operatorname{Inv}(w)$. Now we add:

Definition 12.5. We say $i \in \{1, \dots, r\}$ is a **descent** of $w \in W$ if $\ell(ws_i) < \ell(w)$. This corresponds to going "downwards" in the Bruhat picture. Thus we can define **Coxter-Eulerian numbers** by

$$A_\Phi(k) = \{w \in W \text{ with } k \text{ descents}\}.$$

(Here we're identifying numbers i with simple roots s_i. In Lie theory, apparently people don't do this.)

Remark 12.6. The Coxter-Eulerian numbers give the so-called h-numbers (defined below) of the permutahedron $\Pi(\lambda)$ we saw earlier.

To define the h-numbers for a polyhedron $\mathcal{P}$, we first define

$$f_i = \#i\text{-dimensional faces of } \mathcal{P}$$

and we let the f-polynomial be $f(x) = \sum_{i \ ge0} f_i x^i$. Then we define the h-polynomial to satisfy

$$f(x-1) = h(x) = \sum_{i \ge 0} h_i x^i.$$

It turns out h_i are symmetric and nonnegative as long as $\mathcal{P}$ is simple.

For the permutahedron $\Pi(\lambda)$ we get $A_\Phi(k)$.

Example 12.7 (h-numbers for A_2, a hexagon)

Let $W = A_2$, so $\Pi(\lambda)$ is a hexagon. This has six vertices, six edges and one 2-dimensional face, so

$$f(x) = 6 + 6x + x^2 \implies h(x) = f(x-1) = 1 + 4x + x^2.$$

These correspond to $1 + 4 + 1 = 6$ permutations on three letters: an identity, four with one descent and one with two descents.

§13 March 10, 2017

Let Φ be a crystallographic root system. This lecture we'll generalize the formula $|S_n| = n!$ to a formula for $|W|$ for any Weyl group W.

§13.1 The root poset

Definition 13.1. We define the **root poset** to be the partial ordering on the positive roots Φ^+ with the relation $\geq$ where $\alpha \geq \beta$ if $\alpha - \beta$ is a nonnegative combination of positive (equivalently, simple) roots.

Thus the simple roots are the bottom of the poset.

Proposition 13.2 (The Highest Root)

For irreducible root systems the root poset has a unique maximal element θ called the **highest root**.

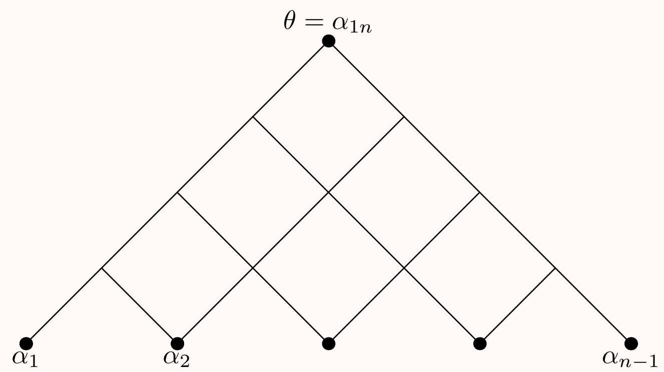
Example 13.3 (Root poset for A_{n-1})

Take $W = A_{n-1}$. Then

$$\Phi^+ = \{\alpha_{ij} = e_i - e_j\}$$

with simple roots $\alpha_i = \alpha i, i+1$. Then $\alpha_{ij} \geq \alpha_{i'j'} \iff i \leq i' < j' \leq j$.

Here is a picture of the poset for $n = 5$:



Remark 13.4. In fact, we get the height function for the poset by: when $\alpha = c_1\alpha_1 \cdots + c_r\alpha_r$ we get

$$\operatorname{ht}(\alpha) = c_1 + \cdots + c_r.$$

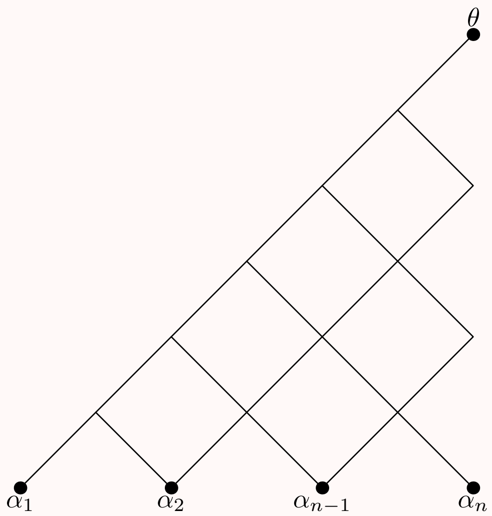
This implies the poset is graded.

Example 13.5 (Root poset for B_n)

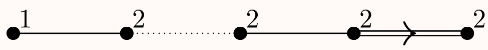
In this example we have

$$\Phi^+ = \{e_i \pm e_j \mid 1 \le i < j \le n\} \cup \{e_i \mid 1 \le i \le n\}$$

The simple roots are $\alpha_i = e_i - e_{i+1}$ and $\alpha_n = e_n$. Here is the picture of the poset for $n = 3$; it is "half" the A_n picture.



In the context of Kostant game, θ corresponds to the following endpoint.



§13.2 Coxeter number

We now define some "magic numbers" of Weyl groups.

Definition 13.6. A **Coxeter element** of W is an element of the form $c = s_1 \dots s_r \in W$.

Lemma 13.7 (Coxeter elemens are conjugate)

Fixing W, all Coxeter elements are conjugate to each other (as we vary the order in which the s_i are multiplied).

Thus we can define

Definition 13.8. The **Coxeter number** h is the order of the Coxeter element (well-defined since they're all conjugate).

Definition 13.9. The **exponents** of W are positive integers $0 < m_1 \le \dots \le m_r < h$ such that the eigenvalues of a Coxeter element $c \in \mathrm{GL}(V)$ (actually $c \in \mathrm{O}(V)$) are

$$\exp\left(2\pi i \cdot \frac{m_j}{h}\right) \qquad j = 1, \dots, r.$$

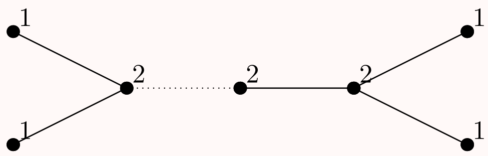
Finally,

Definition 13.10. The **index of connection** f is the determinant of the Cartan matrix.

Now, let $\theta = a_1\alpha_1 + \cdots + a_r\alpha_r$ and set $a_0 = 1$. Then $(a_0, \dots, a_r)$ is an additive function on the nodes of the extended Dnykin diagram.

Example 13.11 ($\widetilde{D}_n$)

Suppose we have the Dynkin diagram $\widetilde{D}_n$.



Then we choose

$$(a_0, \dots, a_r) = \left(1, 1, \underbrace{2, \dots, 2}_{r-3}, 1, 1\right).$$

Theorem 13.12 (h and f determined by θ)

We have

$$\begin{aligned} h &= a_0 + a_1 + \cdots + a_r = \operatorname{ht}(\theta) + 1 \\ f &= \#\left\{i \mid a_i = 1\right\}. \end{aligned}$$

Theorem 13.13 (Exponents determined by root poset; Kostant)

Let $\lambda = (m_r, m_{r-1}, \dots, m_1)$ be a partition and $\lambda^* = (k_1, \dots, k_{m_r})$ its conjugate partition. Then k_i is the number of positive roots of height i.

In particular, $m_r = h - 1$ and $k_1 \ge k_2 \ge \cdots \ge k_{h-1}$.

Example 13.14 (Exponents for A_n)

For $W = A_n$, we have $k_1 = r$, $k_2 = r - 1$, …, $k_{r-1} = 1$. Thus $\lambda^* = (r, r-1, r-2, \dots, 1)$. This is self-dual, so we obtain

$$\lambda = (r, r-1, \dots, 1)$$

hence the exponents are 1, 2, …, r.

Example 13.15 (Exponents for B_n)

For $W = B_n$, we have

$$\lambda^* = (r, r-1, r-1, r-2, r-2, \dots, 1, 1).$$

Taking the dual gives

$$\lambda = (1, 3, 5, \dots, 2r-1)$$

hence the exponents are 1, 3, 5, …, $2r - 1$.

Actually, the exponents satisfy the following properties.

- $m_1 = 1$ and $m_r = h - 1$.
- $m_i = m_{r-i+1} = h$ for all i.

Now the number of elements in the root poset is equal to

$$\left|\Phi^+\right| = m_1 + \cdots + m_r = \frac{rh}{2}$$

so we obtain

Theorem 13.16 (Number of roots)
We have
$$|\Phi| = rh.$$

In fact the following theorem is true too.

Theorem 13.17 (Order of the Weyl group)
For any W, we have the following two formulas:

$$\begin{aligned} |W| &= \prod_{i=1}^{r}(1 + m_i) \\ &= f \cdot r! \cdot a_1 \dots a_r. \end{aligned}$$

§14 March 13, 2017

Last time we saw Weyl's formula

$$|W| = f \cdot r! \cdot a_1 \dots a_r$$

where θ is the highest root $a_1\alpha_1 + \dots + a_r\alpha_r$ and $f = \det A_\Phi = \#\{1 \le i \le r \mid a_i = 1\}$ is the index of connection. This lecture we'll prove this formula, using the so-called *affine Weyl group* (which is infinite!).

§14.1 Root lattice and weight lattice

In fact we well define four lattices in this section: the (co-)root lattice, the (co-)weight lattice.

Definition 14.1. We say $\lambda \in V$ is an **integral weight** if $(\lambda, \alpha^\vee) \in \mathbb{Z}$ for each $\alpha \in \Phi$.

The **fundamental weights**, denoted $\omega_1, \dots, \omega_r$, are the dual basis to the basis of simple co-roots $\alpha_i^\vee$ $(i = 1, \dots, r)$, in the sense that

$$(\omega_i, \alpha_j^\vee) = \begin{cases} 1 & i = j \\ 0 & \text{else.} \end{cases}$$

Definition 14.2. The **weight lattice** P is the $\mathbb{Z}$-lattice of all the integral weights, generated by ω_i.

Definition 14.3. The **root lattice** Q is the $\mathbb{Z}$-lattice of all roots $\alpha \in \Phi$ (or equivalently, just the simple roots $\alpha_1, \dots, \alpha_r$).

Proposition 14.4 ($Q \subset P$)
The root lattice in contained inside the weight lattice.

Proof. It suffices to show each simple root α_i is an integral weight. Assume $\alpha_i = c_1\omega_1 + \dots + c_r\omega_r$, so $(\alpha_i, \alpha_j^\vee) = c_j$ for each j, which is an entry of the Cartan matrix, hence an integer. $\square$

Remark 14.5 (Cartan matrix corresponds to α_i in w_j)**.** In fact, this shows that when we express α_i as linear combinations of ω_i, we end up with the Cartan matrix. For example in A_2 with $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ we will find $\alpha_1 = 2\omega_1 - \omega_2$ and $\alpha_2 = -\omega_1 + 2\omega_2$.

Definition 14.6. The co-root lattice and co-weight lattice are defined by

- The **co-root lattice** $Q^\vee$ is defined by $\alpha_1^\vee, \dots, \alpha_r^\vee$.
- The **co-weight lattice** $P^\vee$ is defined by $\omega_1^\vee, \dots, \omega_r^\vee$.

Remark 14.7. This is a bit confusing since two types of duality are going on:

- The duality going from Q to P corresponds to taking the transpose of the Cartan matrix.
- The between Q and $P^\vee$ is the (Euclidean) dual basis (with respect to inner product). Ditto for $Q^\vee$ and $P^\vee$.

Proposition 14.8 (Index of connection)
$f = [P : Q] = [P^\vee : Q^\vee]$.

Proof. Since $f = \det A$, follows by Remark 14.5. □

§14.2 A picture

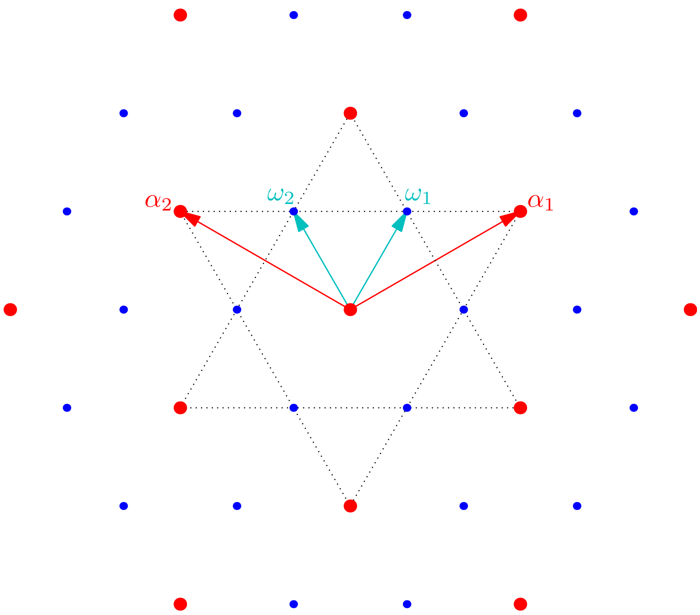
Example 14.9 (Lattice in A_2)
Let $V = \{(x_1, x_2, x_3) \mid x_1 + x_2 + x_3 = 0\}$ be the usual ambient subspace of $\mathbb{R}^3$. Then we have

$$\begin{aligned}\alpha_1 &= \alpha_1^\vee = (1, -1, 0)\\ \alpha_2 &= \alpha_2^\vee = (0, 1, -1).\end{aligned}$$

In that case, one can check

$$\begin{aligned}\omega_1 &= \omega_1^\vee = (1, 0, 0) - \frac{1}{3}(1, 1, 1)\\ \omega_2 &= \omega_2^\vee = (1, 1, 0) - \frac{2}{3}(1, 1, 1).\end{aligned}$$

Here is the picture:



We see visibly, $f = 3$ which follows as

$$\begin{aligned}\alpha_1 &= 2\omega_1 - \omega_2\\ \alpha_2 &= -\omega_1 + 2\omega_2\end{aligned}$$

and hence

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

hence $f = \det A = 3$.

§14.3 Affine Weyl group

Definition 14.10. Let $\alpha \in \Phi$, and $k \in \mathbb{Z}$. We define[2] the **affine hyperplanes** by

$$H_{\alpha,k} = \{\lambda \in V \mid (\lambda, \alpha) = k\}$$

and thus the affine reflection $s_{\alpha,k}$ the reflection with respect to $H_{\alpha,k}$.

Thus we have

$$\begin{aligned} s_(\alpha, k)(\lambda) &\overset{\text{def}}{=} s_\alpha(\lambda) + k\alpha^\vee \\ &= \lambda - (\lambda, \alpha)\alpha^\vee + k\alpha^\vee \\ &= \lambda - ((\lambda, \alpha) - k)\,\alpha^\vee. \end{aligned}$$

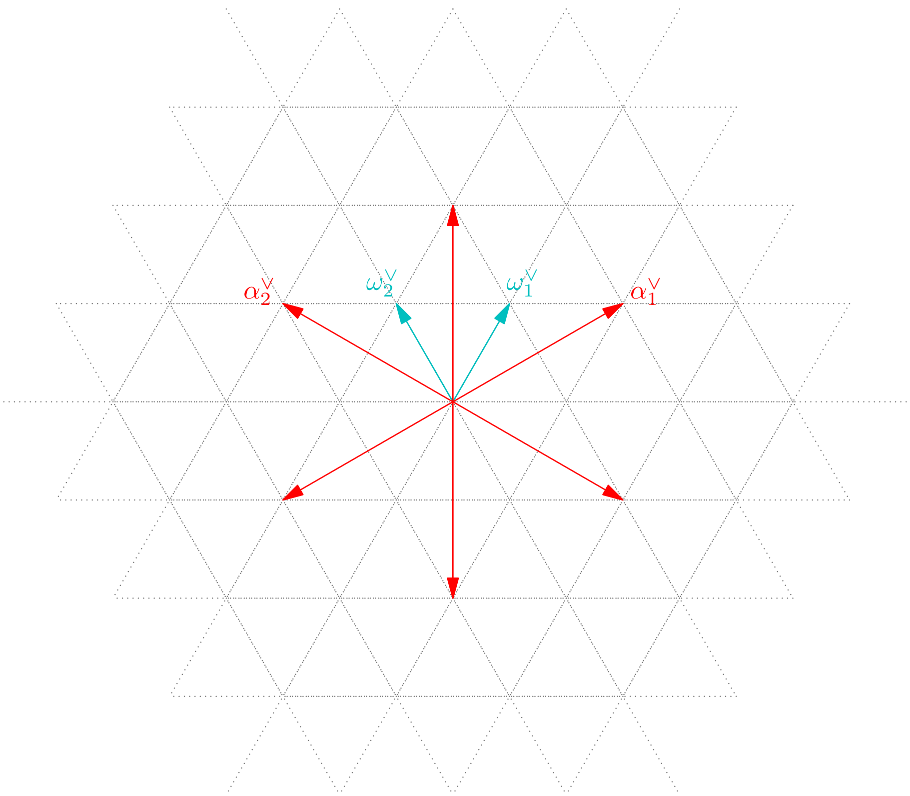
Definition 14.11. The **affine Coxeter arrengement** is the hyperplane arrangement consisting of the (infinitely many!) hyperplane arrangements $H_{\alpha,k}$. The **affine Weyl group** W_{aff} is the group generated by all $s_{\alpha,k}$.

Lemma 14.12 (Affine Weyl group is semidirect product)
We have $W_{\text{aff}} = W \ltimes Q^\vee$.

Proof. For $\beta \in Q^\vee$, we let sh_β be the shift $\lambda \mapsto \lambda + \beta$. We claim $W_{\text{aff}} = \{\mathrm{sh}_\beta \circ w\}$. Clearly $W_{\text{aff}} \subseteq \{\mathrm{sh}_\beta \circ w\}$; on the other hand $\mathrm{sh}_{\alpha^\vee} = s_{\alpha,1} \circ s_\alpha$. □

Definition 14.13. The regions in the affine Weyl arrangement are called **alcoves**.

Here is a picture of an affine Weyl arrangement, with the hyperplanes drawn in dotted lines.:



[2] Some people prefer to use $\alpha^\vee$ instead of α in the definition of $H_{\alpha,k}$.

Lemma 14.14 (Invariance under W_{aff})
The affine Coxeter arrangement W_{aff} is invariant under the action of W_{aff}.

Lemma 14.15 (W_{aff} simply transitive)
W_{aff} acts simply transitively on the alcoves.

Proof. Same proof as we did for W acting on chambers, with the word "chamber" replaced by "alcove" everywhere. □

Thus as before we like to pick a **fundamental alcove** A_0. Like before all of them are equivalent, but typically we like to select the alcove A_0 inside the fundamental chamber C_0 which touches the origin. It turns out that

$$A_0 = \{x \in V \mid (\alpha_i, x) \geq 0 \text{ and } (\theta, x) \leq 1\}.$$

and in particular A_0 is a simplex.

§15 March 15, 2017

We continue the proof of Weyl's formula using W_{aff}.

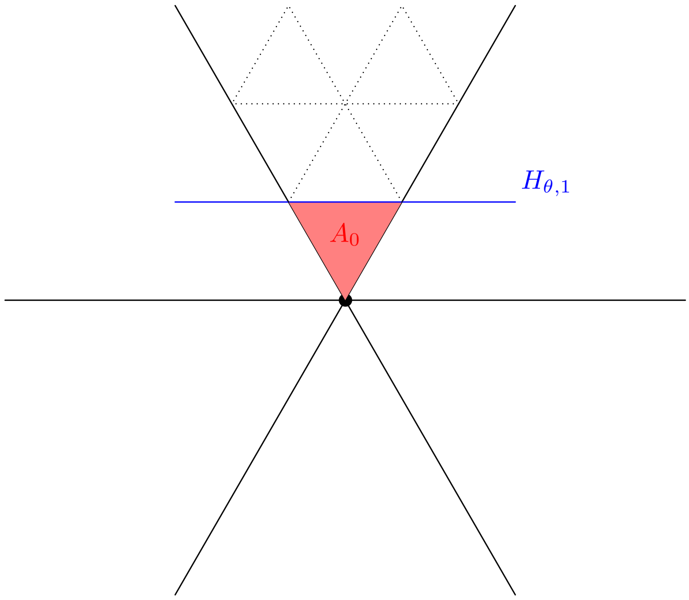
§15.1 Recap

As before W is a Weyl group with weight lattice P and root lattice $Q \subset P$. We set

$$W_{\text{aff}} = W \ltimes Q^\vee.$$

We let C_0 be the fundamental Weyl chamber, (the cone generated by fundamental weights $\omega_1, \dots, \omega_r$).

Then we let A_0 be the fundamental alcove, cut out by $H_{\theta,1}$.



We have the following characterization of the chamber.

Lemma 15.1 (Characterize A_0)

We have the characterization

$$\begin{aligned} A_0 &= C_0 \cap \{x \mid (x, \theta) = 1\} \\ &= \{x \mid (\alpha_i, x) \le 0 \ \forall i, (\theta, x) \le 1\}. \end{aligned}$$

Equivalently, A_0 is the convex of hull of 0 and $\omega_i^\vee / a_i$, where $\theta = \sum a_i \alpha_i$ is the highest root as usual.

Since W_{aff} acts simply transitively, we have

Theorem 15.2 (Alcoves $\leftrightarrow W_{\text{aff}}$)

All alcoves are of the form $w(A_0)$, $w \in W_{\text{aff}}$, in a one-to-one correspondence.

Theorem 15.3 (Presentation of W_{aff})

W_{aff} is generated by reflection with respect to the walls of A_0. In fact, if we set s_1, …, s_r as before and $s_0 = s_{\theta,i}$ the relations of the presentation are

- $s_i^2 = 1$ for $i = 0, \dots, r$.
- $\underbrace{s_i s_j s_i \dots}_{m_{ij}} = \underbrace{s_j s_i s_j \dots}_{m_{ij}}$ for $0 \le i < j \le r$.

Proof. Essentially the same as the proof last time. □

Remark. One computational comment from the proof from last time: if two words on s_i are equal, then one can show they are equal only be deleting $s_i s_i$ and using the Coxeter relation (i.e. we never need to introduce a double $s_i s_i$). This more or less follows directly from the geometric picture.

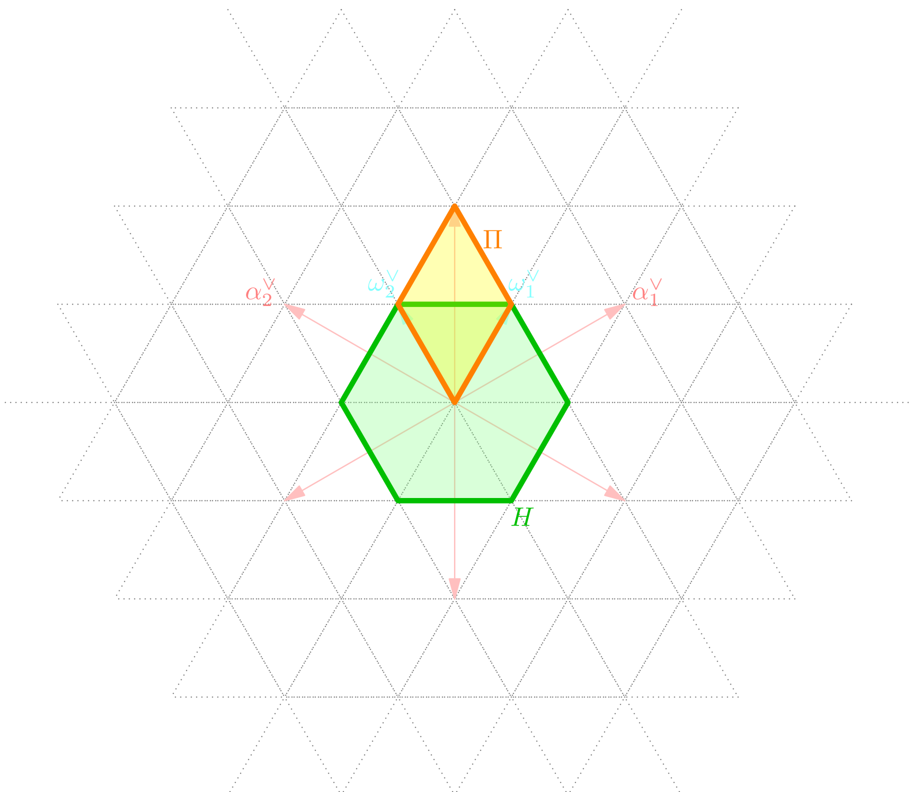
§15.2 Proof of Weyl's formula

We now put together what we know about W_{aff} in order to prove Weyl's formula. The proof will be geometric.

Consider two polytopes (both parallelpiped):

$$\Pi = \{x \in V \mid 0 \le (x, \alpha_i) \le 1 \ \forall i = 1, \dots, r\}.$$
$$H = \left\{x \in V \mid -1 \le (x, \alpha) \le 1 \ \forall \alpha \in \Phi^+\right\}.$$

Here is a picture in A_2, with Π and H shaded in orange and green (respectively).



Each of Π and H consists of finitely many alcoves, since each of the boundaries lies along the grid cut out by $H_{\alpha,i}$. Now we observe that

- Π is really the parallelpiped generated by $\omega_1^\vee, \dots, \omega_r^\vee$. Meanwhile A_0 is the parallelpiped generated by $\omega_1^\vee/a_1, \dots, \omega_r^\vee/a_r$. Consequently, we obtain

$$\frac{\operatorname{Vol}\Pi}{\operatorname{Vol}A_0} = r!a_1 \dots a_r.$$

- Since H consists of all alcoves adjacent to the origin, it has $|W|$ alcoves, meaning

$$\frac{\operatorname{Vol}H}{\operatorname{Vol}A_0} = |W|.$$

So it remains to compare H and A_0. For this we use the following lemma.

Lemma 15.4 (Π, H are fundamental domains of $P^\vee$, $Q^\vee$)
We have that:

- Π is the fundamental domain of the coweight lattice $P^\vee$.
- H is the fundamental domain of the coroot lattice $Q^\vee$.

Proof. Equivalently, if we take Π and translate by $P^\vee$ we tile the whole space, while if we take H and translate by $Q^\vee$ we also tile the whole space.

The first one is more or less obvious. For the second one, follows from $W_{\text{aff}} = W \ltimes Q^\vee$. □

Thus, we have that

$$\frac{|W|}{r!a_1 \dots a_r} = \frac{\operatorname{Vol}H}{\operatorname{Vol}P} = [P^\vee : Q^\vee] = f.$$

This implies Weyl's formula.

§15.3 Example: A_{n-1}

Claim that in the case $W = A_{n-1}$, W_{aff} corresponds to affine permutations, i.e. permutations $w : \mathbb{Z} \to \mathbb{Z}$ for which $w(i+n) = w(i) + n$ subject to the additional constraint $w(1) + \dots + w(n) = 1 + \dots + n$. More about this example coming next lecture.

§16 March 17, 2017

§16.1 The example A_{n-1}

Let $r = n-1$, so we have A_{n-1}. We consider the roots as living in the vector space

$$V = \left\{ (x_1, \dots, x_n) \in \mathbb{R}^n \mid \sum x_i = 0 \right\}$$

and for which we have the dual space

$$V^* = \mathbb{R}^n / (1, \dots, 1)\mathbb{R} \simeq V.$$

The simple roots $\alpha_i = (0, \dots, 0, 1, -1, 0, \dots 0)$ and the fundamental weights ω_i are given modulo $(1, \dots, 1)$ by

$$\omega_i = \left(\underbrace{1, \dots, 1}_{i}, 0, \dots, 0 \right) \equiv \left(\underbrace{1, \dots, 1}_{i}, 0, \dots, 0 \right) - \frac{i}{n}(1, \dots, 1) \in V^*.$$

Then the root lattice is given by

$$Q = \left\{ (x_1, \dots, x_n) \in \mathbb{Z}^n \mid \sum x_i = 0 \right\}.$$

and the weight lattice is

$$P = p(\mathbb{Z}^N)$$

where p is the projection onto the $\sum x_i = 0$ plane used earlier,

$$\begin{aligned} p : \mathbb{R}^n &\longrightarrow V \\ (x_1, \dots, x_n) &\longmapsto (x_1, \dots, x_n) - \frac{\sum x_i}{n}(1, \dots, 1). \end{aligned}$$

§16.2 Affine permutations, and cylindrical wiring diagrams

In the case $W = A_{n-1}$ we are describing, W_{aff} becomes the so-called affine permutations.

Definition 16.1. Fix $n \ge 1$. A map $w : \mathbb{Z} \to \mathbb{Z}$ is an **affine permutation** if

(i) w is a bijection.

(ii) $w(i+n) = w(i) + n$ for all i.

(iii) $w(1) + \dots + w(n) = 1 + 2 + \dots + n$.

These permutations from a group under function composition, which we call the **George group** after George Lusztig.

We introduce the following notation. Given w an affine permutation, we identify w with

$$w = [w(1), \dots, w(n)]$$

and then

$$\overline{w} = (w(1) \mod n, \dots, w(n) \mod n).$$

Obviously $\overline{w}$ is a bijection on $\mathbb{Z}/n$.

Proposition 16.2

The George group is isomorphic to W_{aff} where $W = A_{n-1}$.

Explicitly, s_i corresponds to the permutation for which

$$s_i(k) = \begin{cases} k+1 & k \equiv i \pmod n \\ k-1 & k \equiv i+1 \pmod n \\ k & \text{else} \end{cases}$$

or in our bracket notation

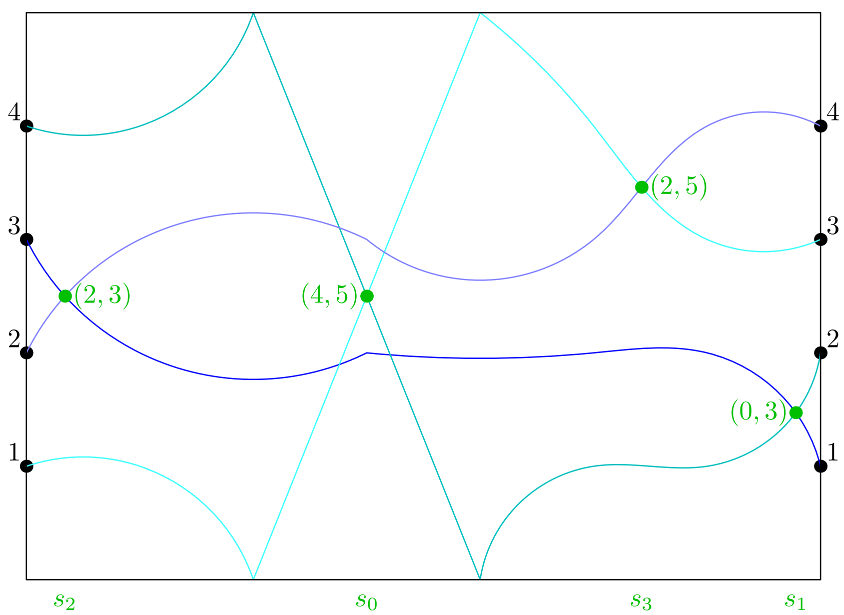
$$\begin{aligned} s_1 &= [2,1,3,4,\dots,n] \\ s_2 &= [1,3,2,4,\dots,n] \\ &\vdots \\ s_{n-1} &= [1,2,\dots,n-2,n,n-1] \\ s_0 &= [0,2,3,\dots,n-1,n+1]. \end{aligned}$$

Example 16.3 (Examples of computation with s_i)

Let $n = 4$. We can compute the following entries:

$$\begin{aligned} \text{id} &= [1,2,3,4] \\ s_1 &= [2,1,3,4] \\ s_1 s_3 &= [2,1,4,3] \\ s_1 s_3 s_0 &= [-1,1,4,6] \\ s_1 s_3 s_0 s_2 &= [-1,4,1,6]. \end{aligned}$$

Like before, we can draw this with a **wiring diagram**, except we will this time have them on a cylinder rather than a plane. Since I am not skilled with 3D diagrams, here is a hacked-together one: you should imagine the straight lines as on the "back" of the cylinder.



Definition 16.4. An **affine inversion** is a pair $i < j$ such that $w(i) > w(j)$.

To see the decomposition $W_{\text{aff}} = W \rtimes Q^\vee$, we can write

$$w = [w(1), \dots, w(n)] \mapsto \left(\overline{w} \in S_n, x \in Q^\vee\right)$$

where $x = (x_1, \dots, x_n)$ is defined by $w(i) = \overline{w(i)} + x_i \cdot n$, since $\overline{w}(i) \equiv w(i) \pmod n$.

§16.3 Alcoves in the example

Now, how do these correspond to alcoves? The claim is the following.

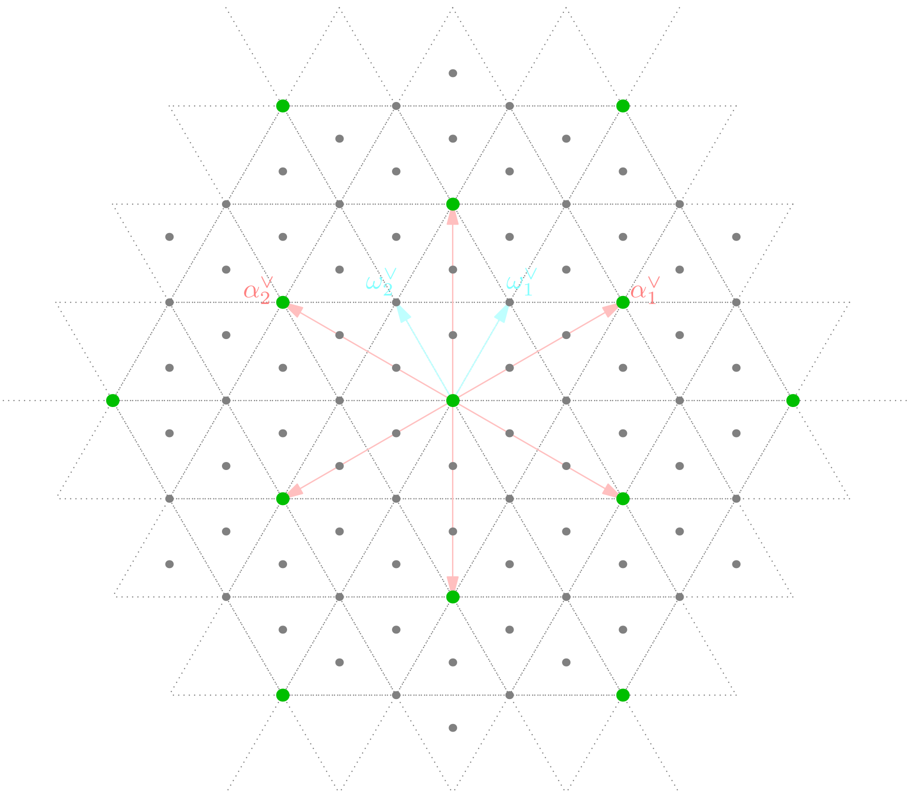
Proposition 16.5 (Alcoves in A_{n-1})
Let $A = w(A_0)$ be the alcove corresponding to $w \in W = A_{n-1}$. Then the center of mass of the alcove is

$$\frac{1}{n}\left(w(1), \dots, w(n)\right).$$

In fact, the following is true.

Lemma 16.6 (Kostant)
Let W be *any* Weyl group. For every alcove there is exactly one point of the lattice $\frac{1}{h}Q^\vee$ strictly inside of A, where h is the Coxeter number.

Here is the example for A_2, with Coxeter number $h = 3$. The points of the co-root lattice $Q^\vee$, are marked in green, and the points of $\frac{1}{h}Q^\vee$ are marked in grey (unless they are grey already). We indeed see exactly one point inside each alcove.



Proof of Kostant's lemma. Consider a point $\lambda \in \frac{1}{h}Q^\vee$. Then λ lie in an alcove exactly if

$$\lambda \in H_{\alpha,k} \iff (\lambda, \alpha) \notin \mathbb{Z} \qquad \forall \alpha \in \Phi.$$

Thus the conditions are $h\lambda \in Q^\vee$ and $(h\lambda, \alpha) \not\equiv 0 \pmod{h}$ for all α. (Why does this finish?) $\square$

§16.4 Bruhat orders

We again draw the weak Bruhat order for S_3:

$$\begin{tikzcd}
 & s_1s_2s_1 = s_2s_1s_2 \arrow[dl, "1"'] \arrow[dr, "2"] & \\
s_1s_2 \arrow[d, "2"] & & s_2s_1 \arrow[d, "1"] \\
s_1 \arrow[dr, "1"] & & s_2 \arrow[dl, "2"'] \\
 & 1 &
\end{tikzcd}$$

We now define the **strong Bruhat order** by letting $u \lessdot w$ if $w = us_\alpha$, with the weight of the edge being $h(\alpha)$. Here it is for S_3:

$$\begin{tikzcd}
 & s_1s_2s_1 = s_2s_1s_2 \arrow[dl, "1"'] \arrow[dr, "1"] & \\
s_1s_2 \arrow[d, "1"] \arrow[drr, "2"] & & s_2s_1 \arrow[d, "1"] \arrow[dll, "2"'] \\
s_1 \arrow[dr, "1"] & & s_2 \arrow[dl, "2"'] \\
 & 1 &
\end{tikzcd}$$

It turns out that in both of these, if we take all the saturated paths and then the product of the labels, we get $\binom{n}{2}!$. For example, in the weak one we have $1 \cdot 2 \cdot 1 + 2 \cdot 1 \cdot 2 = 6$ and in the strong one we get $1 \cdot 1 \cdot 1 + 1 \cdot 2 \cdot 1 + 1 \cdot 2 \cdot 1 + 1 \cdot 1 \cdot 1 = 6$.

§17 March 20, 2017

(Logistic announcements: problem set 1 is up. There are lots of problems; solving 3-4 of them should be sufficient.)

§17.1 Reduced decomposition

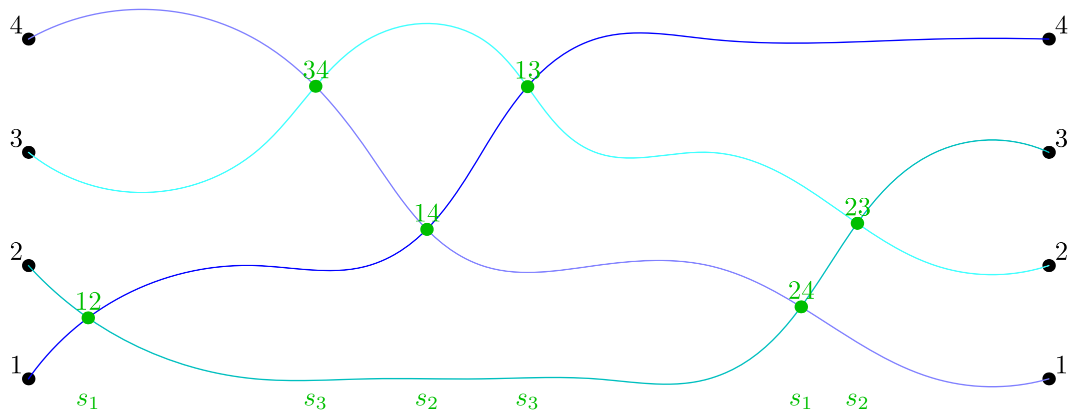
We'll be mostly discussing reduced decompositions, mostly for the longest element θ.

For example, for the element

$$w_0 = \begin{pmatrix} 1 & 2 & \dots & n \\ n & n-1 & \dots & 1 \end{pmatrix} \in S_n$$

and $\ell(w_0) = \# \operatorname{Inv}(w_0) = |\Phi^+|$.

As before we can draw this with a wiring diagram.



Theorem 17.1 (Stanley)

The number of reduced decompositions of the longest permutation $w_0 \in S_n$ is equal to the number of standard Young Tableau of the staircase shape $(n-1, n-2, \dots, 1)$. By the hook-length formula, this equals

$$\frac{\binom{n}{2}!}{1^{n-1}3^{n-2}5^{n-3}\dots(2n-1)^1}.$$

§17.2 Commutation classes

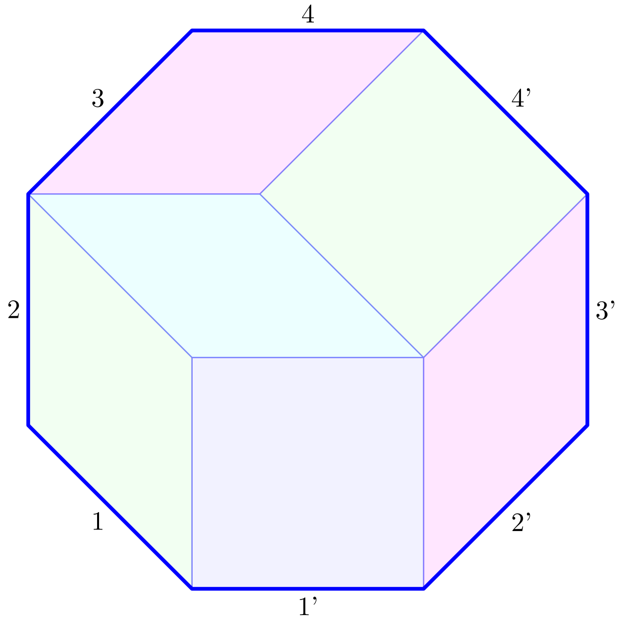
Reduced decompositions probably aren't unique, and often you can commute two elements with each other; for example in the above picture the elements s_1 and s_3 could have been listed in any other.

Definition 17.2. A **commutation class** of a reducible decomposition is an equivalence class obtained by the move

$$s_i s_j = s_j s_i.$$

For S_n, commutation classes are in bijection with pseudo-line arrangements with n pseudolines. They are also in bijection with so-called **rhombus tilings** of a $2n$-gon. Without going into too much detail, here's the earlier example with $n = 4$.

draw paths



§17.3 Inversion sets

Definition 17.3 (Stembridge)**.** An element $w \in W$ is **fully commutative** if there is only commutative class of reducible decompositions for w.

Theorem 17.4

A permutation $w \in S_n$ is fully commutative if and only if w is 321-avoiding. Hence there are $\frac{1}{n+1}\binom{2n}{n}$ avoiding.

Definition 17.5. A **root triple** is $(\alpha, \beta, \gamma) \in \Phi^+$ is a triple such that

$$\alpha + \gamma = \beta.$$

Recall also that $\operatorname{Inv}(w) = \{\alpha \in \Phi^+ \mid w(\alpha) \in \Phi^-\}$.

Theorem 17.6 (Description of inversion sets)

A subset $I \subset \Phi^+$ is an inversion set of some $w \in W$ if and only if for every root triple (α, β, γ) we have

$$\alpha, \gamma \in I \iff \beta \in I.$$

Theorem 17.7 (Description of fully commutative sets)

Assume W is simply laced. The element $w \in W$ is fully commutative if and only if its inversion set contains no root triples.

Remark 17.8. The earlier theorem about $S - n$ follows from the description of fully commutative sets. Indeed 321 pattern in w corresponds to $i < j < k$ such that s_{ij}, s_{ik}, s_{jk} are in $\operatorname{Inv}(w)$, and $s_{ij} + s_{jk} = s_{ik}$.

Thus root triples generalize 321-avoiding permutations.

§17.4 Order ideals

Definition 17.9. An **upper order ideal** of $(\Phi^+, <)$ is a subset $I \subseteq \Phi^+$ such that if $\alpha \in I$ and $\beta > \alpha$ then $\beta \in I$. Such an ideal is **abelian** if it does not contain a root triple.

Proposition 17.10

For A_{n-1} the number of such ideals is $C_n = \frac{1}{n+1}\binom{2n}{n}$. The number of abelian ideals is 2^{n-1}.

§17.5 Recap

We can think of all the classes of objects as the set of root triples which avoid certain signs. For W simply laced:

Class	Forbidden root triples	For A_{n-1}	Other types
Inversion set	$+-+$, $-+-$	$n!$	$\lvert W\rvert$
Fully commutative	$+-+$, $-+-$, $+++$	C_n	(to be announced)
Upper order ideals	$+-+$, $+--$, $--+$	C_n	Coxeter-Catalan numbers
Abelian ideals	$+-+$, $+--$, $--+$, $+++$	2^{n-1}	$2^{\operatorname{rank} W}$

By the signs I mean e.g. that when $+-+$ is forbidden, then if (α, β, γ) is a root triple it is not the case that $\alpha, \gamma \in \Phi^+$, $\beta \in \Phi^-$.

§17.6 A bijection

We will give a bijective proof that inversion sets correspond exactly to upper order ideals.

Definition 17.11 (Edelman-Green)**.** A **balanced tableau** is a filling of the staircase Young diagram $\lambda = (n-1, n-2, \dots, 1)$ by $1, 2, \dots N = \binom{n}{2}$ without repetition such that: for every hook H in λ the corner entry of H is the median of all entries in H.

Example 17.12 (A balanced tableau for $n = 5$)

Here is a balanced tableau for $n = 5$.

$$\begin{bmatrix} 6 & 7 & 3 & 10 \\ 4 & 5 & 1 & \\ 8 & 9 & & \\ 2 & & & \end{bmatrix}$$

For example, the hook at 7 has entries $3 < 5 < 7 < 9 < 10$.

Right now this looks specific to A_n because it involves Young tableau. But we will now generalize this to any Weyl group. Label the entries of the tableau by (i, j), $1 \le i < j \le n$, in the following way:

$$\begin{bmatrix} 15 & 25 & 35 & 45 \\ 14 & 24 & 34 & \\ 13 & 23 & & \\ 12 & & & \end{bmatrix}$$

Now, we give the following weaker condition which also implies the median property.

Lemma 17.13 (Balanced tableaus)
A filling T of $\lambda = (n-1, n-2, \dots, 1)$ is balanced if and only if for any indices $i < j < k$ the entries a, b, c in boxes (i, j), (i, k), and (j, k) satisfy either $a < b < c$ or $a > b > c$.

Thus we can generalize the notion of balanced tableau to any root system.

Definition 17.14. A balanced Φ-tableau is a bijection

$$T : \Phi^+ \to \{1, \dots, N\}$$

such that given any root triple (α, β, γ) the number $T(\beta)$ is the median of $\{T(\alpha), T(\beta), T(\gamma)\}$

This will give us a bijection to reduced decomposition. We claim it as follows: if

$$w = s_{i_1} \dots s_{i_N} = s_{\beta_N} \dots s_{\beta_1}.$$

Then set $T(\beta_i) = i$.

§18 March 22, 2017

§18.1 Tableaus

As before we identify the boxes of the Staircase Young diagram with the pairs

$$\binom{[n]}{2} = \{(i,j) \mid 1 \le i < j \le n\}.$$

Then recall that a balanced tableau is a bijective map

$$T : \binom{[n]}{2} \to \{1, \dots, \binom{n}{2}\}$$

such that $T(i,k)$ is between $T(i,j)$ and $T(j,k)$ (equivalent to the hook property).

This generalizes to give us the following definition for a general root system.

Definition 18.1. Let Φ be a crystallographic root system. A balanced Φ-tableau is a bijection

$$\Phi^+ \to \{1, \dots, |\Phi^+|\}$$

such that for any root triple $\alpha, \beta, \gamma \in \Phi^+$ such that $\beta = \alpha + \gamma$ we have $T(\alpha) < T(\beta) < T(\gamma)$ or $T(\alpha) > T(\beta) > T(\gamma)$.

Theorem 18.2 (Dyer)

Balanced Φ-tableaux are in canonical bijection with reduced decomposition of the longest element $w_0 \in W$.

The bijection is explicitly given as follows: given a reduced decomposition $w_0 = s_{i_1} \dots s_{i_N} = s_{\beta_N} \dots s_{\beta_1}$ in the usual way ($\beta_1 = \alpha - i_1$, $\beta_2 = s_{i_1}(\alpha_{i_2})$, and so on), then the corresponding tableau is given by $T(\beta_i) = i$.

One can also think of this as a **reflection ordering**, corresponding to an ordering

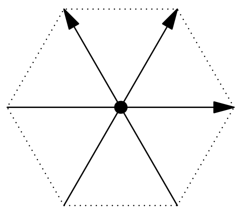
$$\beta_1 < \beta_2 < \dots < \beta_n$$

of the set of positive roots (by T), satisfying the "balanced" condition.

Remark 18.3. The balanced condition can be rewritten as follows. Consider any rank 2 root subsystem Ψ in Φ; for crystallographic Φ we have three possibilities (four possibilities if we count $A_2 \times A_2$, but we'll ignore it).

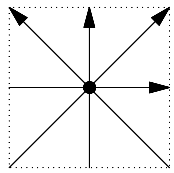
- In A_2, we have three positive roots, say γ_1, γ_2, γ_3 in that order; we will require that

$$T(\gamma_1) < T(\gamma_2) < T(\gamma_3) \quad \text{or} \quad T(\gamma_1) > T(\gamma_2) > T(\gamma_3).$$

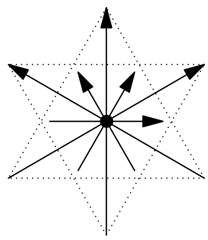


- In B_2 we have four positive roots γ_1, γ_2, γ_3, γ_4 in that order, and we will require

$$T(\gamma_1) < T(\gamma_2) < T(\gamma_3) < T(\gamma_4) \quad \text{or} \quad T(\gamma_1) > T(\gamma_2) > T(\gamma_3) > T(\gamma_4).$$



- In G_2 we have six positive roots γ_1, γ_2, γ_3, γ_4, γ_5, γ_6 in that order. We will require
$$T(\gamma_1) < T(\gamma_2) < T(\gamma_3) < T(\gamma_4) < T(\gamma_5) < T(\gamma_6)$$
or
$$T(\gamma_1) > T(\gamma_2) > T(\gamma_3) > T(\gamma_4) > T(\gamma_5) > T(\gamma_6).$$

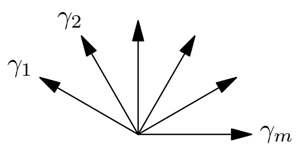


Remark 18.4. Dyer's theorem also holds for non-crystallographic root systems, if we replace it with the Ψ condition above: if Ψ has m positive roots $\gamma_1, \dots, \gamma_m$ in that order, we require $T(\gamma_1) < \dots < T(\gamma_m)$ or $T(\gamma_1) > \dots > T(\gamma_m)$.

§18.2 The easy direction

We prove the easy direction. Given
$$w_0 = s_{i_1} \dots s_{i_n}$$
hence giving an ordering $\beta_1 < \dots < \beta_N$, we have
$$\operatorname{Inv}(s_{i_1} \dots s_{i_k}) = \{\beta_1, \dots, \beta_k\} \qquad k = 1, \dots, N.$$
Now let $H = \{x \mid (\lambda, x) = 0\}$ be the hyperplane separating Φ^+ and Φ^-, and let $H^+ = \{x \mid (\lambda, x) \ge 0\} \supset \Phi^+$ and $H^- = \{x \mid (\lambda, x) \ge 0\} \supset \Phi^-$. Then we can also write the equivalent definition
$$\operatorname{Inv}(w) = \text{roots in } H^+ \cap w^{-1}(H^-).$$
We also now define
$$\begin{aligned} \text{Non-Inv}(w) &\overset{\text{def}}{=} \Phi^+ \setminus \Phi^+ \\ &= \left\{\beta \in \Phi^+ \mid w(\beta) \in \Phi^+\right\} \\ &= \text{roots in } H^+ \cap w^{-1}(H^+). \end{aligned}$$
Both of the sets $H^+ \cap w^{-1}(H^-)$ and $H^+ \cap w^{-1}(H^+)$ are "convex", being the intersection of two half planes.

Now, consider a subsystem Ψ as before, with positive roots $\gamma_1, \dots, \gamma_m$ (here $m \in \{3, 4, 6\}$ in the crystallographic case).



Then for convexity reasons, $\operatorname{Inv}(w) \cap \{\gamma_1, \dots, \gamma_m\}$ is either of the form $\{\gamma_1, \dots, \gamma_i\}$ or $\{\gamma_i, \gamma_{i+1}, \dots, \gamma_m\}$.

This implies that $T(\gamma_1)$, $T(\gamma_2)$, $\dots$, $T(\gamma_m)$ is monotone.

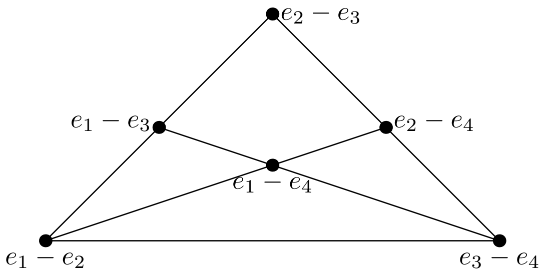
§18.3 The hard direction

The hard direction is to show that the monotonicity condition is sufficient, since the remarks above show it's necessary. For now we will give some comments showing this is indeed hard.

For this we will use A_3. To draw a picture of A_3 in the plane (despite the fact that $A - 3$ lives in $\mathbb{R}^3$), we take the affine hyperplane

$$\widetilde{H} = \{x \mid (\lambda, x) = 1\}$$

and then take the intersection of the ray for each positive root with the plane $\widetilde{H}$. These gives the six positive roots of A_3, drawn in the following picture:



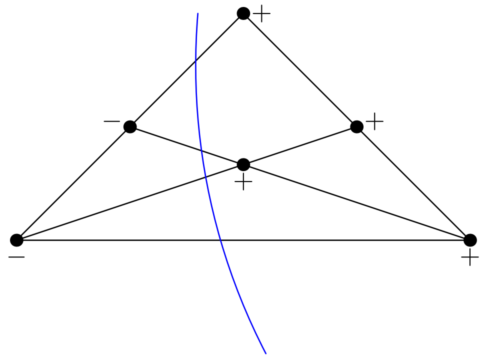
As before the Inversions of a w are then given by drawing the plane $w^{-1}(H)$, and then taking the inversions to be those on one side of the line. Thus the condition is that

For any collinear points, the inversions are separated from the non-inversions.

But the situation is more complicated in general. Consider n lines in general position, giving $\binom{n}{2}$ intersections. Then we would hope:

Claim 18.5. Given an assignment of $+$ and $-$ such that on each line the $+$ and $-$'s are separated, is it always possible to separate them by a line?

For example:



Here's the caveat. For general line arrangements, while we can draw a curve separating the $+$ and $-$ signs, but it is not true pure geometrically; there exists line arrangements in $\mathbb{R}^2$ for which it's impossible to *straighten* the line. The miracle is that for root systems, it turns out it's in fact always possible to straighten the line, not only in the rank three case but generally for any root system in higher dimensions.

§18.4 Edelman-Greene Correspondence

§19 March 24, 2017

Missed class for spring break. Covered Edelman-Greene algorithm.

§20 April 3, 2017

Recall from March 20 lecture the bijection between wiring diagrams and rhombus tilings.

In this lecture we'll generalize this to get the so-called **higher Bruhat orders** $B(n,k)$, generalizing the weak Bruhat order. They were introduced by Manin-Schekhtman, studied by Kapronav-Voevodsky, Ziegler,

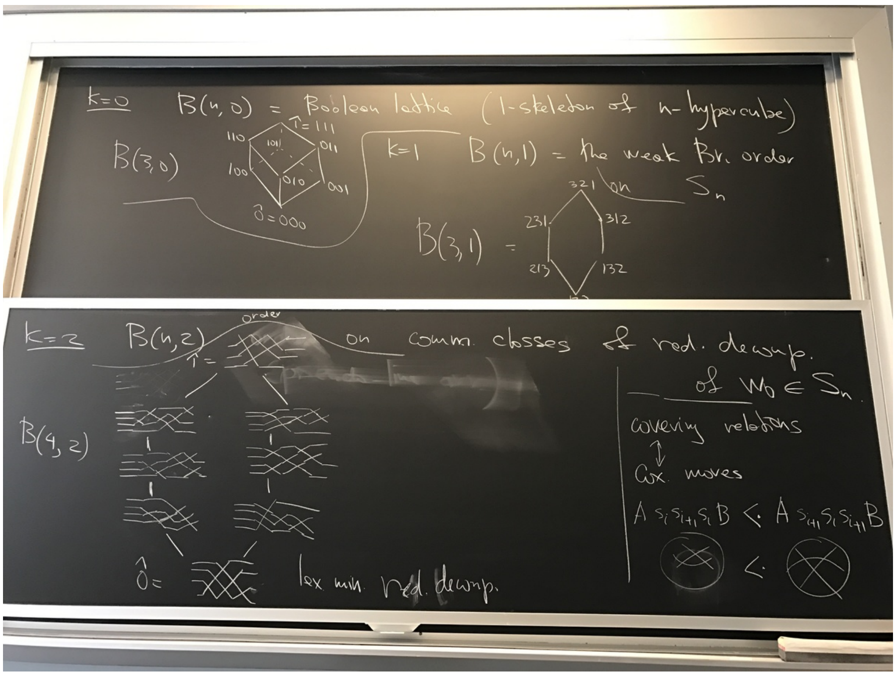
(All these constructions are specific to type A_n.)

§20.1 Examples

Before we define these higher Bruhat orders, we give motivating examples for $k = 0$ and $k = 1$.

- When $k = 0$, $B(n,0)$ is the Boolean lattice, i.e. the 1-skeleton of the n-hypercube.
- When $k = 1$, $B(n,1)$ is the weak Bruhat order on S_n.
- When $k = 2$, $B(n,2)$ is the partial order on commutation classes of reduced decompositions of the longest element $w_0 \in S_n$ (equivalently, on ordering on pseudo-line arrangements). Covering relation corresponds to Coxeter moves: $As_i s_{i+1} s_i B \lessdot As_{i+1} s_i s_{i+1} B$.

Here are pictures of all of these:



Exercise 20.1. The Hasse diagram of $B(n,2)$ is a graph with vertices of degree $\geq n-2$. Equivalently every arrangement of n pseudo-lines contains at least n triangles.

§20.2 Definition

The original definition of $B(n,k)$ was recursive: one may let $B(n,k+1)$ denote the order on saturated chains in $B(n,k)$ modulo certain "commutation relations". However the official definition for this course is not recursive.

Definition 20.2. We define $B(n,k)$ to consist of subsets $S \subseteq \binom{[n]}{k+1}$ (so sets of sets) satisfying the following "separation condition":

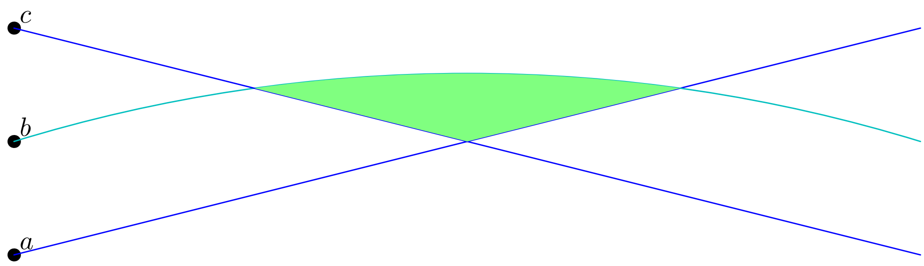
> For every $J = \{j_1 < \cdots < j_{k+2}\} \subseteq [n]$, let $J_i = J \setminus \{j_i\}$. Then there must exist $r \in \{0, \dots, k+2\}$ such that
>
> - $J_1, \dots, J_r \in S$ and $J_{r+1}, \dots, J_{k+2} \notin S$, or
> - $J_1, \dots, J_r \notin S$ and $J_{r+1}, \dots, J_{k+2} \in S$, or
>
> Equivalently, for $a < b < c$ we forbid the patterns $(J_a \in S, J_b \notin S, J_c \in S)$ and $(J_a \notin S, J_b \in S, J_c \notin S)$.

Finally,the covering relation $S \lessdot S'$ means $S' = S \cup \{*\}$ (i.e. $|S'| = |S| + 1$ and $S' \supset S$).

Remark 20.3. The covering relation is not the same as the inclusion ordering.

The idea is that the elements of S should be thought of as "higher inversions". Examples:

- If $k = 0$, we have $S \subseteq \binom{[n]}{1} \cong [n]$, and the separation condition is trivial. Hence the Boolean lattice.
- If $k = 1$, we have $S \subseteq \binom{[n]}{2}$, we can interpret S as the set of inversions of a permutation (the separation discussion being what we discussed in the past already).
- Now let $k = 2$, and considering a wiring diagram. The idea will bet hat a "higher inversion" corresponds to wires $a < b < c$ such that bc intersects and then ac intersects.



§21 April 5, 2017

Recall the definition of $B(n,k)$ we gave in terms of sets. We're going to give characterizations of this again.

§21.1 Saturated chains

Let's try to check the recursive definition in terms of saturated chains coincides with the official one.

Proposition 21.1

This $B(n,k)$ has a unique element $\widehat{0} = \varnothing$ and a unique maximal element $\widehat{1} = \binom{[n]}{k+1}$.

Thus a saturated chain looks like

$$\varnothing = S_0 \lessdot S_1 \lessdot S_2 \lessdot \cdots \lessdot S_N = \binom{[n]}{k+1}$$

where $N = \binom{n}{k+1}$.

As we go from $\widehat{0}$ to $\widehat{1}$, we are adding in sets $J_1, \dots, J_{k+2}$ are added from left-to-right or right-to-left (in the separation). Now we let $\widetilde{S}$ be the set of all J which were added right-to-left.

Claim. $\widetilde{S}$ is an element of $B(n,k+1)$.

Now we know about rhombus tilings from before; the correct analogy in general is **cyclic zonotopes**. The theorem is going to be that:

Theorem 21.2 (Ziegler)

The elements higher Bruhat order $B(n,k)$ are in bijection with (fine) zonotopal tilings of the cyclic zonotope $Z(n,k)$.

To do this we will have to define

- Cyclic polytopes, then
- Cyclic zonotopes, and finally
- Fine zonotopal tilings.

§21.2 Cyclic polytopes

We define the **moment map** $\mathbb{R} \to \mathbb{R}^d$ by $t \mapsto (t, t^2, t^3, \dots, t^d)$ and let $C(n,d)$ denote the convex hull of the points $(x(t_1), \dots, x(t_n))$. For example, when $d = 2$ this is a parabola, and so the convex hull is necessarily an n-gon.

More generally:

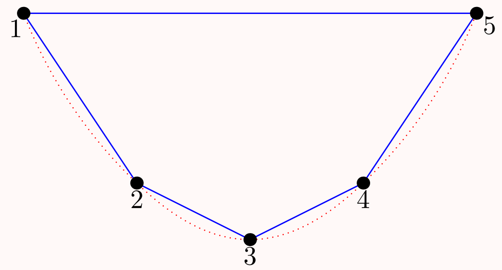
Theorem 21.3 (Gale evenness condition)

The combinatorial structure (i.e. the structure of faces) of $C(n,d)$ does not depend on the choice of $t_1 < \cdots < t_n$.

Moreover, $C(n,d)$ is a simplicial polytope. Finally, for any $I \subseteq [n]$, the points $(x(t_i) \mid i \in I)$ form a facet of $C(n,d)$ if and only if any two elements of $[n] \setminus I$ are separated by an even number of elements of$I$.

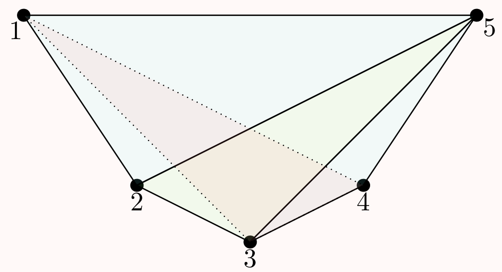
Example 21.4 ($C(5,2)$)

Let $d = 2$. Then we get an n-gon, and the facets are $\{1,2\}$, $\{2,3\}$, $\dots$, $\{n-1,n\}$, $\{1,n\}$.



Example 21.5 ($C(5,3)$)

Here is a picture of the four faces of $C(5,3)$, which are 125, 235, 345, 134.



§21.3 Proof of Gale evenness condition

We'll project the polytope onto the plane $x_0 = 1$, and thus let

$$\widetilde{x}(t) = (1, t, t^2, \dots, t^d) \in \mathbb{R}^k$$

where $k = d + 1$. Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & \dots & 1 \\ t_1 & t_2 & \dots & t_n \\ t_1^2 & t_2^2 & \dots & t_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ t_1^d & t_2^d & \dots & {}^d_n \end{bmatrix}$$

Proposition 21.6

Any $k \times k$ minor of A is positive.

Proof. It's a Vandermonde determinant! $\square$

Now, how can we determine a facet? Let $I = \{i_1, \dots, i_d\} \subseteq [n]$. Then I corresponds to a facet of $C(n,d)$ if and only if all other $\widetilde{x}(t_j)$ for $j \neq I$ lies on the same side of the hyperplane spanned by $\widetilde{(x_i)}$. In other words we require that all determinants

$$D_j = \det \begin{bmatrix} 1 & 1 & \dots & 1 & 1 \\ t_{i_1} & t_{i_2} & \dots & t_{i_d} & t_j \\ t_{i_1}^2 & t_{i_2}^2 & \dots & t_{i_d}^2 & t_j \\ \vdots & \vdots & \ddots & \vdots & \vdots \end{bmatrix}$$

have the same sign for all $j \neq I$. But because again we have Vanerdomde determinants, the sign of D_j is determine just by the position of j relative to the (i_d). More explicitly, if $t_{i_k} < t_j < t_{i_{k+1}}$ then the sign of D_j is $(-1)^{k-d}$.

This directly implies the Gale evenness condition, because it means that every $j \notin I$ ought to move the same number of columns.

§21.4 Cyclic zonotopes

We define the cyclic zonotope $Z(n,k)$ as follows. As before, let $\widetilde{x}(t) = \left\{1, t, t^2, \dots, t^{k-1}\right\}$.

Now recall the definition of the Minkowsky sum:

Definition 21.7. Given sets A and B in $\mathbb{R}^2$, we let $A + B = \{a + b \mid a \in A, b \in B\}$.

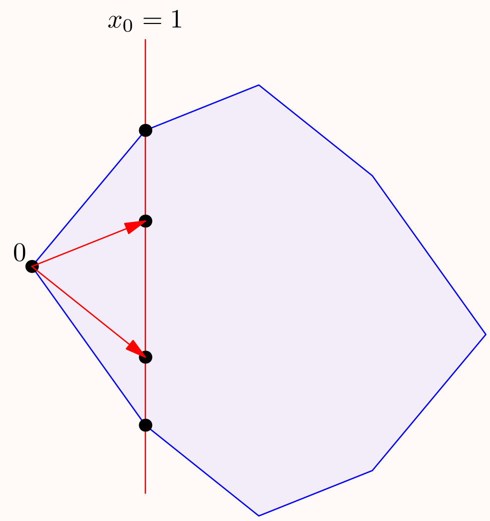
Then:

Definition 21.8. Pick $t_1 < t_2 < \dots < t_n$ as usual. The **cyclic zonotope** $Z(n,k)$ is the Minkowsky sum of the n line segments joining the origin from 0 to $\widetilde{(x_i)}$, for $i = 1, \dots, n$.

Exercise 21.9. Describe the facets of $Z(n,k)$ and show that they don't depend on the choice of t_i.

Example 21.10 ($Z(n,2)$)

Here is a picture of $Z(4,2)$. In general, $Z(n,2)$ is a centrally symmetric $2n$-gon.



The analogy is: generalizing n-gons to higher dimensions correspond to cyclic polytopes; generalizing $2n$-gons are the cyclic zonotopes.

§21.5 Zonotopal tilings

We now have the correct generalization of rhombus tilings.

Definition 21.11. A **(fine) zonotopal tiling** of $Z = Z(n, k)$ is a subdivision of Z into parallelepipeds called **tiles** such that

- The edges of each parallelepiped tile are parallel translations of some edges of the zonotope Z.
- Z is the union of the tiles.
- Every pair of tiles T and T' intersects at the common face of these tiles (possibly empty).

Thus for $k = 2$ we get rhombus tilings.

§22 April 7, 2017

§22.1 Vinberg with integer entries

Theorem 22.1

If A is an $n \times n$ matrix with integer entries, then there exists $v > 0$ with *integer* entries such that:

- If A is finite type, then $v > 0$ and $Av > 0$.
- If A is affine type, then $v > 0$ and $Av = 0$.
- If A is indefinite type, then $v > 0$ and $Av < 0$.

(The previous result was shown with A a real matrix, with $v \in \mathbb{R}^n$ instead.)

Proof. In the finite case, we can easily pick v in $\mathbb{Q}^n$ with this property; then scale v upwards.

In the affine case, we need Dirichlet approximation theorem in the following way.

Theorem 22.2 (Dirichlet)

Let $\alpha_1, \dots, \alpha_d$ be real numbers and pick some integer $M \ge 1$. Then there exists integers $p, q_1, \dots, q_d \in \mathbb{Z}$ such that $1 \le p \le M$ and

$$\left| \alpha_i - \frac{q_i}{p} \right| \le \frac{1}{pM^{1/d}} \qquad \forall i.$$

Then let α_i be the entries of v, $C = \max_j \sum_i |Aij|$ and then pick $M = (2C)^d$. Then $A\vec{q} = 0$ with q as in Dirichlet, so $A(p\vec{q}) = 0$ and $p\vec{v} \in \mathbb{Z}^n$. $\square$

§22.2 Pseudoline arrangements have $n - 2$ arguments

Proposition 22.3

In a pseudoline arrangement with n pseudolines, it has at least $n - 2$ triangular regions.

Consider the obvious planar graph on $\binom{n}{2}$ vertices, which has $E = n^2 - 2n$ and $F = \frac{n^2-3n+2}{2}$.

We use a so-called "discharging" argument: we put one unit of charge on each edge e. Then for each edge $e = vw$, we let x be the intersection of the pseudolines through v and w, and discharge the edge into the face on the same side of e as x.

Claim 22.4. A face has at most 3 units of charge, with equality if and only if it's a triangle.

§23 April 10, 2017

§23.1 A remark on moment

Earlier we used the moment curve

$$x(t) = \left(t, t^2, \dots, t^d\right)$$

and define the cyclic polytope

$$C(n,d) = \operatorname{Hull}\left(x(t_1), \dots, x(t_n)\right)$$

for any choice $t_1 < t_2 < \cdots < t_n$.

However, we claim that we don't really need the moment curve itself to study the combinatorial structure of $C(n,d)$. We can replace it as follows:

Theorem 23.1

Let $A = [v_1, \dots, v_n]$ be a $k \times n$ matrix, where $k = d+1$. Assume that:

- All maximal $k \times k$ minors are positive
- All endpoints of v_i lie in some affine hyperplane (this condition is cosmetic, since one can scale).

Then the convex hull of v_1, ..., v_n is combinatorially equivalent to $C(n,d)$.

Proof. Our proof of Gale's evennes condition used only positivity of matrix minors. □

In fact, the following converse holds.

Theorem 23.2

If P is combinatorially equivalent to $C(n,d)$ with a given order of the vertices v_1, ..., v_n, then all maximal minors of $A = [v_1, \dots, v_n]$ are positive.

§23.2 Positive Grassmannian

In what follows the base field always $\mathbb{R}$.

We define the **positive Grassmannian** $\mathbf{Gr}^{>0}(k,n)$ as the set of $k \times n$ (real) matrices with positive matrix minors, modulo the action of $\mathrm{GL}(k)$. This is a very rich structure that one can teach a whole course on. For now we'll just talk about:

> This is essentially the same thing as "moduli spaces of cyclic polytopes".

Remark 23.3. There exists an even more complicated object, the **amplituhedron**, developed my physicists. We may talk about this more later, but the point is the positive Grassmannian is just the tip of the iceberg.

§23.3 Cyclic zonotopes

Recall the cyclic zonotope $Z(n,k)$ is produced from the cyclic polytoppe $C(n,k-1)$ by taking Minkowski sums.

(Vertices correspond to subsets of $\{1, 2, \dots, n\}$.)

Exercise 23.4. Show that vertices of $Z(n, 3)$ correspond to *cyclic intervals*. What about $Z(n, k)$?

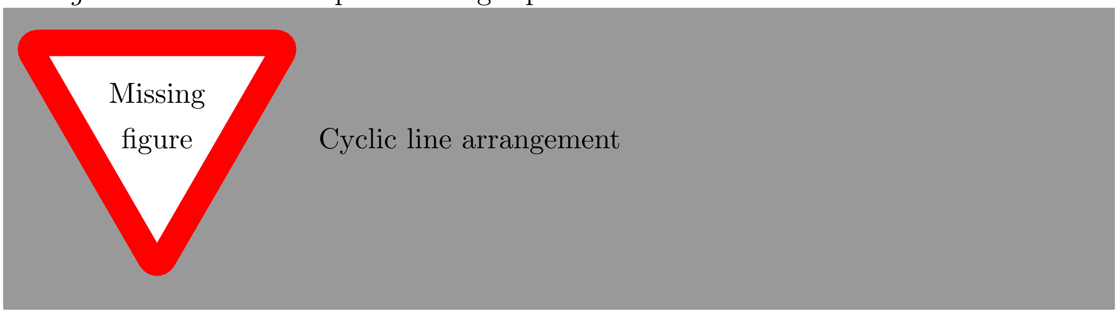
Exercise 23.5. Any tiling of $Z(n, k)$ has exactly $\binom{n}{k}$ tiles.

§23.4 $B(n, n-3)$

Recall:

- $B(n, n-1)$ has just two elements.
- $B(n, n-2)$ has a Hasse diagram which gives a $2n$-gon.

So let's start with the interesting case $B(n, n-3)$. These correspond to subsets of $\binom{[n]}{n-2}$ satisfying the separation condition. We can think of it in terms of *cyclic line arrangements* which we separate using a pseudoline.



§23.5 Generalization of $B(n, n-3)$

Let $\mathcal{A}$ be any pseudoline arrangement (corresponding to commutation classes of reduced decompositions of the longest word $w_0 \in S_n$). Let $B(\mathcal{A})$ denote all admissible subsets S of vertices of A (all pseudoline elements in S are separated from elements not in S), with the poset structure $S \lessdot S'$ if $S' = S \cup \{*\}$.

Proposition 23.6

The poset $B(\mathcal{A})$ has a unique minimal element $\widehat{0} = \varnothing$ and a unique maximal element $\widehat{1}$. In particular, the Hasse diagram of this poset is connected.

The proof is based on the following lemma.

Lemma 23.7

Given any pseudoline arrangement and a directed pseudoline, consider the vertices to the left of this pseudoline. Then as long as there is at least one vertex on the left, then there is a triangle using a vertex on the left and an edge along the directed pseudoline.

This is not true in higher dimensions (possibly not even in three dimensions).

§24 April 12, 2017

Today: Schubert calculus! We'll be working over $\mathbb{C}$ today for convenience.

§24.1 Invariant algebra

Let $\Phi \subseteq V$ be a root system of rank r and W a Weyl group. The **symmetric algebra** is

$$\operatorname{Sym}(V) = \mathbb{C}[V^\vee] = \mathbb{C}[y_1, \dots, y_r]$$

is the polynomial ring on the dual space $V^\vee$.

Then W acts on this set of polynomials $\mathbb{C}[V^\vee]$. We'll consider $\mathbb{C}[V^\vee]^W$ the subalgebra of W-invariant polynomials.

Theorem 24.1 (Chevalley)
We have $\mathbb{C}[V^\vee]^W = \mathbb{C}[f_1, \dots, f_r]$ where $f_i$are homogeneous, algebraically independent polynomials with degrees $\deg f_i = m_i + 1$. (Here m_i are the exponents of Φ.)

Example 24.2 (Fundamental theorem of symmetric polynomials)
Let $\Phi = A_{n-1}$ hence $W = S_n$ as usual ($V = \{x \in \mathbb{R}^n \mid x_1 + \dots + x_n\}$). Then

$$\mathbb{C}[V^\vee]^W = \mathbb{C}[x_1, \dots, x_n]^{S_n} / (x_1 + \dots + x_n)$$

is the algebra of symmetric polynomials.

The fundamental theorem of symmetric polynomials then gives us $\mathbb{C}[x_1, \dots, x_n]^{S_n}$ is generated by homogeneous independent polynomials.

Of course there are many choices of bases. Two common examples:

- the elementary symmetric ones $e_k = \sum_{i_1 < \dots < i_k} x_{i_1} \dots x_{i_k}$, or
- the complete homogeneous polynomials $h_k = \sum_{i_1 \le i_2 \le \dots \le i_k} x_{i_1} \dots x_{i_k}$.

Note that we've modded out by $e_1 = h_1$ already in the context of the theorem. The exponents of A_{n-1} are $\{1, 2, \dots, n-1\}$ and hence the f_i have degrees $\{2, 3, \dots, n\}$.

So we see that while the choice of $\{f_i\}$ is not unique at all, the degrees are determined completely.

§24.2 Coinvariant algebra

Definition 24.3. Let I_W denote the ideal in $\mathbb{C}[V^\vee]$ generated by W-invariant polynomials with no constant term. Then we define the **coinvariant algebra** by

$$C_W = \mathbb{C}[V^\vee]/I_W.$$

Equivalently, I_W is the ideal generated by f_1, ..., f_r in the context of Chevalley's theorem.

Corollary 24.4 (Corollary of Chevalley theorem)
C_W is a finite dimensional algebra graded by degree:

$$C_W = \bigoplus_{k\ge 0} C_{W,k}.$$

Its Hilbert series satisfies the relation

$$\operatorname{Hilb}(C_W) = \sum_{i\ge 0} q^i \cdot \dim C_{W,i} = \prod_{i=1}^{r} [m_i+1]_q$$

where as usual $[n]_q \overset{\text{def}}{=} 1+q+\dots+q^{n-1} = \frac{1-q^n}{1-q}$.

Proof. We have

$$\begin{aligned}\operatorname{Hilb}(C_W) &= \frac{\operatorname{Hilb}(\mathbb{C}[V^*])}{\operatorname{Hilb}(I_W)} \\ &= \frac{(1-q)^{-r}}{\prod_i (1-q^{m_i+1})^{-1}}\end{aligned}$$

as desired. $\square$

Now compare this to the earlier formula

$$\sum_{w\in W} q^{\ell(w)} = \sum_{i=1}^{r} [m_i+1]_q.$$

Hence we deduce $\dim C_W = |W|$.

Example 24.5 (A_{n-1})
In this case we have

$$\operatorname{Hilb}(C_W) = [2]_q[3]_q \dots [n]_q$$

which is sometimes denoted $[n]_q!$ for obvious reasons.

Example 24.6 (A_2)
Let $W = S_3$. In this case,

$$\begin{aligned} C_W &= \mathbb{C}[x,y,z]/\left\langle x+y+z, xy+yz+zx, xyz\right\rangle \\ &= \mathbb{C}[x,y]/\left\langle xy-(x+y)^2, xy(x+y)\right\rangle. \end{aligned}$$

One choice of linear basis: $\{1, x, y, x^2, xy, x^2y\}$. This matches

$$\operatorname{Hilb} C_W = [2]_q[3]_q = (1+q)(1+q+q^2) = 1+2q+2q^2+q^3.$$

Exercise 24.7. For A_{n-1}, one choice of basis for $\mathbb{C}[x_1,\dots,x_{n-1}]/\left\langle e_1,\dots,e_n\right\rangle$ is those elements of the form $x_1^{a_1}x_2^{a_2}\dots x_{n-1}^{a_{n-1}}$ where $a_1 \le n-1$, $a_2 \le n-2$, …, $a_{n-1} \le 1$.

(Possible hint: Gröbner basis.)

§24.3 Geometrical background

Here is some geometrical context for the invariant and coinvariant algebra. In what follows it's important that our ground field is $\mathbb{C}$.

Definition 24.8. The **flag manifold** over $\mathbb{C}$ is the manifold whose points are complete flags over $\mathbb{C}$:

$$\mathbf{Fl}_n = \left\{ \{0\} = V_0 \subsetneq V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_n = \mathbb{C}^n \right\}.$$

(Here $\dim V_i = i$ for each i.) It is isomorphic to $\mathrm{SL}(n)/B$ where B is the group of upper triangular matrices.

We can generalize this to any root system in a way that we won't define now: there is a so-called **generalized flag manifold** G/B where G is a semisimple Lie group and B is Borel group. The previous flag manifold corresponds to type A_{n-1}.

The connection is:

Theorem 24.9 (Borel's Theorem)

We have an isomorphism of the cohomology ring of G/B to the coinvariant algebra:

$$H^\bullet(G/B) \cong \mathbb{C}[V^\vee]/I_W.$$

This isn't really a theorem since we haven't defined G/B. But you can think of it as a justification for studying $\mathbb{C}[V^\vee]/I_W$.

Now $H^\bullet(G/B)$ turns out to have a "natural" linear basis of Schubert classes, so this isomorphism should imply that there's one distinguished nice linear basis $\mathbb{C}[V^\vee]/I_W$.

So, all this motivates the construction of Schubert classes.

§24.4 Schubert classes

There are two algebraic/combinatorial ways to construct the Schubert classes $\sigma_w \in \mathbb{C}[V^\vee]/I_W$.

- The "bottom-to-top" approach (using so-called Monk-Chevalley formula) is a recursive construction starting from the identity element and going *upwards* along the *strong Bruhat order*.
- The "top-to-bottom" approach (Bernstein-Gelfand-Gelfand, see also Demazure) starts from the longest element and goes *down* the edges of the *weak Bruhat order*.

Some historical remarks: the Monk-Chevalley formula came first and was geometric. Then Bernstein-Gelfand-Gelfand deduced from Monk-Chevalley that their algebraic method should exist. Then Lascoux-Schützenberger used this in order to obtain the so-called Schubert polynomials using the work of Bernstein-Gelfand-Gelfand.

This gives us one nice choice of (f_i). They are not the only ones; for example, the Kostant polynomials also behave well.

The key of the BGG approach is the **divided difference operators**. Given $\alpha \in W$, we define

$$\partial_\alpha : \mathbb{C}[V^\vee] \to \mathbb{C}[V^\vee]$$

by

$$f(x) \mapsto \frac{f(x) - s_\alpha f(x)}{\alpha}.$$

This is well-defined since α divides $f(x) - s_\alpha(f(x))$, since it's symmetric with respect to the reflection about α, so the numerator is actually divisible by α as claimed.

§25 April 14, 2017

§25.1 Divided differences

As before we define the **divided difference operators**.

Definition 25.1. Given $\alpha \in W$, we define

$$\partial_\alpha : \mathbb{C}[V^\vee] \to \mathbb{C}[V^\vee]$$

by

$$f(x) \mapsto \frac{f(x) - s_\alpha f(x)}{\alpha}.$$

Let $\partial_i \overset{\text{def}}{=} \partial_{\alpha_i}$.

This is well-defined since α divides $f(x) - s_\alpha(f(x))$, since it's symmetric with respect to the reflection about α, so the numerator is actually divisible by α as claimed.

Example 25.2 (∂_i in A_{n-1})

In type A_{n-1},

$$\partial_i : f(x_1, \dots, x_n) \mapsto \frac{f(x_1, \dots, x_n) - f(x_1, \dots, x_{i-1}, x_{i+1}, x_i, \dots, x_n)}{x_i - x_{i+1}}.$$

This is a polynomial since the numerator remains unchanged when we interchange x_i and x_{i+1}, so is divisible by $x_i - x_{i+1}$.

The ∂_i satisfy "nil-Coxeter relations" similar to the original Coxeter relations in W.

Lemma 25.3 (nil-Coxeter relations)

We have

1. $\partial_i^2 = 0$
2. $\underbrace{\partial_i \partial_j \partial_i \dots}_{m_{ij}} = \underbrace{\partial_j \partial_i \partial_j \dots}_{m_{ij}}$

Proof. For (1), in what follows 1 denotes the identity operator. Note that

$$\partial_i = \frac{1}{\alpha_i}(1 - s_i) = (1 + s_i) \circ \frac{1}{\alpha_i}$$

since α_i^{-1} anti-commutes with s_i (this is an identity of operators on $\mathbb{C}[V^\vee]$). Thus

$$\partial_i \partial_i = \frac{1}{\alpha_i}(1 - s_i)(1 + s_i)\frac{1}{\alpha_i} = \frac{1}{\alpha_i}(1 - s_i^2)\frac{1}{\alpha_i} = 0.$$ □

Part 2 is an exercise (main idea is to "open up" parentheses).

§25.2 Divided difference via Weyl group

Definition 25.4. Henceforth, if $w = s_{i_1} \dots s_{i_\ell}$ is a reduced decomposition then we let $\partial_w \overset{\text{def}}{=} \partial_{i_1} \dots \partial_{i_\ell}$.

Lemma 25.5

∂_w is well-defined in the sense that it depends only on w and not on the choice of reduced decomposition.

Proof. This follows from the fact that the nil-Coxeter relations above give a presentation. Our approach earlier was via alcove walks; another proof is by the so-called exchange lemma. □

Lemma 25.6

Let $w = s_{i_1} \dots s_{i_\ell} = s_{j_1} \dots s_{j_\ell}$ be a reduced decomposition of w. Then there exists $r \in \{1, \dots, \ell\}$ such that

$$w = s_{i_1} \dots s_{i_{r-1}} s_{i_{r+1}} \dots s_{i_\ell} s_{j_\ell}.$$

Consequently, for all $w \in W$ we have a well-defined action $\partial_w : \mathbb{C}[V^\vee] \to \mathbb{C}[V^\vee]$. These ∂_i function much like derivative operators: they drop the degree of the polynomial by 1. Actually, we in fact have the following "product formula".

Lemma 25.7 (Product rule aka Leibniz formula)

We have

$$\partial_i(fg) = \partial_i(f)g + s_i(f)\partial_i(g).$$

Proof. Easy exercise (plug and chug). □

Corollary 25.8

If f is W-invariant, then $\partial_i(fg) = f\partial_i(g)$.

Thus, ∂_i (and hence ∂_w) preserve the ideal I_W, so they induced well-defined operators on the coinvariant algebra $\mathbb{C}[V^\vee]/I_W$.

§25.3 BGG

Theorem 25.9 (Bernstein-Gelfand-Gelfand)

The **Schubert classes** $\sigma_w \in \mathbb{C}[V^\vee]/I_W$ are given by

$$\sigma_w = \partial_{w^{-1}w_0}(\sigma_{w_0}).$$

These form a linear basis of $\mathbb{C}[V^\vee]/I_W$.

(You can take this as the definition of Schubert classes, if you like.)

Remark 25.10. One may obtain polynomial representations of these Schubert classes in the following way:

1. Take *any* $f_{w_0} \in \mathbb{C}[V^\vee]$ of degree $\ell(w_0)$.
2. Set $f_w = \partial_{w^{-1}w_0}(f_{w_0})$.
3. Rescale them so that $f_1 = 1$.

The basis of $\mathbb{C}[V^\vee]/I_W$ does not depend on the initial choice, but the actual representative polynomials do.

§25.4 Choices of basis polynomials

A good choice of f_{w_0} is the **Kostant polynomial**

$$P_{w_0} = \frac{1}{|W|} \prod_{\alpha \in \Phi^+} \alpha.$$

For example, in type A_{n-1} this gives $\frac{1}{n!} = \prod_{i<j}(x_i - x_j)$. These then obey the relation $P_{ws_i} = \partial_i P_w$ if $\ell(ws_i) = \ell(w) - 1$.

Exercise 25.11. Prove that $P_1 = 1$ in this case, so the "scaling" step is not needed.

Remark 25.12. The above polynomials don't have $\mathbb{N}$ coefficients.

In type A_{n-1} a particularly nice starting choice is

Definition 25.13 (Schubert polynomials)**.** We let

$$\mathfrak{S}_{w_0} = x_1^{n-1} x_2^{n-2} \dots x_{n-1}.$$

Then $\mathfrak{S}_{ws_i} = \partial_i \sigma_w$ if $\ell(ws_i) = \ell(w) - 1$. These are very nice combinatorially and have $\mathbb{N}$-coefficients.

§26 April 19, 2017

Last time we defined the single-variable Schubert polynomials $\mathfrak{S}_w(x)$ by stipulating $\mathfrak{S}_{w_0} = x_1^{n-1} \dots x_{n-1}^1$ and $\mathfrak{S}_{ws_i} = \partial_i \mathfrak{S}_w$ for $\ell(ws_i) = \ell(w) - 1$.

§26.1 Double Schubert polynomials

We define the double Schubert polynomials as follows.

Definition 26.1. We let $\mathfrak{S}_w(x, y) = \mathfrak{S}_w(x_1, \dots, x_n, y_1, \dots, y_n)$ according to the relations

$$\mathfrak{S}_{w_0}(x, y) = \prod_{\substack{i+j \le n \\ i,j \in [n]}} (x_i - y_j)$$

and $\mathfrak{S}_{ws_0}(x, y) = \partial_i \mathfrak{S}_w(x, y)$ if $\ell(ws_i) = \ell(w) - 1$.

These satisfy the following properties.

Proposition 26.2 (Properties of $\mathfrak{S}_w(x, y)$)
We have:

- $\mathfrak{S}_w(x) = \mathfrak{S}_w(x, 0)$ is a linear basis for the co-invariant algebra.
- $\mathfrak{S}_w(x, -y)$ has nonnegative integer coefficients.
- $\mathfrak{S}_w(x, -y) = \mathfrak{S}_{w^{-1}}(y, x)$ (symmetry).
- Let $w \in S_n$ and $\widetilde{w} \in S_{n+1}$ by fixing $n + 1$. Then $\mathfrak{S}_w(x, y) = \mathfrak{S}_{\widetilde{w}}(x, y)$.

These will be proven later; see the next lecture.

§26.2 Nil-Hecke Algebra

Definition 26.3. We define the **nil-Hecke algebra** NH_n over $\mathbb{C}$ (say) to be the algebra generated by $u_1, \dots, u_{n-1}$ with the following relations:

- $u_i^2 = 0$,
- $u_i u_j = u_j u_i$ for $|i - j| \ge 2$,
- $u_i u_{i+1} u_i = u_{i+1} u_i u_{i+1}$.

Definition 26.4. We let $h_i(x) = 1 + xu_i \in \mathrm{NH}_n[x]$.

The h_i satisfy the following relations:

Lemma 26.5 (Yang-Baxter relations)
The following relations hold in $\mathrm{NH}_n[x, y]$.

$$\begin{aligned} h_i(x)h_i(y) &= h_i(x + y) \\ h_i(x)h_j(y) &= h_j(y)h_i(x) \qquad |i - j| \ge 2 \\ h_i(x)h_{i+1}(x + y)h_i(y) &= h_{i+1}(y)h_i(x + y)h_{i+1}(x). \end{aligned}$$

Thus x and y "commute with everything".

§27 April 21, 2017

Definition 27.1. Recall that for $w = s_{i_1} \dots s_{i_\ell}$ then we have $u_w = u_{i_1} \dots u_{i_\ell}$ (noting that we get 0 for any non-reduced decomposition).

§27.1 Main Theorem

Theorem 27.2 (BJS, FS, FK)

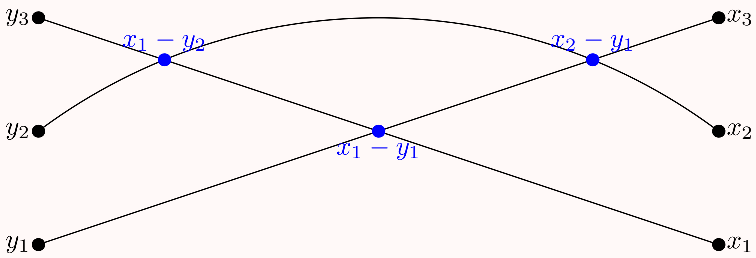
We have

$$\Phi_n \stackrel{\text{def}}{=} \prod_{i=1}^{n-1} \prod_{j=n-i}^{1} h_{i+j-1}(x_i - y_j) = \sum_{w \in S_n} \mathfrak{S}_w(x,y) u_w$$

where $\mathfrak{S}_w(x,y)$ is the double Schubert polynomial.

Example 27.3 (Theorem with $n = 3$)

Let $n = 3$. Consider the wiring diagram for $s_1 s_2 s_1$:



We obtain that:

- This diagram gives $\mathfrak{S}_{s_1 s_2 s_1} = (x_1 - y_2)(x_1 - y_1)(x_2 - y_1)$.
- Undoing the $x_1 - y_2$ crossing gives $\mathfrak{S}_{s_1 s_2} = (x_1 - y_1)(x_2 - y_1)$.
- Undoing the $x_2 - y_1$ crossing gives $\mathfrak{S}_{s_1 s_2 s_1} = (x_1 - y_2)(x_1 - y_1)$.
- Undoing the $x_1 - y_1$ crossing gives an invalid double crossing (corresponding to $s_1 s_1$ which isn't reduced).
- Undo both $x_1 - y_2$ and $x_2 - y_1$ to get $\mathfrak{S}_{s_1} = x_1 - y_1$.
- Undoing $x_1 - y_1$ and either of $x_1 - y_2$ or $x_2 - y_1$ gives $\mathfrak{S}_{s_2} = (x_1 - y_2) + (x_2 - y_1)$.
- Undoing all three crossings gives $\mathfrak{S}_1 = 1$.

Proof. Let $\Phi_n = \sum_w f_w(x,y) u_w$. We want to show f_w gives Schubert polynomials.

The top element is clearly as desired: $f_{w_0} = \prod_{i+j \le n} (x_i - y_j) = \mathfrak{S}_{w_0}(x,y)$.

Now, consider the following claim.

Claim. $\partial_i(\Phi_n) = \Phi_n \cdot u_i$.

In fact this lemma is equivalent to: $\partial_i(f_w) = f_{ws_i}$ if $\ell(ws_i) = \ell(w) - 1$, which is the defining relation for the double Schubert polynomials. We have the following relations from nil-Coxeter algebra:

$$u_v u_i = \begin{cases} u_{vs_i} & \ell(vs_i) = \ell(v) + 1 \\ 0 & \text{otherwise.} \end{cases}$$

Also, consider the following calculation:

$$\begin{aligned} \partial_i(\Phi_n) &= \Phi_n u_i \\ \iff \frac{1}{x_i - x_{i+1}}(1 - s_i)(\Phi_n) &= \Phi_n u_i \\ \iff (1 - s_i)\Phi_n &= \Phi_n \cdot u_i(x_i - x_{i+1}) \\ \iff s_i(\Phi_n) &= \Phi_n \left(1 + (x_{i+1} - x_i)u_i\right) \\ &= \Phi_n h_i(x_{i+1} - x_i). \end{aligned}$$

since x_i commutes with u_i. So it suffices to prove $s_i(\Phi_n) = \Phi_n h_i(x_{i+1} - x_i)$. We invoke the Yang-Baxter relations below. □

Lemma 27.4

$s_i(\Phi_n) = \Phi_n h_i(x_{i+1} - x_i)$.

Example 27.5 ($n = 3$, $i = 2$)

We have

$$\begin{aligned} \Phi_n h_i(x_{i+1} - x_i) &= \underbrace{h_2(x_1 - y_2)h_1(x_1 - y_1)h_2(x_2 - y_1)}_{=\Phi_3} h_2(x_3 - x_2) \\ &= h_2(x_1 - y_2)h_1(x_1 - y_2)h_2(x_3 - y_1) = s_2(\Phi_3). \end{aligned}$$

Example 27.6 ($n = 3$, $i = 2$)

We have

$$\begin{aligned} \Phi_n h_i(x_{i+1} - x_i) &= \underbrace{h_2(x_1 - y_2)h_1(x_1 - y_1)h_2(x_2 - y_1)}_{=\Phi_3} h_1(x_2 - x_1) \\ &= h_2(x_1 - y_2)h_2(x_2 - x_1)h_1(x_2 - y_1)h_2(x_1 - x_1) \\ &= h_2(x_2 - y_2)h_1(x_2 - y_1)h_2(x_1 - x_1) = s_1(\Phi_3). \end{aligned}$$

An outline of the general proof. We have

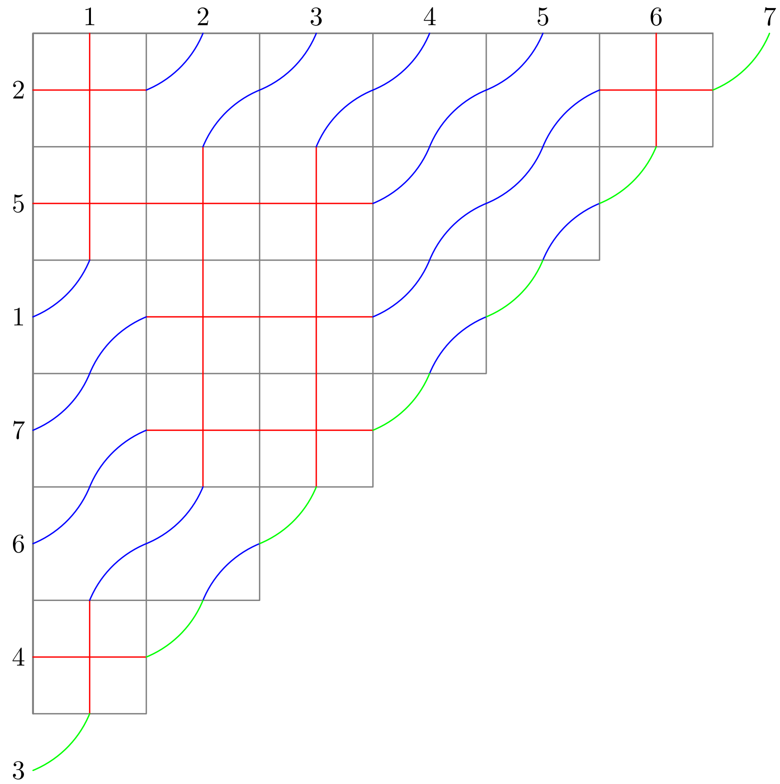
$$\begin{aligned} \Phi_n = h_{n-1}(x_1 - y_{n-1})h_{n-2}(x_1 - y_{n-2})h_{n-1}(x_2 - y_{n-2}) \dots \\ h_1(x_1 - y_1)h_2(x_2 - y_1) \dots h_{n-1}(x_{n-1} - y_1) \cdot h_i(x_{i+1} - x_i). \end{aligned}$$

Then commute the h_i until it bumps into something, and keep throwing Yang-Baxter relations at it.

§27.2 A Word on RC graphs

Usually, RC graphs are drawn in a staircase fashion in the following way (equivalent to what we've been already doing, but. . .).

Draw a staircase as below. In each cell we place either a cross or two bends. Also, we put a single bend at the edge of each staircase.



The polynomials we get out of this correspond to the crosses:

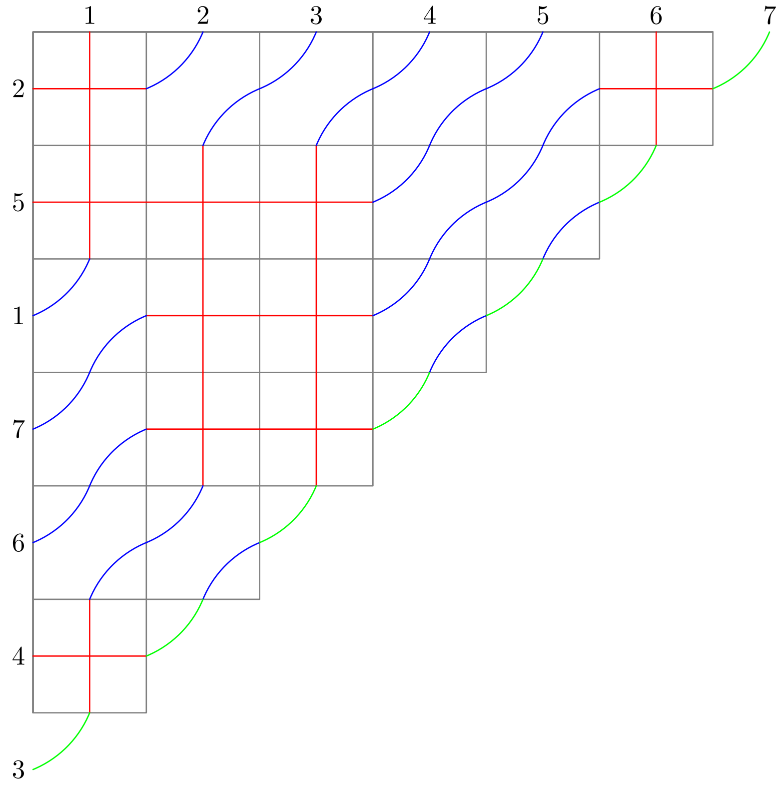
- $(x_1 - y_1)(x_1 - y_0)$
- $(x_2 - y_1)(x_2 - y_2)$
- $(x_2 - y_3)$
- $(x_3 - y_2)(x_3 - y_3)$
- $(x_4 - y_2)(x_4 - y_3)$
- $(x_0 - y_1)$.

The condition is that we avoid double crossings.
Call them strands, pipes, wires, etc . . .

§28 April 24, 2017

§28.1 RC graphs

We reproduce the graph from the previous lecture:



This is an **RC-graph**, which is required to not have double intersections within the strands.

Definition 28.1. For a given permutation $w \in S_n$, we let $\mathrm{RC}(w)$ denote the set of all RC-graphs for that $w \in S_n$.

For $D \in \mathrm{RC}(w)$, we let

$$x^D \overset{\text{def}}{=} \prod_{i=1}^{n} x_i^{\#\text{cross in row } i}.$$

For example, the diagram from last time, we have

$$w = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 5 & 1 & 7 & 6 & 4 & 3 \end{pmatrix}$$

we have $x^D = x_1^2 x_2^3 x_3^2 x_4^2 x_6^1$.

The content from last lecture essentially shows that:

$$\begin{aligned} \mathfrak{S}_w(x) &= \sum_{D \in \mathrm{RC}(w)} x^D \\ \mathfrak{S}_w(x, y) &= \sum_{D \in \mathrm{RC}(w)} \prod_{(i,j) \text{ crosses}} x_i - y_j \end{aligned}$$

(Note that $\mathfrak{S}_w(x) = \mathfrak{S}_w(x, 0)$ by definition.) These combinatorial definition implies three properties about the Schubert polynomials.

Corollary 28.2 (Properties of Schubert polynomials)

The Schubert polynomials satisfy the following properties.

- (Positivity) $\mathfrak{S}_w(x, -y)$ is a polynomial in $x_1, \dots, x_{n-1}$ and $y_1, \dots, y_{n-1}$ with *nonnegative integer coefficients.*
- (Symmetry) $\mathfrak{S}_w(x, -y) = \mathfrak{S}_{w^{-1}}(y, x^{-1})$.
- (Stability) Consider the map $S_n \hookrightarrow S_{n+1}$ by $w \mapsto \widetilde{w}$ with $\widetilde{w}$ fixing $n+1$. Then $\mathfrak{S}_w(x, y) = \mathfrak{S}_{\widetilde{w}}(x, y)$.

Proof. First two are obvious. Stability follows by noting that to avoid a double crossing, the only way that a strand from $n+1$ to reach $n+1$ is for it to never encounter a crossing. $\square$

§28.2 Cauchy formula

Theorem 28.3 (Cauchy formula for Schubert polynomials, Lascoux 1982)

$$\mathfrak{S}_w(x, y) = \sum_{\substack{u,v \\ w=v^{-1}u \\ \ell(w)=\ell(u)+\ell(v)}} \mathfrak{S}_u(x)\mathfrak{S}_v(-y).$$

Hence this expresses double Schubert polynomials in terms of the ordinary Schubert values.

In particular, if w is the longest permutation, we deduce that

$$\sum_{\substack{i+j \le n \\ i,j \ge 1}} (x_i - y_j) = \mathfrak{S}_w(x, y) = \sum_{w \in S_n} \mathfrak{S}_w(x)\mathfrak{S}_{w_0 w}(-y).$$

Remark 28.4 (Digression on Schur polynomials)**.** This is reminiscent of Schur polynomials $s_\lambda(x)$ (which turn out to be a special case of Schubert polynomials, which we'll define later). To be concrete, the *dual Cauchy formula for Schur polynomials* states

$$\sum_\lambda s_\lambda(x) s_{\lambda'}(y) = \prod_{i,j} (1 + x_i y_j)$$

where λ and λ' are the dual. (The usual Cauchy formula for Schur polynomials states

$$\sum_\lambda s_\lambda(x) s_\lambda(y) = \prod_{i,j} (1 - x_i y_j)^{-1}$$

instead.)

We now prove Cauchy's formula. Define $\Phi_n(x, y) = \Phi_n(x_1, \dots, x_{n-1}, y_1, \dots, y_{n-1})$ as in previous lectures.

Lemma 28.5

Cauchy's formula is equivalent to

$$\Phi_n(x, y) = \Phi_n(0, y)\Phi_n(x, 0).$$

Proof. Recall the notation $u_w = u_{i_1} \dots u_{i_\ell}$ if $w = s_{i_1} \dots s_{i_\ell}$ in the nil-Coxeter algebra. Then

$$u_v u_w = \begin{cases} u_{v \cdot w} & \ell(v \cdot w) = \ell(v) + \ell(w) \\ 0 & \text{else.} \end{cases}$$

So one sees this just by writing out definition of $\Phi_n(x, y)$. $\square$

Thus we prove $\Phi_n(x, y) = \Phi_n(0, y)\Phi_n(x, 0)$. This will be repeated application of Yang-Baxter relations.

Example 28.6 ($n = 2$)

Let's check the above formula for $n = 2$. Recall that $\Phi_2(x, y) = h_1(x_1 - y_1)$ and so it's equivalent to show

$$h_1(x_1 - y_1) = h_1(-y_1)h_1(x_1)$$

which is more or less obvious.

Example 28.7 ($n = 3$)

Let's check the above formula for $n = 3$. Recall that We wish to show

$$\begin{aligned} \Phi_3(x, y) &\overset{\text{def}}{=} h_2(x_1 - y_2)h_1(x_1 - y_1)h_2(x_2 - y_1) \\ &= h_2(-y_2)h_1(-y_1)h_2(-y_1)h_2(x_1)h_1(x_1)h_2(x_2). \end{aligned}$$

Again keep using Yang-Baxter relations.

§28.3 Linear space of Schubert polynomial

We let

$$\mathbf{Stair}_n = \left\{ x_1^{a_1} x_2^{a_2} \dots x_{n-1}^{a_{n-1}} \mid a_i \le n - i \right\}$$

be the $n!$ monomials which divide $x_1^{n-1} x_2^{n-2} \dots x_1$, and let $\langle \mathbf{Stair}_n \rangle$ denote their span (which is an $n!$-dimensional space). Obviously, if $D \in \mathrm{RC}(w)$ we have $D \in \langle \mathbf{Stair}_n \rangle$. We now make the following claim about the linear bases of $\langle \mathbf{Stair}_n \rangle$.

Theorem 28.8 (Basis of $\langle \mathbf{Stair}_n \rangle$)

The space $\langle \mathbf{Stair}_n \rangle$ has the following three linear bases.

- $x_1^{a_1} \dots x_{n-1}^{a_{n-1}}$, i.e. the elements of $\langle \mathbf{Stair}_n \rangle$ (by definition).
- $\mathfrak{S}_w(x)$ for $w \in S_n$.
- Products

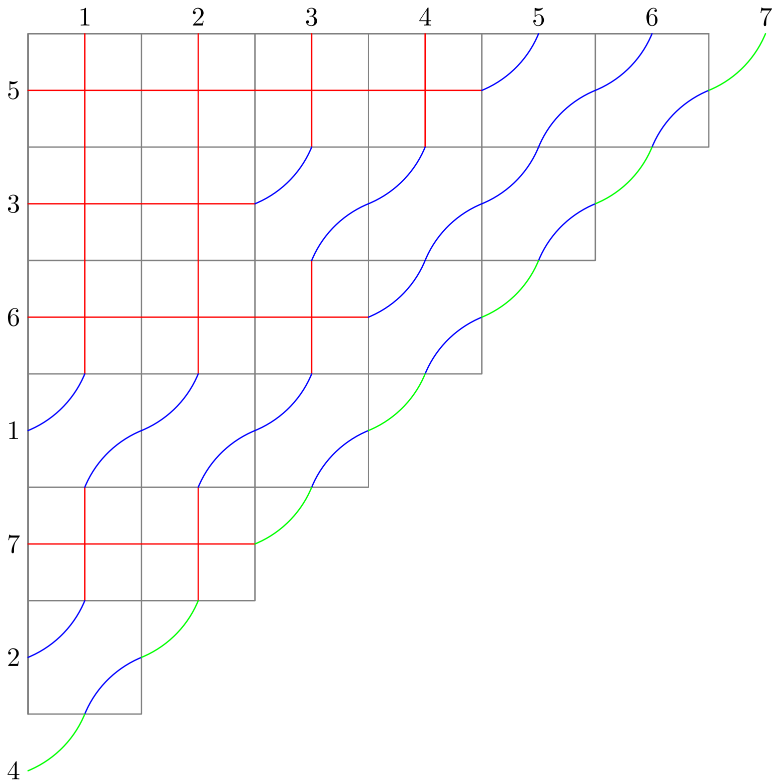
  $$e_{i_1}(x_1)e_{i_2}(x_1, x_2)e_{i_3}(x_1, x_2, x_3) \dots e_{i_{n-1}}(x_1, x_2, \dots, x_{n-1})$$

  of elementary symmetric polynomials. Here $i_k \in \{0, 1, \dots, k\}$ for each k.

Moreover the images of these polynomials in the coinvariant algebra also form a basis.

In order to do this, we will need for each w to find $D \in \mathrm{RC}(w)$ for which x^D is lexicographically minimal.

It's easier to describe the construction in reverse. Let $\vec{a}$ be a vector for which $x^{\vec{a}} \in \langle \mathbf{Stair}_n \rangle$. Then we can define an RC diagram by taking all the crosses to be left-justified. For example, if $\vec{a} = (4, 2, 3, 0, 2, 0)$ we get the following:



The claim with this particular procedure is that:

Claim 28.9. When we do this procedure with any given vector $\vec{a}$ (satisfying $a_i \le n - i$), each permutation w arises exactly once.

In fact we have $a_i = \#\{j > i \mid w_j < w_i\}$ equal to the number of inversions with smaller component i. The bijection is called the **Lehmer code**.

Claim 28.10. Let w be the permutation obtained by this procedure from $\vec{a}$. Consider another RC-graph D with the same number of crossings in each row, and let w' be the associated permutation. Then $w' \le w$ in the *lexicographical* order on permutations.

Using these claims we will construct the **Kostka-Schubert matrix** $(K_{w,a})$, which is an $n! \times n!$ matrix, such that the (w, a)th entry is equal to $\#\left\{D \in \mathrm{RC}(w) \mid x^D = x^a\right\}$. Thus this serves as a transition matrix from the $\mathfrak{S}_w(x)$ basis to $x^{\vec{a}}$ basis. In fact the claim is that the $K_{w,a}$ is an upper-triangular matrix once sorted in lexicographical order.

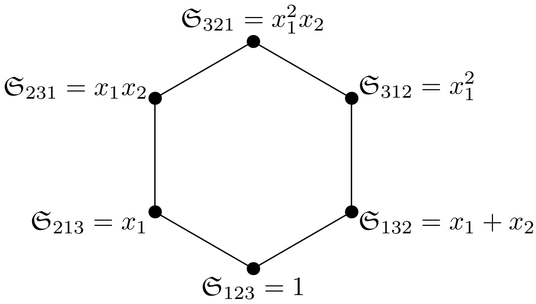
§29 April 26, 2017

Recall $K = (K_{w,a})$ as defined last lecture, with the property that

$$\mathfrak{S}_w = \sum_a K_{w,a} x^a.$$

§29.1 The $n = 3$ example

The poset of Schubert polynomials is:



The entries of K can be arranged as follows, with columns indexed by vectors a and rows indexed by permutations.

$$K = \begin{array}{c|cccccc}
 & 00 & 01 & 10 & 11 & 20 & 21 \\ \hline
123 & 1 & 0 & 0 & 0 & 0 & 0 \\
132 & 0 & 1 & 1 & 0 & 0 & 0 \\
213 & 0 & 0 & 1 & 0 & 0 & 0 \\
231 & 0 & 0 & 0 & 1 & 0 & 0 \\
312 & 0 & 0 & 0 & 0 & 1 & 0 \\
321 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}$$

As claimed last time, the claim is that K is upper triangular with respect to the lexical order above. Moreover, the entries on the main diagonal correspond to the lexically minimal term x^a.

Thus in particular K is invertible, and so $\mathfrak{S}_w$ is a linear basis of $\langle \mathbf{Stair}_n \rangle$.

Problem 29.1. Find a combinatorial (subtraction-free) formula for K^{-1}.

§29.2 Infinite permutations

From $S_1 \subset S_2 \subset S_3 \subset \dots$ let us consider the limit

$$S_\infty = \varinjlim S_n$$

consisting of those infinite permutations $w : \mathbb{Z} \to \mathbb{Z}$ which fix all co-finitely many points (i.e. $w(i) = i$ for all sufficiently large i, in terms of w).

Then, it follows we can define polynomials $\mathfrak{S}_w$ for $w \in S_\infty$.

Corollary 29.2

$\mathfrak{S}_w$ for $w \in S_\infty$ forms a linear basis of $\mathbb{C}[x_1, x_2, \dots]$.

Remark 29.3. Not clear how to write $K_{w,a}$ for S_∞, since the usual lexicographic order isn't great here: one would have to write infinitely many things like $01\dots$, $02\dots$, et cetera, before writing $100\dots$. It would be interesting to figure out what the correct order is.

§29.3 Geometrical background

Consider the two manifolds we define earlier:

- $\mathbf{Fl}_n$, the complete flags of $\mathbb{C}^n$.
- $\mathbf{Gr}_{k,n}$, the manifold of k-dimensional linear subspaces in $\mathbb{C}^n$, for $k \le n$.

There is a natural map

$$\begin{aligned} p : \mathbf{Fl}_n &\twoheadrightarrow \mathbf{Gr}_{k,n} \\ (V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n) &\mapsto V_k. \end{aligned}$$

This gives a map of cohomology rings

$$p^\bullet : H^\bullet(\mathbf{Gr}_{k,n}) \hookrightarrow H^\bullet(\mathbf{Fl}_n).$$

We now give a description of the two bases of these maps. First, notation for the ideal:

Definition 29.4. Let I_n denote the ideal generated by homogeneous symmetric polynomials of positive degree. (This notation will be retained in next lectures.)

- $H^\bullet(\mathbf{Gr}_{k,n})$ has a linear basis of *Schubert polynomials* σ_λ, where λ is partition whose Young diagram fits inside a $k \times (n-k)$ rectangle. (That is, $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ where $n - k \ge \lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_k$.) This space has dimension $\binom{n}{k}$.
- $H^\bullet(\mathbf{Fl}_n) \cong \mathbb{C}[x_1, \dots, x_n]/I_n$ has a linear basis of *Schur polynomials* σ_w, where $w \in S_n$. This space has dimension $n!$.

Both of these cohomology rings can be thought of as quotients for the cohomology ring. So the bottom line is the we have a map

$$\lambda \mapsto w(\lambda) \in S_n$$

for which $\mathfrak{S}_{w(\lambda)}$ is $s_\lambda(x_1, \dots, x_k)$.

Let's now describe this map taking partitions to permutations. First, we describe the permutations achievable as the image of $\mathbf{Gr}_{k,n}$.

Definition 29.5. A permutation $w \in S_n$ is (n,k)-**Grassmanian** if $w_1 < w_2 < \cdots < w_k$ and $w_{k+1} < w_{k+2} < \cdots < w_n$.

Lemma 29.6

Grassmanian permutations are in bijection with Young diagrams λ which fit inside $k \times (n-k)$ rectangles.

Given a partition λ, the permutation $w = w(\lambda)$ is such that:

The Lemer code of $w(\lambda)$ coincides with $(\lambda_k, \lambda_{k-1}, \dots, \lambda_1, 0, \dots, 0)$.

Example 29.7

Let $n = 9$ and $k = 4$. An example of a $(9, 4)$-Grassmanian permutation is

$$w = 2\,5\,6\,9 \mid 1\,3\,4\,7\,8.$$

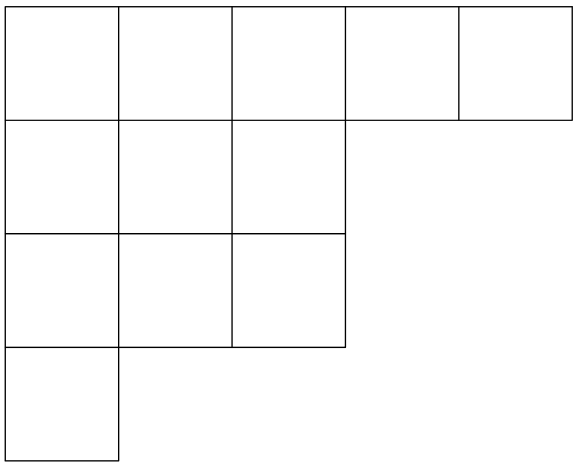
In that case the Lemer code of w is $(1, 3, 3, 5, 0, 0, 0, 0)$, so the corresponding Young diagram is $\lambda = (5, 3, 3, 1)$.

Remark 29.8. An equivalent, perhaps more concrete, way to describe the bijection is that

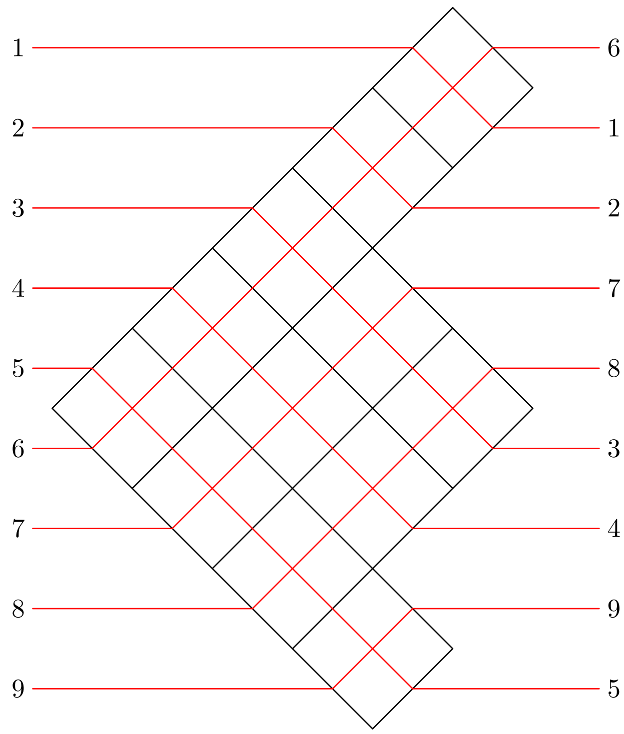
$$\lambda = (w_k - k, w_{k-1} - (k - 1), \dots, w_1 - 1).$$

§29.4 Wiring diagrams of Grassmanian permutations

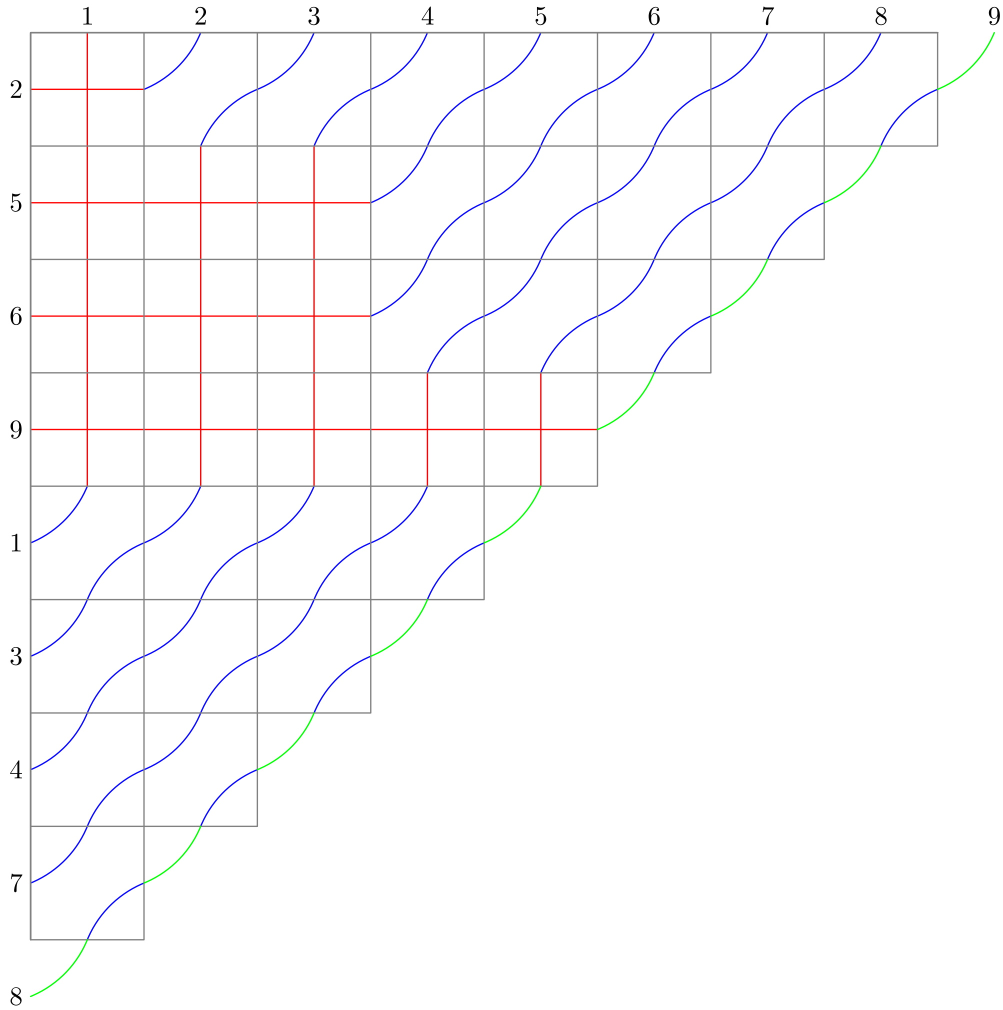
Main claim: these diagrams are totally commutative. Let's take the Young diagram for $(5, 3, 3, 1)$ again:



We add crosses into it and rotate it by 45 degrees.



One can even embed this diagram as follows. Rotate and reflect, them embed in upper left: Here is the previous example:



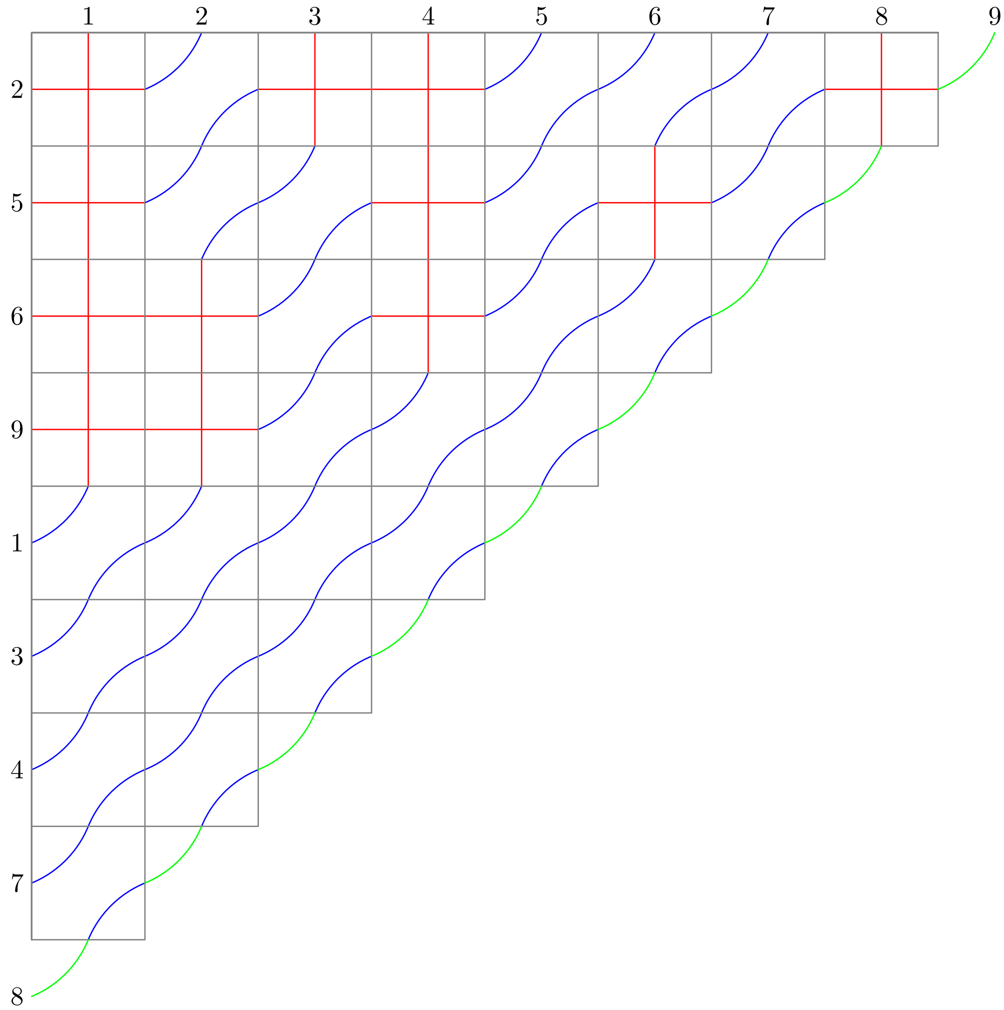
This is not the only RC graph for the permutation, although it is the lexicographically minimal one. But:

Claim 29.9. The RC-graph for $w = w(\lambda)$ are in bijection with semi standard Young tableau with entries in $\{1, 2, \dots, k\}$.

§30 April 28, 2017

§30.1 The correspondence

We claim that RC-diagrams of Grassmanian are in bijection with **semi-standard Young tableaux**. Here is the bijection. Consider the following RC diagram for the same permutation as last time.



This is obtained by taking the original embedded guy and then letting the red crosses "float" northeast.

We can then fill an inverted Young tableau by recording the row number in which the intersections occur after floating:

$$\begin{bmatrix} 1 & & & & \\ 2 & 1 & 1 & & \\ 3 & 3 & 2 & & \\ 4 & 4 & 3 & 2 & 1 \end{bmatrix}$$

This is increasing strictly downwards, and decreasing weakly to the right. To obtain a semi-standard Young tableau we can then just invert the entries and flip it:

$$\begin{bmatrix} 1 & 1 & 2 & 3 & 4 \\ 2 & 2 & 3 & & \\ 3 & 4 & 4 & & \\ 4 & & & & \end{bmatrix}$$

Now this is weakly increasing to right, and strictly increasing down. The associated polynomial is $x_1^2 x_2^3 x_3^3 x_4^4$, with x_i having $\#i$ as an exponent.

§30.2 Schur symmetric polynomial

Definition 30.1. We define

$$s_\lambda(x) = s_\lambda(x_1, \dots, x_k) \overset{\text{def}}{=} \sum x_1^{\#1} x_2^{\#2} \dots x_k^{\#k}$$

where the sum is taken over all semi-standard Young tableau of shape λ with entries in $\{1, \dots, k\}$.

The bijection earlier gives:

Corollary 30.2
If $w = w(\lambda)$, then $\mathfrak{S}_{w(\lambda)} = s_\lambda(x_1, \dots, x_k)$.

Examples:

- $\mathfrak{S}_{s_k} = x_1 + x +_2 + \dots + x_k = e_1(x_1, \dots, x_k)$.
- $\mathfrak{S}_{s_{k-i+1} s_{k-i+2} \dots s_k} = e_i(x_1, \dots, x_k)$.
- $\mathfrak{S}_{s_{k+i-1} s_{k+i-2} \dots s_k} = h_i(x_1, \dots, x_k)$ (complete homogeneous polynomial).

§30.3 Symmetry of $H^\bullet(\mathbf{Fl}_n) = \mathbb{C}[x_1, \dots, x_n]/I_n$

Those familiar with this area may know of the so-called **involution ω** defined by

$$\begin{aligned} x_i &\mapsto -x_{n-i+1} \\ s_i &\mapsto s_{n-i} \end{aligned}$$

Thus if $w = s_{i_1} s_{i_2} \dots s_{i_\ell}$, and $\omega(w) = s_{n-i_1} \dots s_{n-i_\ell} = w_0 w w_0$. In terms of Young diagram, this corresponds to conjugation of λ.

Proposition 30.3 (ω on Schubert polynomials)
ω sends σ_w to $\sigma_{w_0 w w_0}$.

Corollary 30.4
$\mathfrak{S}_w(x_1, \dots, x_n) \equiv \mathfrak{S}_{w_0 w w_0}(-x_n, -x_{n-1}, \dots, -x_1) \pmod{I_n}$.

Corollary 30.5
$s_\lambda(x_1, \dots, x_k) \equiv s_{\lambda'}(x_n, -x_{n-1}, \dots, -x_{k+1}) \pmod{I_n}$.

Corollary 30.6
$e_i(x_1, \dots, x_k) = h_i(-x_n, -x_{n-1}, \dots, -x_{k+1}) \pmod{I_n}$.

Example 30.7

When $i = 1$ in the last corollary, we have $e_1(x_1, \dots, x_k) = x_1 + \dots + x_k$, and $h_1(-x_n, -x_{n-1}, \dots, -x_{k+1}) = -x_n - x_{n-1} - \dots - x_{k+1}$. But I_n contains $x_1 + \dots + x_n \equiv 0$.

Exercise 30.8. Check the last corollary algebraically from definitions, for all i.

§30.4 Monk's formula

There are many variations; here is one.

Recall that $\mathfrak{S}_w$ forms a basis of $\mathbb{C}[x_1, \dots]$ for $w \in S_\infty$.

Theorem 30.9 (Monk's formula)

$$\underbrace{\mathfrak{S}_{s_k}}_{=x_1+\dots+x_k} \cdot \mathfrak{S}_w = \sum_{\substack{i,j \\ i \le k < j \\ \ell(wt_{ij}) = \ell(w)+1}} \mathfrak{S}_{wt_{ij}}$$

where t_{ij} is the transposition $(i\ j)$.

The sum is exactly the covering relations in the strong Bruhat order on S_∞. (We really want S_∞, not just S_n! So e.g. when computing $\mathfrak{S}_{s_3}\mathfrak{S}_{315462}$ it's possible that some new terms will have 7 in them.)

§31 May 1, 2017

Was ill today. To be written.

§32 May 3, 2017

Was ill today. To be written.

§33 May 4, 2017

Was ill today. To be written.

§34 May 8, 2017

§34.1 Generalizing Schubert calculus

This week we will introduce some generalizations of the Schubert calculus we talked about in the past.

Schubert calculus essentially studies cohomology rings $H(G/B)$ (which in the A_n case is $H(\mathbf{Fl}_n)$); this ends up being the ω-invariant algebra.

Some possible generalizations:

- The T-equivariant cohomology $H_T(G/B)$, which gives the double Schubert polynomials.
- The quantum cohomology groups $\mathrm{QH}(G/B)$, which count intersections of Schubert varieties. (The name "quantum" here is misleading, not the same as in "quantum groups".) This lead to quantum Schubert polynomials.
- K-theory: this is $K(G/B)$. This is the Grothendiek rings, which replaces Schubert polynomials with Grothendiek rings.

(You can put all three directions together in so-called "quantum equivariant K-theory".)

Today we'll talk about the K-theory.

§34.2 K-theory of G/B

Let Φ, W, α_i, ω_i $(1 \le i \le r)$ be as always and let $P = \langle \omega_1, \dots, \omega_r \rangle_{\mathbb{Z}}$ be the weight lattice. We define $\mathbb{Z}[P]$ as the group algebra of P, with

- linear basis e^λ, $\lambda \in P$, and
- multiplication by $e^\lambda \cdot e^\mu = e^{\lambda+\mu}$.

In other words,
$$\mathbb{Z}[P] = \mathbb{Z}[e^{\pm\omega_1}, \dots, e^{\pm\omega_r}]$$
becomes the ring of Laurent polynomials in r variables e^{ω_i}. Now, W acts on $\mathbb{Z}[p]$ via $w(e^\lambda) = e^{w(\lambda)}$.

Theorem 34.1 (Grothendiek Ring)
We have
$$K(G/B) \simeq \mathbb{Z}[P]/J$$
where J is the ideal generated by expressions of the form $f - f(1)$ (i.e. those polynomials with vanishing constant term).

Here when $f = \sum c_\lambda e^\lambda$ we mean $f(1) = \sum c_\lambda$.

Example 34.2 (Example when $W = A_1$)
Let's take $W = A_1$, so $\mathbb{Z}[P] = \mathbb{Z}[e^{\pm\omega_1}]$, and $s_1(e^{\omega_1}) = e^{-\omega_1}$. Thus
$$f = \sum_{j \in \mathbb{Z}} c_j e^{j \cdot \omega_1}.$$
In that case $J = \left\langle e^{j\omega_1} + e^{-j\omega_1} - 2 \right\rangle$. So $K(G/B)$ would be generated by linear span of 1 and e^{ω_1}.

§34.3 Linear basis of $K(G/B)$

In general, $\dim K(G/B) = |W|$.

In fact the ring is isomorphic to the coinvariant algebra. Despite the fact that $K(G/B)$ is isomorphic to the coinvariant algebra, there is something new here. It turns out $K(G/B)$ has a linear basis of the form $[\mathcal{O}_w]$, $w \in W$. We won't discuss the geometry of this basis, but these are actually structure sheafs. The main question is to understand $[\mathcal{O}_u][\mathcal{O}_v]$.

In order to define $\mathcal{O}_w$, we need to define the Demazure operators (analogous to the divided difference operator before).

Definition 34.3. The ith **Demazure operator** is defined by

$$\begin{aligned} D_i &: \mathbb{Z}[P] \to \mathbb{Z}[P] \\ f &\mapsto \frac{f - e^{-\alpha_i} s_i(f)}{1 - e^{-\alpha_i}}. \end{aligned}$$

Proposition 34.4

The Demazure operators satisfy the 0-Hecke relations:

(1) $D_i^2 = D_i$.

(2) $D_i D_j D_i \dots = D_j D_i D_j \dots$ (each m_{ij} times, as usual).

Exercise 34.5. Check the relations.

Thus again given $w = s_{i_1} \dots s_{i_\ell}$ reduced, we may define

$$D_w \overset{\text{def}}{=} D_{i_1} \dots D_{i_\ell}$$

which depends only on w and not the choice of reduced decomposition.

Theorem 34.6 (Kostant-Kumar)

We have the relation

$$[\mathcal{O}_w] = D_{w^{-1}}([\mathcal{O}_1])$$

with $1 \in W$ being the identity of W.

Thus $[\mathcal{O}_w]$ is a K-theoretic analogue of $\sigma_{w_0 w}$. Anyways, since we haven't given the geometric definition of $\mathcal{O}_w$ yet, we can just take this as a definition.

§34.4 Specializing to A_{n-1}

We now specialize to type A_{n-1}.

We introduce variables $z_1, \dots, z_n$ such that $z_1 z_2 \dots z_n = 1$ and

$$\begin{aligned} e^{\omega_i} &= z_1 z_2 \dots z_i \\ e^{\alpha_i} &= \frac{z_{i+1}}{z_i}. \end{aligned}$$

We then have

$$K(\mathbf{Fl}_n) = \mathbb{Z}[z_1^{\pm 1}, \dots, z_n^{\pm 1}]/J.$$

Here J_n is generated by $f - f(1)$ for symmetric Laurent polynomials f. The claim is that:

Theorem 34.7

We have an isomorphism of the $n!$-dimensional rings

$$\begin{aligned}\mathbb{Z}[z_1^{\pm 1}, \dots, z_n^{\pm 1}]/J_n &\to \mathbb{Z}[x_1, \dots, x_n]/I_n \\ z_i &\mapsto 1 - x_i.\end{aligned}$$

So the Schubert classes σ_w correspond to Schubert polynomials, while $[\mathcal{O}_w]$ correspond to so-called Grothendiek polynomials under this isomorphism.

Under this isomorphism the Demazure operators gain the following shape:

$$\begin{aligned}f \mapsto \frac{f - \frac{z_i}{z_i+1} s_i(f)}{1 - \frac{z_i}{z_{i+1}}} \\ = \frac{z_{i+1} f - z_i s_i(f)}{z_{i+1} - z_i}.\end{aligned}$$

These are called **isoberic divided differences**.

So the interesting thing happening is that the expression $1 - \frac{z_i}{z_{i+1}}$ that we get in K-theory becomes $z_{i+1} - z_i$ in cohomology.

§34.5 Construction of Grothendiek polynomial

In the language of the x_i, one first notices that D_i preserves the ideal J_n, then writes

$$\pi_i(f) \overset{\text{def}}{=} \partial_i \left((1 - x_{i+1}) f \right)$$

where ∂_i is the usual divided difference. These satisfy the relations

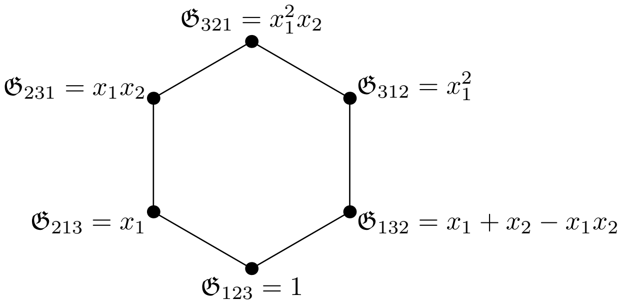
(1) $\pi_i^2 = \pi_i$.

(2) $\pi_i \pi_j = \pi_j \pi_i$

(3) $\pi_i \pi_{i+1} \pi_i = \pi_{i+1} \pi_i \pi_{i+1}$.

When then constructs the Grothendiek polynomial $\mathfrak{G}_w$ (Gothic G) by replacing ∂_i with π_i, verbatim. The interesting bit is that π_i is not homogeneous in degree n.

For example, with $n = 3$ we get



so the only difference in $\mathfrak{G}_{132} = x_1 + x_2 - x_1 x_2$. There is also an RC-graph interpretation, but the big change is that Monks' rule becomes very interesting. We'll see this in future lectures.

§35 May 10, 2017

Quick logistical announcement: problem set 2 is up. Reproduced in its entirety below.

> Optional problem set. Turn in by May 17, 2017.
>
> Solve any number of exercises or problems that were given in lectures during the semester, or prove any claim(s) that was stated in lectures without proof.

§35.1 Definition of Grothendiek polynomials

These were introduced by Lascoux Schützenberger. They represent $[\mathcal{O}_u]$, classes of structure sheaf of Schubert varieties in $K(\mathbf{Fl}_n) \simeq \mathbb{Z}[x_1, \dots, x_n] / \langle e_1, \dots, e_n \rangle$.

This is actually a special case of a construction that works for any Weyl group, vie the Demazure operators. In type A_{n-1} they are the **isobaric divided difference operators**

$$\begin{aligned} \pi_i : \mathbb{Z}[x_1, \dots, x_n] &\to \mathbb{Z}[x_1, \dots, x_n] \\ f &\mapsto \partial_i \left((1 - x_{i+1}) f \right) \end{aligned}$$

where $\partial_i = \frac{f - s_i(f)}{x_i - x_{i+1}}$ is as usual. Then, the Grothendiek polynomials

$$\mathfrak{G}_w \qquad w \in S_n$$

are defined in analogy to Schubert polynomials with π_i in place of ∂_i, meaning that

$$\begin{aligned} \mathfrak{G}_{w_0} &= x_1^{n-1} x_2^{n-2} \dots x_{n-1} \\ \mathfrak{G}_{ws_i} &= \pi_i(\mathfrak{G}_w) \quad \text{ if } \ell(ws_i) = \ell(w) - 1. \end{aligned}$$

See the examples as before.

These Grothendiek satisfy the following properties.

(1) The Schur polynomial $\mathfrak{S}_w$ is the lowest homogeneous component (i.e. the component of degree $\ell(w)$) of $\mathfrak{G}_w$.

(2) $\mathfrak{G}_w$ are stable under the embedding $S_n \hookrightarrow S_{n+1}$. Hence $\mathfrak{G}_w$ can be defined for $w \in S_\infty$.

(3) (Positivity) We have $\widehat{\mathfrak{G}}_w \overset{\text{def}}{=} (-1)^{\ell(w)} G_w(-x_1, -x_2, \dots)$ has nonnegative integer coefficients.

§35.2 Grothendiek pipe dreams

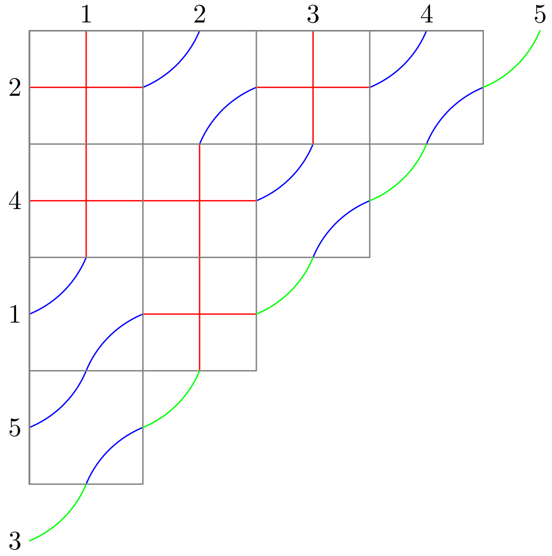
This will be almost the same as with Schur polynomials except instead of the nil-Coxeter relation $u_i^2 = 0$, we have 0-Hecke $u_i^2 = u_i$.

The wiring pictures will also be the same, except now the strings may intersect multiple times. This means that there are $2^{\binom{n}{2}}$ permitted diagrams this time: there are no longer any constraints on the diagram.

The weight of the diagram is

$$x^D = \prod_i x_i^{\#\text{cross in } i\text{th row}}.$$

Here is the picture for a diagram D, for which $x^D = x_1^2 x_2^2 x_3^1 x_4^1$.



There is one more difference in reading the permutation from the wiring diagram — the relation $u_i^2 = u_i$ means that if two strands intersect twice, then we have to ignore the second cross (and thus turn at a sharp right angle at the cross). So for example, in the above example,

$$w(D) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 1 & 5 & 3 \end{pmatrix}$$

Theorem 35.1 (Fomin-Kirillov)
We have

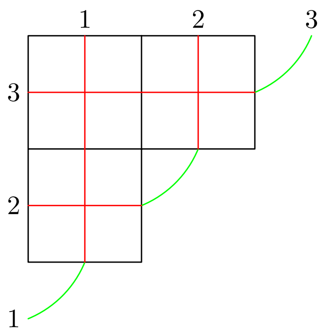
$$\widehat{\mathfrak{G}}_w = \sum_{D: w(D) = w} x^D.$$

Proof. Basically exactly the same. Exercise: do so. □

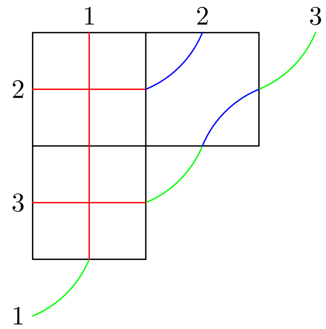
§35.3 Extended Example for $n = 3$

We draw all the $2^3 = 8$ diagrams for $n = 3$:

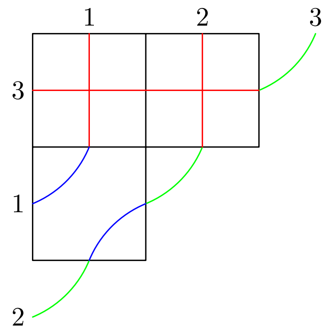
- The permutation 321 and monomial $x_1^2 x_2$.



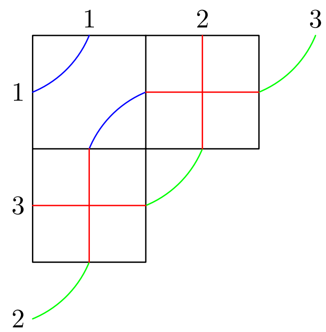
- The permutation 231 and monomial $x_1 x_2$.



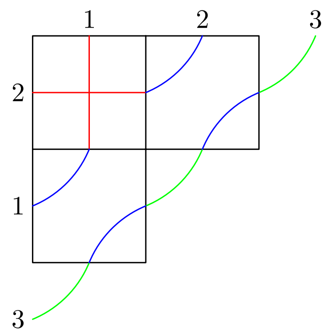
- The permutation 312 and monomial x_1^2.



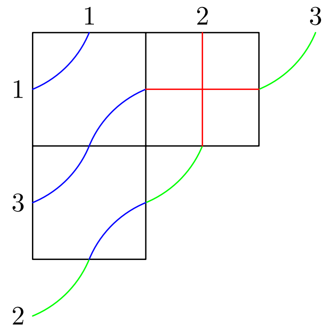
- The permutation 132 and monomial $x_1 x_2$. (This is the new term that wasn't there before; note the double crossing, and how 3 and 2 don't switch at the southwest intersection.)



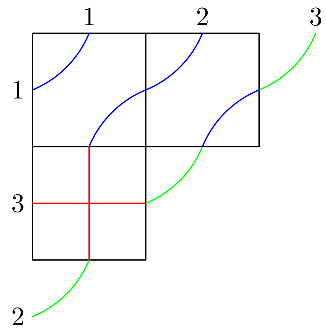
- The permutation 213 and monomial x_1.



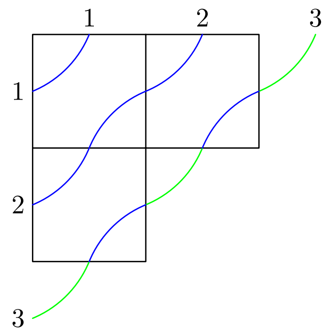
- The permutation 132 and monomial x_1.



- The permutation 132 and monomial x_2.



- The permutation 123 and monomial 1.



§35.4 Monk's Formula for Grothendiek polynomials

Let $K(G/B) \simeq \mathbb{Z}[P]/J$, with

$$\mathbb{Z}[P] = \left\{ \sum_{\lambda \in P} a_\lambda e^\lambda \right\}$$
$$J = \left\langle \sum a_\lambda e^\lambda \mid a_\lambda = a_{w(\lambda)} \forall \lambda \in P, w \in W \text{ and } \sum a_\lambda = 0 \right\rangle .$$

Here $\sum a_\lambda = 0$ really is $f(1) = 0$.

When we stated Monk's formula for Schur polynomials, we saw formulas for $\sigma_{s_i} \cdot \sigma_w$ and $[\lambda] \cdot \sigma_w$; these are equivalent since $\sigma_{s_i} = [w_i]$. We will use the latter form.

Consider the map

$$\begin{aligned} H(G/B) &\to K(G/B) \\ \sigma_w &\mapsto [\mathcal{O}_{w_0 w}] \\ [\lambda] &\mapsto [\mathcal{L}_\lambda] = e^\lambda . \end{aligned}$$

Here $\mathcal{L}_\lambda$ is a line bundle (which we haven't defined); then we have

$$e^\lambda \cdot [\mathcal{O}_w] = \sum_u c_{\lambda,u,w} [\mathcal{O}_u]$$

and the point of Monk's formula is to compute the coefficients $c_{\lambda,u,w}$.

This formula is complicated: the first part is based on so-called alcove path model.

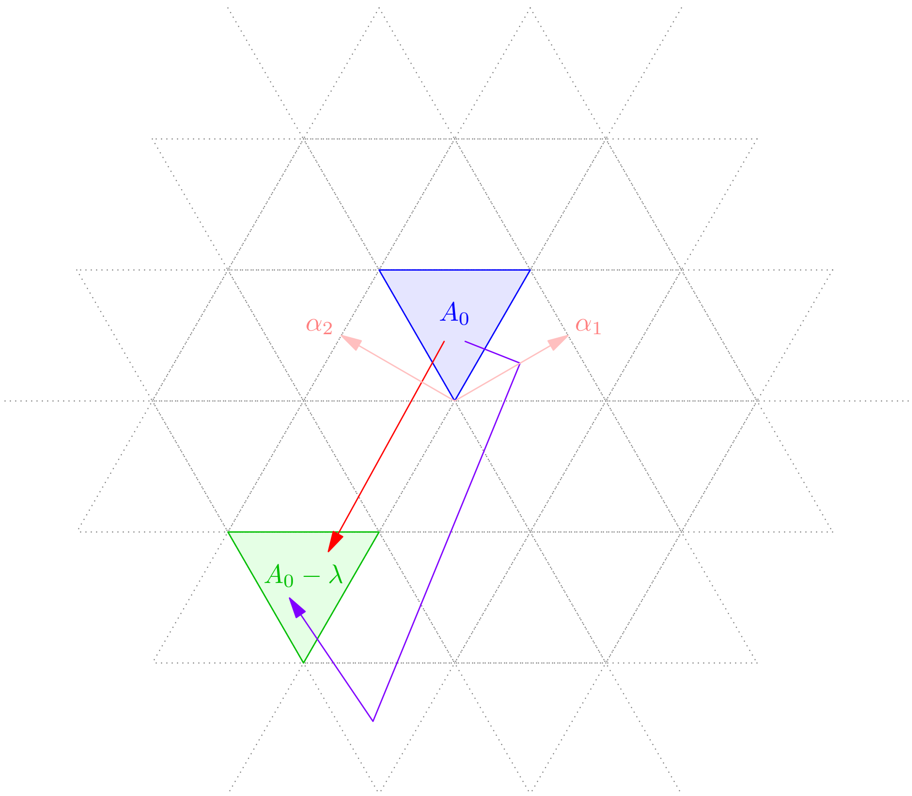
§35.5 Alcove path model

Consider the affine Coxeter arrangement for the dual root system $\Phi^\vee$, which is given by

$$H_{\alpha,k} = \left\{ (x, \alpha^\vee) = k \mid \alpha \in \Phi, k \in \mathbb{Z} \right\}$$

The resulting regions are alcoves; the fundamental alcove A_0 is marked.

Let λ be any root, and consider any path p from A_0 to $A_0 - \lambda$ (drawn in red below). A **λ-chain** is then defined as follows: it is a sequence of roots $(\beta_1, \dots, \beta_\ell)$ corresponding to the intersection of p with the hyperplanes. This chain is not unique, and it need not even be shortest.



The sign of the chain is determined by whether we go from the positive side to the negative side of the hyperplane, or vice-versa. For example, in the red path shown, we have a chain

$$(\alpha_2, \alpha_1 + \alpha_2, \alpha_2, \alpha_1 + \alpha_2) .$$

in the purple path, we instead get the chain

$$(\alpha_1, \alpha_1 + \alpha_2, \alpha_2, \alpha_1 + \alpha_2, \alpha_2, \alpha_1 + \alpha_2, -\alpha_1 - \alpha_2, -\alpha_1) .$$

So again these chains need not be unique.

Next, for $\alpha \in \Phi^+$ we define the operator

$$B_\alpha : [\mathcal{O}_w] \mapsto \begin{cases} [\mathcal{O}_{ws_\alpha}] & \text{if } \ell(ws_\alpha) = \ell(w) - 1 \\ 0 & \text{otherwise.} \end{cases}$$

Then define $B_{-\alpha} = -B_\alpha$. Now, the result, due to Leneert and Postnikov, is:

> Pick any λ-chain $(\beta_1, \dots, \beta_\ell)$. We have
>
> $$e^\lambda \cdot [\mathcal{O}_w] = (1 + B_{\beta_\ell}) \left(1 + B_{\beta_{\ell-1}}\right) \dots (1 + B_{\beta_1}) \left([\mathcal{O}_w]\right).$$

§36 May 12, 2017

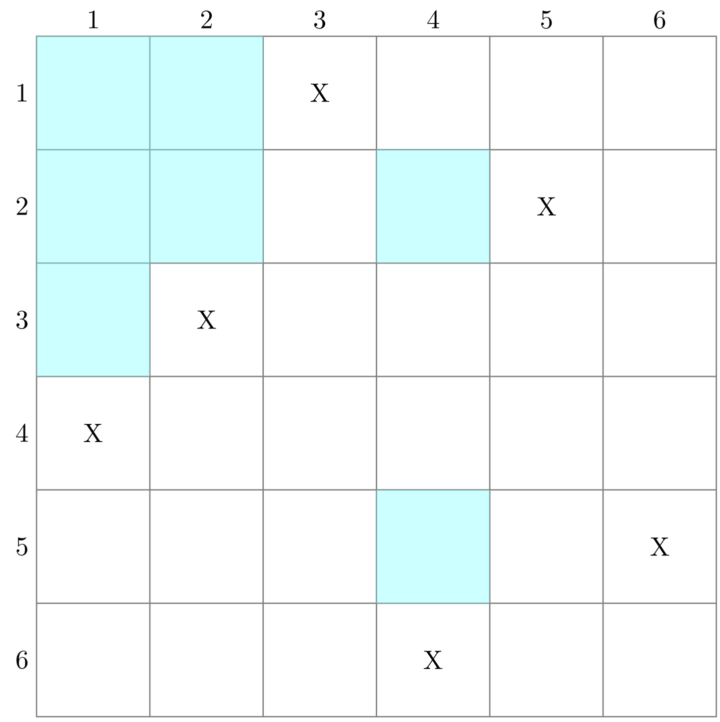
§36.1 Another perspective on pipe dreams

Given $w \in S_n$, we can consider the so-called **Rothe diagram**.

Definition by example: let

$$w = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 5 & 2 & 1 & 6 & 4 \end{pmatrix}$$

We place an X in all the cells, then shade in all the squares which are not either below or to the right of any X. This gives the **Rothe diagram**; the **dominant part** is the connected component in the upper-left (which may be empty if $w(1) = 1$); this will be a Young diagram.



Exercise 36.1. The Rothe diagram coincides with the dominant part (i.e. no other squares) if and only if the permutation is 132-avoiding.

Pick any outer corner (i.e. x adjacent to dominant part), denote it (i, j), so that $w(i) = w(j)$.

Claim 36.2. We have

$$(x_i - y_j)\mathfrak{S}_w(x, y) = \sum_{\substack{w' \gtrdot w \\ w'(i) \neq j}} \mathfrak{S}_{w'}(x, y).$$

Here the covering $\gtrdot$ is in the strong Bruhat order; so it's equivalent to $w'(i) > j$.

Claim 36.3. Given the Rothe diagram for w, all pipe dreams for w have;

- a cross inside the dominant part, and
- a non-crossing in each dominant corner.

By changing one of the non-crossings to a crossing, we get also the recursion

$$(x_i - y_j)\mathfrak{S}_w = \sum_{\substack{w' \gtrdot w \\ w'(i)=j}} \mathfrak{S}_{w'}.$$

Also related: https://arxiv.org/abs/1704.00851.

§36.2 Alcove path model, continued

We now continue where we left off last lecture.

Let $K(G/B) \simeq \mathbb{Z}[P]/J$ as before, with $\mathbb{Z}$-basis $[\mathcal{O}_w]$; we gave last time a formula for multiplication is the algebra. We restate it now:

Theorem 36.4 (K-Chevalley formula)
Pick any λ-chain $(\beta_1, \dots, \beta_\ell)$. We have

$$e^\lambda \cdot [\mathcal{O}_w] = (1 + B_{\beta_\ell}) \left(1 + B_{\beta_{\ell-1}}\right) \dots (1 + B_{\beta_1}) ([\mathcal{O}_w])$$

where we have defined the operator

$$B_\alpha : [\mathcal{O}_w] \mapsto \begin{cases} [\mathcal{O}_{ws_\alpha}] & \text{if } \ell(ws_\alpha) = \ell(w) - 1 \\ 0 & \text{otherwise.} \end{cases}$$

for $\alpha \in \Phi^+$, with $B_{-\alpha} = -B_\alpha$ for $-\alpha \in \Phi^-$.

Here e^λ plays the role of x_i while $[\mathcal{O}_w]$ plays the role of Schubert polynomial.

In type A_{n-1}, let Z be the set of center of mass of all the alcoves, and embed it in the coordinate system as before. Then

$$nZ = \{\mu = (\mu_1, \dots, \mu_n) \in \mathbb{Z}^n \mid \mu_1, \dots, \mu_n \text{ distinct mod } n, \mu_1 + \dots + \mu_n = 1 + \dots + n\}$$

is the set of affine permutations, with n serving the role of the Coxeter number h. Under this bijection, the fundamental alcove is

$$\rho = (n, n-1, \dots, 1).$$

The adjacency is as follows: for alcoves μ, if $\mu_i + 1 \equiv \mu_j \pmod n$ then and $\mu + \alpha_{ij}$ are adjacent, where $\alpha_{ij} = e_i - e_j$ (with edge directed). (The first condition is necessary since otherwise

$$\mu + \alpha_{ij} = (\dots, \mu_i + 1, \dots, \mu_j - 1, \dots)$$

would not have distinct residues modulo n. The implicit claim is that given two adjacent alcoves separated by α, the centers differ by the vector α/h).

Example for $n = 4$:

$$\rho = (4, 3, 2, 1) \xrightarrow{-\alpha_{23}} (4, 2, 3, 1) \xrightarrow{-\alpha_{13}} (3, 2, 4, 1) \xrightarrow{-\alpha_{24}} (3, 1, 4, 2) \xrightarrow{-\alpha_{14}} (2, 1, 4, 3).$$

If $\omega_2 = (1, 1, 0, 0) \pmod{(1, 1, 1, 1)} = (\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$, then $\rho - n\omega_2 = (2, 1, 4, 3)$, so an ω_2-chain is

$$(\alpha_{23}, \alpha_{13}, \alpha_{24}, \alpha_{14}).$$

This gives us an ω_2-chain example. So, we can now apply the formula to

$$e^{\omega_2}[\mathcal{O}_{4321}] = \sum_{u \in W} n_u[\mathcal{O}_u]$$

where n_u is the number of decreasing chains from 4321 to u whose label path forms a subsequence of $(\alpha_{23}, \alpha_{13}, \alpha_{24}, \alpha_{14})$. One can compute the corresponding terms:

- $[\mathcal{O}_{4321}]$ from the empty sequence.
- $[\mathcal{O}_{4231}]$ from α_{23}
- $[\mathcal{O}_{3241}] + [\mathcal{O}_{4132}]$ from $\alpha_{23}\alpha_{13}$ and $\alpha_{23}\alpha_{24}$
- $[\mathcal{O}_{3142}]$ from $\alpha_{23}\alpha_{13}\alpha_{24}$.
- $[\mathcal{O}_{2143}]$ from $\alpha_{23}\alpha_{13}\alpha_{24}\alpha_{14}$.

Monk's formula corresponds to the "linear part of this" i.e. the length-one chain $[\mathcal{O}_{4231}]$ in the above.

§37 May 15, 2017

To be added.

§38 May 17, 2017

In this lecture we'll briefly mention a lot of the topics that we didn't get to.

§38.1 Weyl Characters

In last lecture we discussed the characters

$$\mathrm{ch}(V_\lambda) = \sum_\mu m_\lambda(\mu) e^\lambda.$$

(In type A_{n-1} these are Kostant numbers.) Related topics:

- Weyl's character formula
- Demazure character formula
- Littlemenn path model
- Berenstein-Zelevinsky polytopes
- Alcove path model (Gaussat-Littlemann galleries)

The ultimate goal is to understand the irreducible representations V_λ of $U_q(\mathfrak{g})$. The models we have do a very good job of giving the characters $\mathrm{ch}(V_\lambda)$, but this doesn't tell us the whole V_λ. Related keywords: Lustzsig's canonical basis, Koshiwara's crystal basis.

§38.2 Kostant partition function

For $\lambda \in Q$, we let $K(\lambda)$ denote the number of ways to express λ as a positive linear combination of positive roots. We thus have

$$\prod_{\alpha \in \Phi^+} \frac{1}{1 - e^\alpha} = \sum_{\lambda \in Q^+} K(\lambda) e^\lambda.$$

For example, in type A_{n-1}, we end up with

$$\prod_{1 \le i < j \le n} \left(1 - \frac{x_i}{x_j} \right)^{-1} = \sum_{\substack{(a_1, \dots, a_n) \in \mathbb{Z}^n \\ a_1 + \dots + a_k \ge 0 \ \forall k \\ a_1 + \dots + a_n = 0}} K(a_1, \dots, a_n) x_1^{a_1} \dots x_n^{a_n}.$$

Example 38.1

We compute $K(2, -1, -1)$ in the case A_2. The ways to write $(2, 1, -1)$ as the sum are:

$$\begin{aligned}(2, -1, -1) &= (1, -1, 0) + (1, 0, -1) \\ &= 2(1, -1, 0) + (0, 1, -1)\end{aligned}$$

So $K(2, -1, -1) = 2$.

Example 38.2

We compute $K(1,2,-3)$ in the case A_2.

$$\begin{aligned}(1,2,-3) &= (1,-1,0)+3(0,1,-1)\\ &= (1,0,-1)+2(0,1,-1).\end{aligned}$$

So $K(1,2,-3)=2$.

Theorem 38.3 (Conjectured by Chan-Robbins-Yuen, proved in 1998 by D. Ziellberger)

In type A_{n-1}, we have

$$K\left(1,2,3,\dots,n,-\binom{n+1}{2}\right)=C_1C_2\dots C_n$$

where $C_k=\frac{1}{k+1}\binom{2k}{k}$ is the Catalan number.

Example 38.4

This implies $K(1,2,3,-6)=1\cdot 2\cdot 5=10$. One can visualize this as follows: we can place the coefficeints of the root in the triangular lattice

$$\begin{bmatrix} a & b & c \\ & d & e \\ & & f \end{bmatrix}$$

Then we require that

$$\begin{aligned}a+b+c&=1\\ d+e-a&=2\\ f-(d+b)&=3.\end{aligned}$$

Remark 38.5. It turns out one can express $m_\lambda(\mu)$ as an alternating sum of $K(\widetilde{\mu})$.

It turns out this result is a special case of Morris identity, proved analytically: the one-page proof is `https://arxiv.org/abs/math/9811108v2`. However, apparently no combinatorial proof is known.