18.217 Lecture Notes

Taught by Alex Postnikov

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This is MIT's graduate 18.217, instructed by Alex Postnikov. The formal name for this class is "Combinatorial Theory". All errors are my responsibility.

The permanent URL for this document is http://web.evanchen.cc/coursework.html, along with all my other course notes.

Contents

§1 September 5

No cluster algebras for first lecture, we discuss a predecessor first.

§1.1 Frieze patterns

We start with an example of a frieze pattern of order 5, which consists of four infinite rows.

$$\begin{array}{cccccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
 & 1 & 2 & 2 & 1 & 3 & 1 & 2 & 2 & 1 & 3 \\
 & & 1 & 3 & 1 & 2 & 2 & 1 & 3 & 1 & 2 & 1 \\
 & & & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{array}$$

Definition 1.1. A **frieze pattern** of order n is an arrangement of $n-1$ rows with indents as above such that:

(1) There are $n-1$ rows and the first and last rows are filled with all 1's.

(2) In the diamond

$$\begin{array}{ccc} & b & \\ a & & d \\ & c & \end{array}$$

we always have

$$ad - bc = 1.$$

(3) All entries are positive real numbers.

Theorem 1.2 (Coxeter)
Rows of frieze patterns of order n are periodic with period n.

In fact there is a stronger result that implies this property.

Definition 1.3. A **glide reflection** consists of a reflection of a horizontal axis plus a vertical translation to the right.

PC: https://mathbitsnotebook.com/Geometry/Transformations/glide2.jpg

Theorem 1.4

Frieze patterns are invariant under gliding reflections where the translation is by $n/2$.

Thus one can find a fundamental domain (red triangle in later picture).

Example 1.5

Consider $n = 4$. The Frieze pattern then must have central row x, $2/x$, and so on (alternating). Thus the set of Frieze patterns of order 4 is isomorphic to $\mathbb{R}_{>0}$, and the only possibility giving integer Frieze patterns is $x \in \{1, 2\}$.

§1.2 Clusters

Moreover, consider a lattice path $x_1, \dots, x_{n-3}$ joining 1 to 1.

Proposition 1.6

The rest of the Frieze pattern is uniquely determined by any positive real numbers $x_1, \dots, x_{n-3}$. Moreover, all other entries are uniquely expressed as subtraction-free rational formulas with monomial denominators (Laurent polynomials).

$$
\begin{array}{ccccccccccccccccccccc}
1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 \\
 & & & & & & & & x_1 & & & & & & & & & & & & \\
 & & & & & & & & & x_2 & & & & & & & & & & & \\
 & & & & & & & & & & x_3 & & & & & & & & & & \\
 & & & & & & & & & x_4 & & & & & & & & & & & \\
 & & & & & & & & x_5 & & \frac{x_4+1}{x_5} & & & & & & & & & & \\
 & & & & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1
\end{array}
$$

In general, the idea is that we will have some sort of structure where we want to choose a "cluster" that uniquely determines the rest of the picture. These clusters will be related by "mutations"; for example replacing the red x_5 by the blue $\frac{x_4-1}{x_5}$ as the independent variable.

Let's go back to the example with $n = 5$.

$$
\begin{array}{ccccccccccccccccccccc}
1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 \\
 & x & & \frac{y+1}{x} & & \frac{x+1}{y} & & y & & & & & & & & & & & \\
 & & y & & \frac{x+y+1}{xy} & & x & & & & & & & & & & & & \\
 & & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1
\end{array}
$$

Nicely enough:

Theorem 1.7

For any n, these expressions are in fact *Laurent polynomials*!

Let's then turn our attention to integer polynomials. For the specific example, there are five pairs (x, y) such that x, y, $\frac{x+1}{y}$, $\frac{y+1}{x}$, $\frac{x+y+1}{xy}$ are all integers, namely

$$(x, y) \in \{(1,1), (1,2), (2,1), (3,2), (2,3)\}.$$

And in fact:

Theorem 1.8

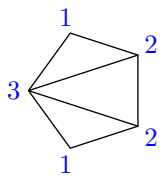
For Frieze patterns of order n, there are exactly

$$C_{n-2} = \frac{1}{n-2}\binom{2n-4}{n-2}$$

integer Frieze patterns of order n.

Moreover there is an explicit bijection between these integer Frieze patterns with traiangulations of an n-gon.

The bijection is given as follows: given a triangulation of an n-gon, count the number of triangles touching each vertex, read off the labels clockwise, and use the numbers as the entries of the first Frieze pattern, periodic every n. Then use the diamond rule to determine the rest of the entries.



1	1	1	1	1	1	1	1	1	1	
1	2	2	1	3	1	2	2	1	3	
1	3	1	2	2	1	3	1	2	2	
1	1	1	1	1	1	1	1	1	1	1

In fact, in this bijection we can even determine the entries of the diagonal under each entry. Label the vertices of n-gon by $\{1, \dots, n\}$ and fix the triangulation; right down the first row.

The rule for $p_{i,j}$ is as follows:

1. Label vertex i by 0.
2. Label all neighbors of i by 1.
3. Henceforth, if a triangle has two labels a and b and the third vertex is empty, write $a + b$ at the empty vertex.

Then the label of j is the entry p_{ij}.

§2 September 7

§2.1 Frieze patterns continued

Recall that once we determine a lattice path, the rest of the frieze pattern is uniquely determined. If we believe the claim that the resulting polynomials are subtraction-free with Laurent denominators, then it follows that one valid choice is to set all the numbers equal to 1.

$$\begin{array}{cccccccccccccccccccccccc}
1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & \\
 & & & & & & & & & & 1 & & & & & & & & & & & & \\
 & & & & & & & & & & & 1 & & & & & & & & & & & \\
 & & & & & & & & & & & & 1 & & & & & & & & & & \\
 & & & & & & & & & & & 1 & & & & & & & & & & & \\
 & & & & & & & & & & 1 & & & & & & & & & & & & \\
 & & & & & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1
\end{array}$$

(We don't consider the 1's in the topmost and bottom row to be part of the lattice path anymore.) There are 2^{n-4} ways to go left/right then, and n places to start the lattice path; modding out by the glide reflection gives

$$\frac{2^{n-4} \cdot n}{2} = 2^{n-5} n$$

frieze patterns obtained from setting a lattice path equal to 1.

Since $2^{n-5} n < C_{n-2} = \frac{1}{n+1}\binom{2n}{n}$, it follows there are some Frieze patterns that don't arise this way. We'll give these on later in the lecture.

§2.2 Notation for Frieze patterns

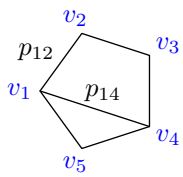
Suppose we label the entries of the Frieze patterns as:

$$\begin{array}{ccccc}
\cdots & p_{12} & p_{23} & p_{34} & \cdots \\
 & p_{13} & p_{24} & & \\
 & & p_{14} & &
\end{array}$$

Extending modulo n, we can then view a Frieze pattern as $(p_{i,j})_{i,j \in \mathbb{Z}}$ satisfying

1. $\det \begin{bmatrix} p_{i,j} & p_{i,j+1} \\ p_{i+1,j} & p_{i+1,j+1} \end{bmatrix} = 1$.
2. $p_{i,i+1} = p_{i,i+(n-1)} = 1$.
3. $p_{i+n,j} = p_{i,j+n} = p_{i,j}$ (we take indices modulo n).
4. $p_{i,j} > 0$ if $i \not\equiv j \pmod{n}$, and $p_{i,j} = 0$ if $i \equiv j \pmod{n}$.
5. $p_{i,j} = p_{j,i}$ (glide reflection).

Thus each $p_{i,j}$ depends only on the unordered pair $\{i, j\}$ modulo in $\mathbb{Z}/n\mathbb{Z}$. In other words, $p_{i,j}$ corresponds to the diagonals and sides of a regular n-gon.



The rule that we gave at last lecture then goes verbatim; given a triangulation T we get $(p_{i,j})$ as follows. For every i:

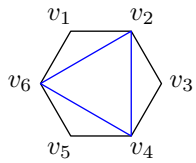
1. Label vertex i by 0.
2. Label all neighbors of i by 1.
3. Henceforth, if a triangle has two labels a and b and the third vertex is empty, write $a+b$ at the empty vertex.

Then the label of j is the entry p_{ij}.

Going backwards is even simpler: $(p_{i,j})$ gives a triangulation T by drawing an edge between $\{i,j\}$ if $p_{i,j}=1$.

§2.3 Missing frieze patterns

Let's return to the question at the beginning of the lecture.



Applying the rule gives the corresponding frieze pattern.

$$\begin{array}{ccccccccccc}
1 & & 1 & & 1 & & 1 & & & & \\
& 3 & & 1 & & 3 & & 1 & & & \\
& & 2 & & 2 & & 2 & & 2 & & \\
& & & 3 & & 1 & & 3 & & 1 & \\
& & & & 1 & & 1 & & 1 & & 1
\end{array}$$

This is an example where there is no lattice path of 1's.

In fact, we can now identify the "missing" patterns:

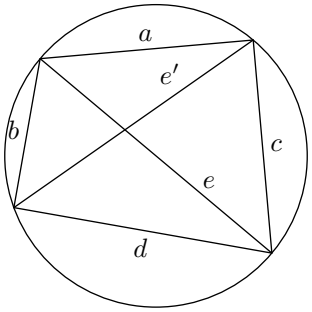
Proposition 2.1

Triangulations which correspond to lattice paths of 1's are those with no internal triangles.

Hence the first $n=6$ example.

§2.4 Ptolemy's theorem

Consider the following picture:



From elementary geometry we have the following result.

Theorem 2.2 (Ptolemy's theorem)
We have
$$ee' = ad + bc.$$

This gives a silly way to remember the following definition.

Definition 2.3. The **Ptolemy algebra** has variables

$$p_{i,j} \qquad 1 \le i < j \le n$$

subject to the relations that whenever $i < j < k < \ell$,

$$p_{i,k} \cdot p_{j,\ell} = p_{i,j} + p_{k,\ell} + p_{i,\ell} + p_{j,k}$$

This has a more modern name: "the coordinate ring of the Grassmannian $\mathrm{Gr}(2, n)$", with $p_{i,j}$ bearing the name "Plücker coordinates".

The choice of reused notation is deliberate:

Theorem 2.4
Frieze patterns of order n correspond to arrays $(p_{i,j})_{1 \le i < j \le n}$ which satisfy Ptolemy's relations and $p_{i,j} > 0$, $p_{12} = p_{23} = \cdots = p_{(n-1)n} = p1n = 1$.

Thus the diamond relations from frieze patterns are actually equivalent to Ptolemy's relations. In particular, the diamond relation is a particular instance of the Ptolemy relations:

$$p_{i,j}p_{i+1,j+1} = p_{i,i+1}p_{j,j+1} + p_{i,j+1}p_{i+1,j} = 1 + p_{i,j+1}p_{i+1,j}.$$

§3 September 10, 2018

§3.1 The coordinate ring of the Grassmannian

We will show that the coordinate ring of the Grassmannian $\mathrm{Gr}(2,n)$ is given by

$$P_n = \mathbb{F}[x_{ij}] / \left\langle x_{ik}x_{j\ell} = x_{ij}x_{k\ell} + x_{i\ell} + x_{jk} \mid 1 \le i < j < k < \ell < n \right\rangle.$$

In what follows we will take the indices modulo n, and set $x_{ii} = 0$ for every i. (Here, we use x_{ij} rather than p_{ij} to emphasize that these are indeterminates.)

Now, let

$$p_{ij} = \det \begin{bmatrix} a_i & a_j \\ b_i & b_j \end{bmatrix}$$

for $i < j$ (this is anti-symmetric, unlike x_{ij} which are symmetric). To check that the p_{ij} satisfy the Ptolemy relations, one can assume WLOG that $a_i = 1$ (by replacing each (a_i, b_i) with $(1, b_i/a_i)$), at which point

$$(b_i - b_k)(b_j - b_\ell) = (b_i - b_j)(b_k - b_\ell) + (b_i - b_\ell)(b_j - b_k)$$

as desired.

Homework:

Problem 3.1. Show that these are the only relations.

§3.2 A proof of the (Euclidean!) Ptolemy's theorem

Let $A_1A_2A_3A_4$ be a cyclic quadrilateral. We will encode by picking a 4×2 matrix

$$\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{bmatrix}$$

and then letting

$$z_i = a_i + b_i\sqrt{-1}.$$

Then, the vertex A_i corresponds to

$$A_i = \frac{z_i^2}{|z_i^2} = \frac{z_i}{\overline{z_i}} \in \mathbb{C}$$

on the unit circle.

$$\begin{aligned} |A_i - A_j| &= \left| \frac{z_i}{\overline{z_i}} - \frac{z_j}{\overline{z_j}} \right| \\ &= \left| \frac{z_i\overline{z_j} - z_j\overline{z_i}}{\overline{z_i}\overline{z_j}} \right| \\ &= \frac{2}{|z_i||z_j|} \left| \det \begin{bmatrix} a_i & a_j \\ b_i & b_j \end{bmatrix} \right|. \end{aligned}$$

So Ptolemy's theorem from Euclidean geometry is equivalent to the Plücker relations.

§3.3 Assigning values to triangluations

We retain the relations and variables for x_{ij}. Let T be a triangulation of an n-gon and let

$$X_T = \{x_{ij} \mid (i,j) \text{ is an edge of } Tj\}.$$

This is called a *cluster*; it includes x_{ij} when $j = i-1$ due to the sides.

Proposition 3.2

Fix the triangulation T.

1. Any x_{ij} is expressed in terms of variables from X_T by subtraction-free rational expressions.
2. The x_{ij} are linearly independent.
3. If we assign a positive value to every x_{ij} in X_T, then there is a unique assignment of values to all x_{ij} which satisfies all the Ptolemy relations.

In fact we will later see that these are not only rational, but *Laurent polynomials*.

Proof.
1. Local swapping argument: given any quadrilateral in the triangulation, one can flip its diagonal and use Ptolemy theorem.
2. We have $\#X_T = n-3+n = 2n-3$, but it is known that $\dim \operatorname{Gr}(2,n) = 2(n-2) = 2n-4$ as a subspace of projective space. Once we add in the extra degree of freedom for rescaling, we account for the discrepancy of 1.
3. Uniqueness follows from (1). Existence left as homework exercise. (Hint: need to check only for a specific triangulation.)

□

We claim now that this proposition explains all the results on Frieze algebras that we had before. We will prove Theorem 1.2 and Theorem 1.4 in the following way.

Theorem 3.3

Frieze patterns are n-periodic and have glide symmetry; they are exactly the arrays (p_{ij}) such that

(1) All Plücker relations are satisfied,

(2) $p_{ij} > 0$,

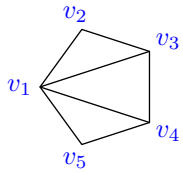
(3) $p_{i,i+1} = p_{i,i+n-1} = 1$.

Indeed the diamond relations are a special case of Plücker relations as we stated before. On the other hand, it is not *exactly* true that the diamond relations imply all the Plücker relations: one can consider the values $p_{ij} = 1$, $p_{k\ell} = 1$, with all other values equal to 0. In that case the general Plücker relations fail.

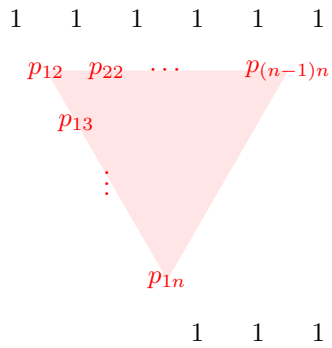
The remedy is to require all values positive.

Claim 3.4 — If $x_{ij} > 0$, then the diamond relations imply all Plücker relations. In other words, Frieze patterns satisfy the Plücker relations.

Proof. It suffices to check it for the special star triangulation T with edges $(1, i)$ for every i.



Repeatedly throw the diamond relations in order to show there is *at most* one way to fill in the entries.



□

Finally, we now officially state the Laurent phenomenon:

Proposition 3.5

In Proposition 3.2 the expressions are all Laurent polynomials

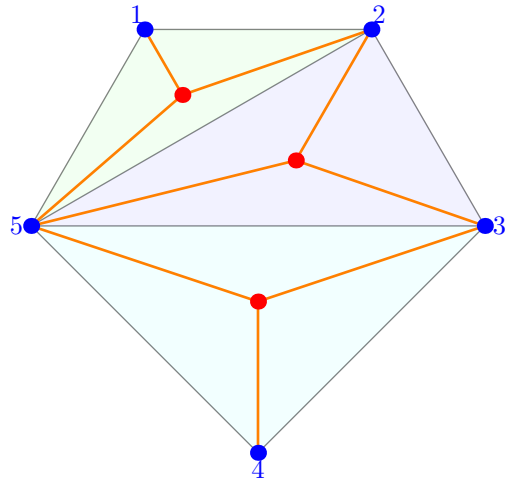
Proof. By induction, using the recursion that was given when we mapped T to integer Frieze patterns. □

§3.4 An explicit description of the formulas

We give an explicit (i.e. not recursive) description of the Laurent polynomials which are arising.

Given a triangulation T, construct a bi-colored graph G such that we have blue vertices $\{1, \dots, n\}$ corresponding to the vertices of the polygon, and red vertices corresponding to the triangles in the triangulation; connect each red vertex to the three blue vertices making up the triangle. (The graph is bipartite for now, but later on they may not be.)

Example below.



This graph obviously has no perfect matching since the number of vertices doesn't match, but:

Definition 3.6. An *almost perfect matching* is one that matches all but two of the vertices.

Then the number of such matchings will actually correspond to p_{ij}. Next lecture we will also give a weighting on such matchings.

§4 September 12, 2018

§4.1 Infinite Frieze patterns

We now consider Frieze patterns with $n \to \infty$, at which point it doesn't make sense to talk about periodicity anymore. We will take a doubly infinite matrix

$$p_{ij} = a_{i+j}$$

for $i, j \in \mathbb{Z}$, so the antidiagonals are constant. Thus the diamond relation becomes the condition that

$$a_{i+2}a_i = a_{i+1}^2 + 1.$$

Example 4.1 (Fibonacci)
If $a_0 = a_1 = 1$, then the terms a_n for $n \geq 0$ are

$$1,\ 1,\ 2,\ 5,\ 13,\ 34,\ \dots.$$

These are alternating Fibonacci numbers: we have $a_n = F_{2n-1}$.

In particular, it turns out that the terms are all integers! Since there is an explicit formula for Fibonacci numbers, this lets us check it directly with computation (exercise).

In general, suppose $a_0 = x$ and $a_1 = y$, and then generate the next terms by the same recursion:

$$x,\ y,\ \frac{y^2+1}{x},\ \frac{(\frac{y^2+1}{x})^2+1}{y},\ \dots$$

Exercise 4.2. Prove that these polynomials are all Laurent polynomials and give a combinatorial interpretation of these polynomials.

Interestingly, there are *other* sequences with these type of properties. For example:

Definition 4.3. The **Somos-4 sequence** is the one satisfying

$$a_{n+4}a_n = a_{n+3}a_{n+1} + a_{n+2}^2.$$

The **Somos-5 sequence** is the one satisfying

$$a_{n+5}a_n = a_{n+4}a_{n+1} + a_{n+3}a_{n+2}.$$

For $k \geq 4$ we can define **Somos-k sequence** analogously.

In general, Somos-k sequences satisfy the same Laurent phenomenon for $k \in \{4, 5, 6, 7\}$.

Exercise 4.4. Prove the same properties for Somos-4 and Somos-5.

§4.2 Laurent polynomial formula

Recall last time we were considering P_n, the algebra $\mathbb{R}[x_{ij}]$ modulo Ptolemy relations. We will

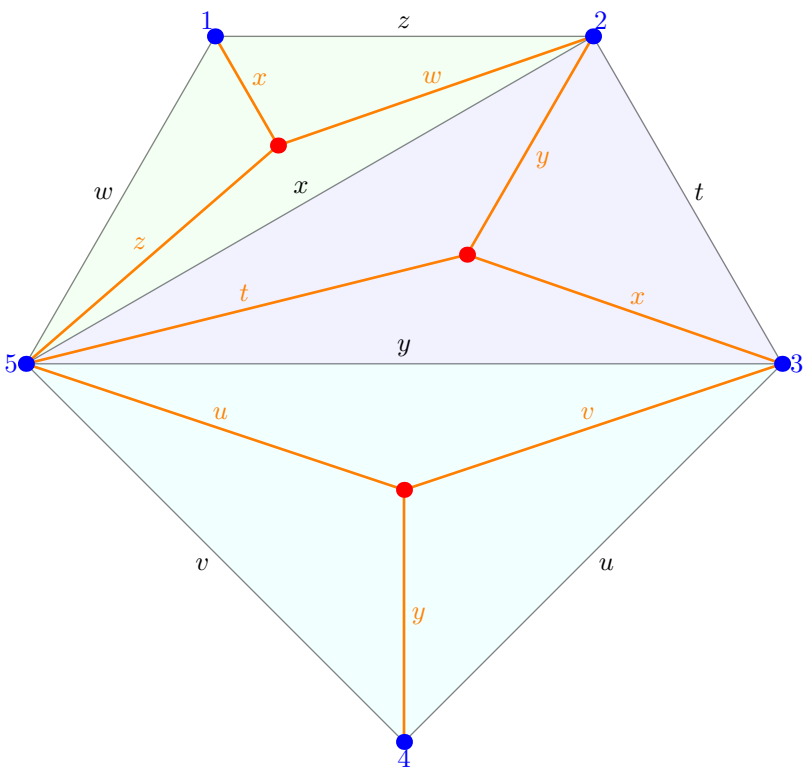
> **Remark 4.5** (Historical)**.** Fomin-Zelevinsky (2001) proved the Laurent phenomenon in the more general context of cluster algebras, which we'll prove later. They conjectured Laurent positivity. Carroll-Price (2002), among others includin Propp and Schiffler, gave positive formulas.

(Note that positive numerator and denominator isn't enough: $\frac{x^3+y^3}{x+y} = x^2 - xy + y^2$. The claim is this never happens in cluster algebra settings.)

§4.3 Explicit interpretation in G_T

Given a triangulation T, we define the graph G_T with red/blue vertices as in the end of lecture 3 (example again below). We will let $\mathcal{M}_{ij}$ denote the set of almost-perfect matchings of G_T that cover all vertices except the blue vertices i and j.

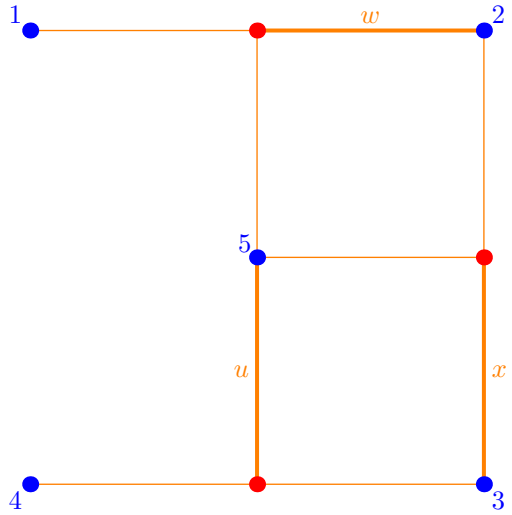
We will also label the edges of the triangulation with variables in the following way: in a triangle with edges x, y, z, the cevian away not touching the edge z will be labeled z, etc. Example below.



Definition 4.6. If M is an almost perfect matching, we define

$$\text{weight}(M) = \frac{\prod_{e \text{ edge of } M} \text{weight}(e)}{\prod_{(i',j') \text{ internal diagonal of } T} x_{i',j'}}.$$

An example of a matching $M \in \mathcal{M}_{1,4}$ is shown below, with $\text{weight}(M) = \frac{wxu}{xy}$. For this particular example, $|\mathcal{M}_{1,4}| = 3$.



Theorem 4.7

$$x_{ij} = \sum_{M \in \mathcal{M}_{ij}} \text{weight}(M).$$

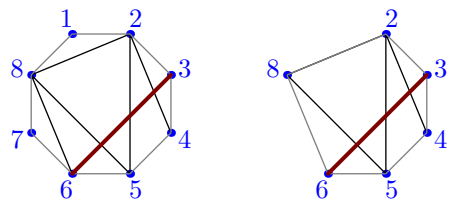
Let $\widetilde{x}_{ij}$ denote the right-hand side; we wish to show $\widetilde{x}_{ij}$.

We start with one observation.

Lemma 4.8

The quantity $\widetilde{x}_{ij}$ depends only on the part T' of the triangulation given by all triangles whose interiors intersect with (i, j).

Proof. An example of T and T' is given below, where $(i, j) = (3, 6)$ is highlighted in brown.



Then, it suffices to check that when we delete a single triangle from the triangulation. When we do this, we lose exactly one red and one blue vertex v (that is not $\{i, j\}$), so the number of matchings is still in obvious bijection, and one can check the weight(M) does not change under the bijection. □

We state two sub-claims.

Claim — If $x_{ij} \in X_T$, then $\widetilde{x}_{ij} = x_{ij}$.

Proof. Follows from lemma, since one can remove *all* triangles. □

Claim — Whenever $i < j < k < \ell$ we have

$$\widetilde{x}_{ik} = \left(\frac{x_{k\ell}}{x_{j\ell}}\right)\widetilde{x}_{ij} + \left(\frac{x_{jk}}{x_{j\ell}}\right)\widetilde{x}_{i\ell}.$$

Proof. Similar, to be finished next time. □

By repeating the second claim, it follows by induction (with the first claim as base case) that $\widetilde{x}_{ij} = x_{ij}$.

§5 September 14, 2018

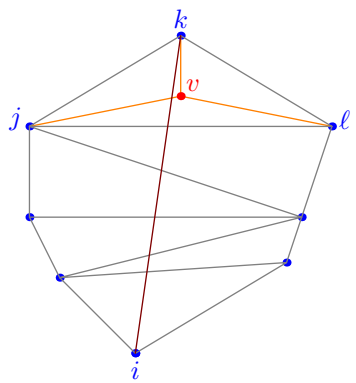
The graph G_T from before is a special case of a **plabic** graph, which means "planar bicolored graph", embedded in a disk (in the sense that the graph is inscribed in a disk).

§5.1 Finishing proof of weighting formula with plabic graphs

Let us prove the claim from last time that if (j, k, ℓ) is a triangle in T, then we have

$$\widetilde{x}_{ik} = \left(\frac{x_{k\ell}}{x_{j\ell}}\right)\widetilde{x}_{ij} + \left(\frac{x_{jk}}{x_{j\ell}}\right)\widetilde{x}_{i\ell}$$

We zoom in on the part T' of the triangulation T that intersects the diagonal joining i and k.



Let T'' be the triangulation T' with $\triangle jk\ell$ deleted. By considering the cases of how v is paired off (either to j or ℓ), we have

$$\begin{aligned}\mathcal{M}_{ik}\left(G_{T'}\right) &= \mathcal{M}_{ik}^{(1)} \sqcup \mathcal{M}_{ik}^{(2)} \\ &= \mathcal{M}_{ij}(G_{T''}) \sqcup \mathcal{M}_{i\ell}(G_{T''}).\end{aligned}$$

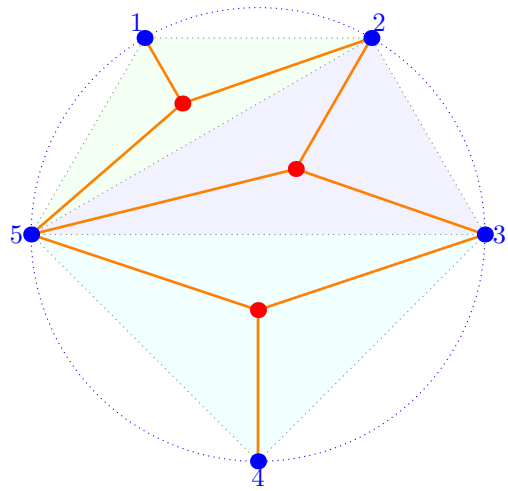
§5.2 Trailer for cluster algebras

We now start setting up for cluster algebras. To give a preview of what's coming up:

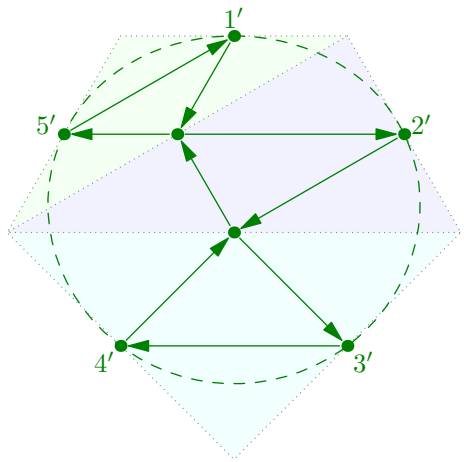
Old terminology	New terminology
triangulation of n-gon	cluster
flipping diagonals	mutation
variables for sides of n-gon	frozen variable
variables for diagonals of n-gon	mutable variable

§5.3 Quivers attached to triangulations

We had previously considered triangulations T. We then constructed a plabic graph G_T earlier. Here it is again (this time with the disk drawn):



We will also define for each triangulation T the quiver Q_T. (A **quiver** is officially a directed graph, but the word will carry different connotations.) We take the midpoint of every edge in the triangulation, and then we draw all the clockwise medial triangles (directed edges). The quiver of the same graph is given below, in deep green, with seven vertices and nine directed edges. We also consider it embedded in a disc, dashed below also in green (this will give us some counterclockwise faces).



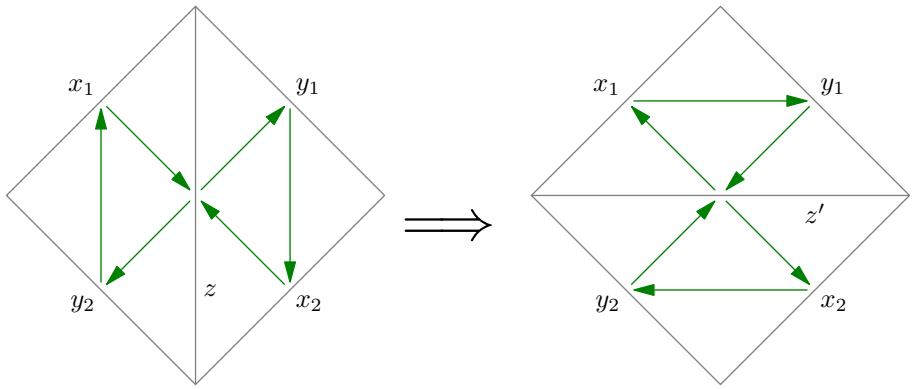
Hence, we have the following dictionary of our three viewpoints.

Triangulation T	Plabic graph G_T	Quiver Q_T
vertices	blue vertices	counterclockwise faces
edges	faces	vertices
faces	red vertices	clockwise faces

Depending on which viewpoint we go, this gives us various generalizations of P_n.

- P_n can be generalized to cluster algebras for trianglutade surfaces, see Fomin-Shaprio-Thusdon 2007 and Musiker-Schiffler-Williams 2009.
- It can also be generalized to positroid cluster algebras, $\mathrm{Gr}^{\geq}(k,n)$. See Postinkov 2006 or Oh-Postinkov-Speyer 2011.
- Cluster algebra for general quivers generalize *both* of these; see Fomin-Zelevinsky 2001.

Let's examine the Ptolemy relation in terms of the quiver.



In the Ptolemy case, we have $z' = \frac{x_1x_2+y_1y_2}{z}$. We have mutated both the quiver and the variables.

In the general situation, if we have edges $x_1 \to z, \dots, x_k \to z$, and $z \to y_1, \dots, z \to y_\ell$, the mutations consists of replacing z by

$$z' = \frac{x_1x_2\dots x_k + y_1\dots y_\ell}{z}$$

and flipping the arrows touching z, and toggling some other arrows between the x's and y's (to be described next lecture).

§6 September 17, 2018

§6.1 Quivers, continued

Recall the construction of Q_T from last lecture. We now define a quiver officially.

Definition 6.1. A **quiver** is a directed graph with no loops or 2-cycles. Its vertex set will be partitioned into two parts

$$V = V_{\text{int}} \sqcup V_{\text{bound}}$$

for **interior vertices** and **boundary vertices** (also called "frozen" vertices).

Each vertex v of a quiver graph Q will correspond naturally to algebraically independent variables x_v, which will be called *mutable* if v is an internal vertex and *frozen* if v is a boundary vertex. We denote $X = (x_v)_{v \in V}$ the collection of variables, which we call an **initial cluster**.

Definition 6.2. Let $v \in V_{\text{int}}$ be an internal vertex of a quiver (Q, X). The **mutation operator** μ_v to get a quiver (Q', X'), in the following way.

Suppose the edges adjacent to v in Q are

- $a_1 \to v$, $a_2 \to v$, $\dots$, $a_k \to v$, and
- $v \to b_1$, $v \to b_2$, $\dots$, $v \to b_\ell$

Then variable x_v is replaced by the variable

$$x_{v'} = \frac{x_{a_1} \dots x_{a_k} + x_{b_1} \dots x_{b_\ell}}{x_v}.$$

Next, the quiver Q is changed in the following way.

1. We flip the direction of all the arrows touching v.
2. If $a \to v \to b$ are edges in Q, then
   - if there is at least one edge $b \to a$, we delete one of those edges;
   - otherwise, we add an edge $a \to b$

   One can imagine this as adding in the edge $a \to b$ and then cancelling any "double edge" (2-cycle).

Lemma 6.3

μ_v is an involution, i.e. $\mu_v^2 = 1$.

Theorem 6.4 (Fomin-Zelevinsky)

After some number of mutations, the new set of variables $\widetilde{x}_v \in \widetilde{X}$ will be some Laurent polynomials in $x_v \in X$.

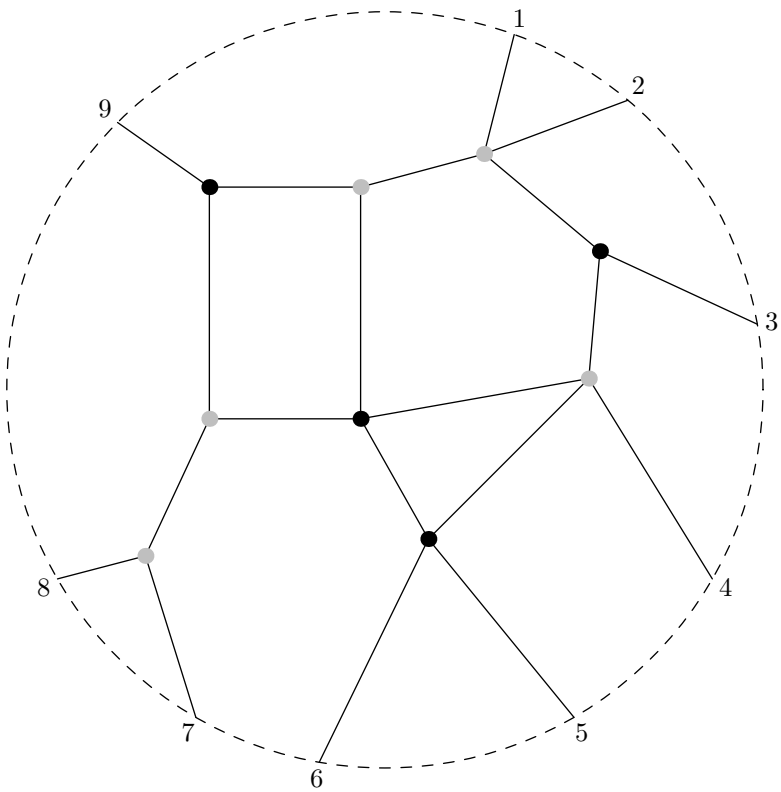
§6.2 Plabic graphs and quivers

WE now also officially define plabic graphs.

Definition 6.5. A **plabic graph** G is a planar graph which is

- drawn inside a disk,
- has n vertices which are connected to the boundary,
- and all internal vertices are colored in two colors (the coloring need not be proper).

Here is an example (where "white" is actually "medium grey"). The marks on the boundary are not considered vertices (in particular they are not colored).



Definition 6.6. A **bipartite plabic graph** is a plabic graph whose associated 2-coloring is a proper coloring (and in particular the graph is bipartite).

Definition 6.7. A **trivalent** (3-valent) plabic graph is one internal vertices are all 3-valent, except for possibly **lollipops**, which are degree 1 vertices attached to the boundary.

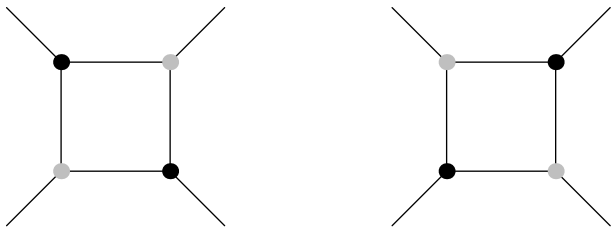
We consider the following two simple moves:

(M1) Contract an internal edge where both vertices are the same color, or un-contract such a vertex

(M2) Remove/add a vertex of any color in the middle of an edge.

We also have the following move:

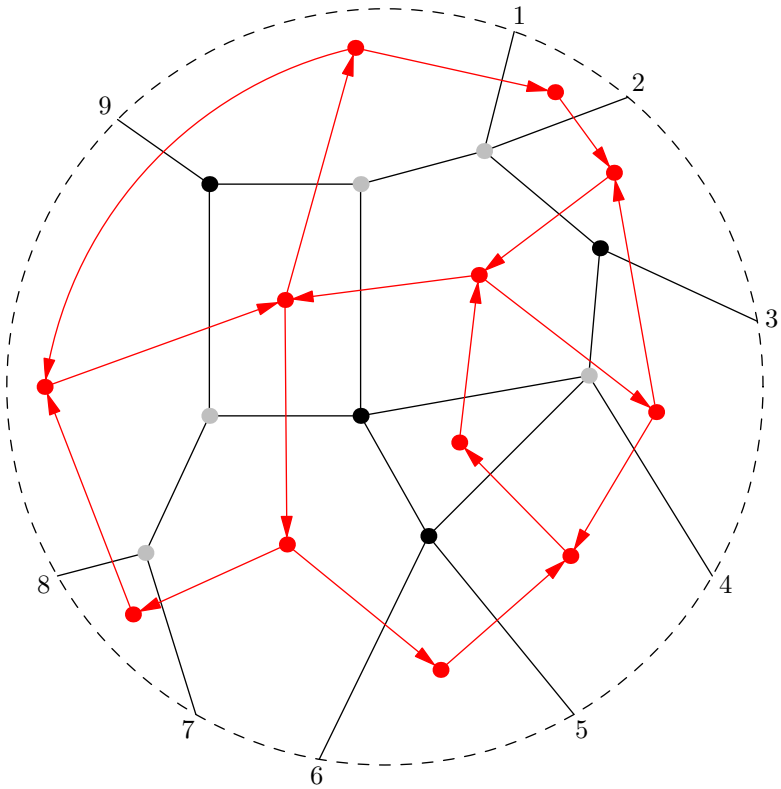
(Msq) Given a square with 3-valent vertices, we swap the colors on the four vertices.



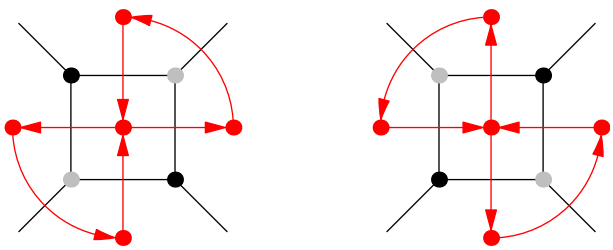
Definition 6.8. A **plabic tiling** is the planar dual of a bipartite plabic graph.

In general, a **plabic quiver** is given by a planar dual where whenever two faces are joined by a black-white edge, we draw an arrow such that the white vertex is on the right.

The plabic quiver of the earlier graph is shown below.



Observe that the square move corresponds to our Ptolemy move on the generated plabic quiver.



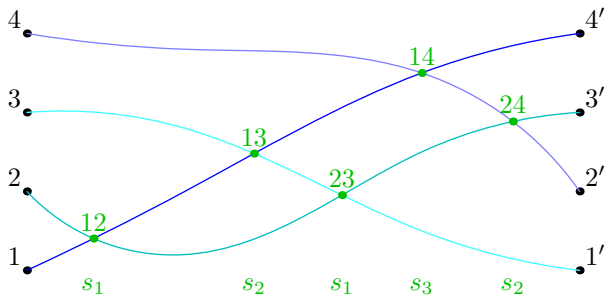
§7 September 19, 2018

§7.1 Wiring diagrams of permutations

Here is just an example when $n = 4$ (hence $r = 3$) Consider the following permutation

$$w = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4' & 3' & 1' & 2' \end{pmatrix}.$$

We can imagine this as a series of "wires", which have some intersections from left to right; this gives us a **wiring diagram**. We label each intersection of wires with s_h where $1 \le h \le r$ is the "height" of the intersection (which is the number of wires below that intersection point, plus one).



The intersections are height 1, 2, 1, 3, 2, and writing these from right to left we obtain

$$\boxed{w = s_2 s_3 s_1 s_2 s_1}.$$

Moreover, it happens that the intersection points correspond exactly to the inversions of the permutation (with the "reduced" condition corresponding to no two paths intersecting twice). As usual, we do this subject to the relations

$$\begin{aligned} s_i s_j &= s_j s_i \qquad |i-j| \ge 2 \\ s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1} \end{aligned}$$

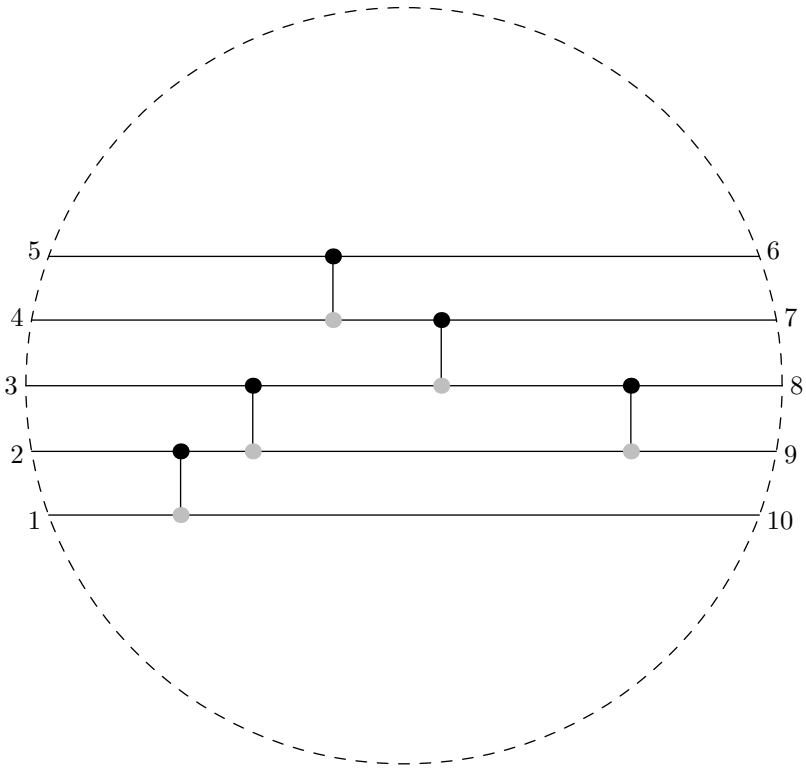
(and also $s_i^2 = 1$ but we won't use this since we will assume there are no double crossing anyways).

§7.2 Wiring diagrams to plabic graphs

We now show how to transform a wiring diagram into plabic graph. Any time we would have a wire intersection, we instead draw a **bridge**, which is a black/white edge. For example, given the permutation

$$w = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4' & 1' & 3' & 5' & 2' \end{pmatrix}$$

the wiring diagram becomes the following plabic graph:

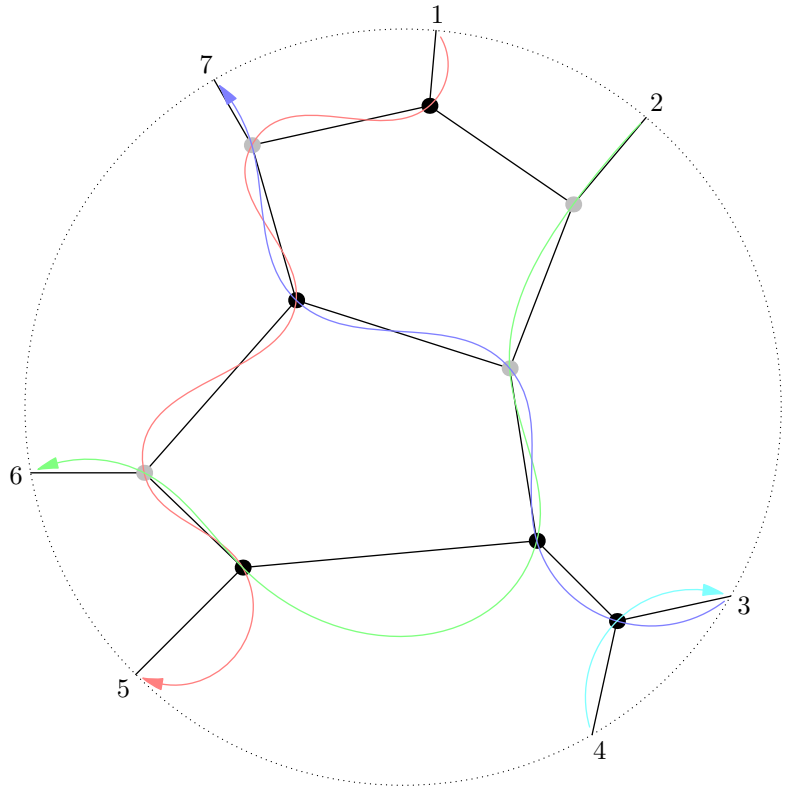


§7.3 Rules of the road

Given a trivalent graph G whose vertices are bicolored black and white, we can get a permutation as well by following so-called "rules of the road": starting from a boundary vertex,

- we turn left at every white vertex,
- and right at every black vertex.

Example on seven boundary points, with four of the strands drawn in through various colors:



This naturally gives a map from every point on the boundary to some other point on the boundary. We denote it by π.

Lemma 7.1

The map π which results is a permutation. Moreover, this permutation π is invariant under the various operations on plabic graphs.

Exercise 7.2. Prove it.

§7.4 Reduced plabic graphs

In the same way that we define a reduced wiring diagram, we want to be able to talk about reduced plabic graphs.

Definition 7.3. In a **reduced plabic graph**, the strands are required to satisfy the following properties:

- The strands are not self-intersecting.
- No closed strands.
- No bad double crossings: there should not be two strands which intersect at edges a and such that both strands point from a to b.

Theorem 7.4 (Postnikov)

Two reduced plabic graphs are move-equivalent if and only if they have the same strand permutation.

§8 September 24, 2018

Last time, we had quivers, which we then specialized to plabic quivers, and today we will restrict to so-called "split-chessboard quivers".

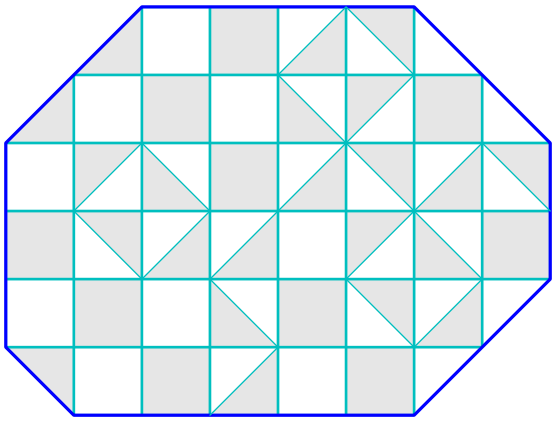
§8.1 Split chessboard pictures

We use the following notation for this class (invented):

Definition 8.1. An **R-simply connected region** in $\mathbb{R}^2$ is a region bounded by piecewise-linear curves with vertical/horizontal/diagonal edges, and all vertices in $\mathbb{Z}^2$.

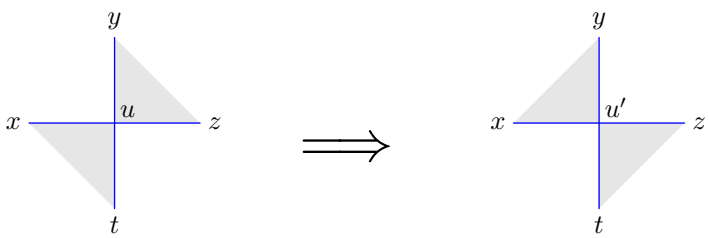
A **split-chessboard tiling** is a tiling by black and white unit squares and unit right triangles (with side lengths 1, 1, $\sqrt{2}$) in a chessboard fashion.

We assign a cluster variable to every lattice point $R \cap \mathbb{Z}^2$ which is mutable iff it is not on the perimeter.



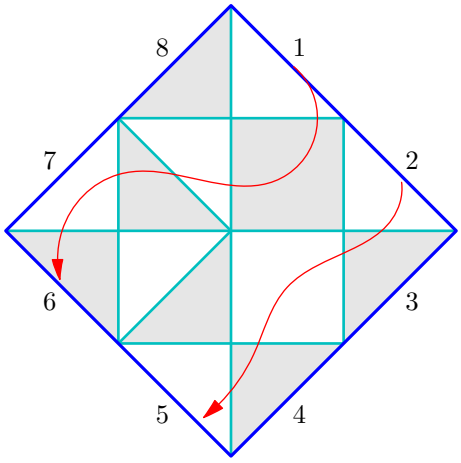
§8.2 Mutations of split chessboards

Now in a move, suppose we have an internal vertex u which is not adjacent to any diagonal; then we can transform as follows:



Here, among the four edges x, yz, zt, tx, we add an edge if it was not present already, and delete if it was present.

Finally, in a similar way we can define **strand permutations** as last time: starting from the edges, we turn left on white and right on black.



§8.3 Reduction of rhombus and domino tilings to chessboard tilings

§9 September 26, 2018

§9.1 Chessboard triangulations

We can think of the previous split-chessboard tilings in the following way.

Definition 9.1. A **chessboard triangulation** is a tiling of a region R by black and white unit right triangles (side lengths 1, 1, $\sqrt{2}$) such that any two *orthogonally* adjacent triangles have different colors.

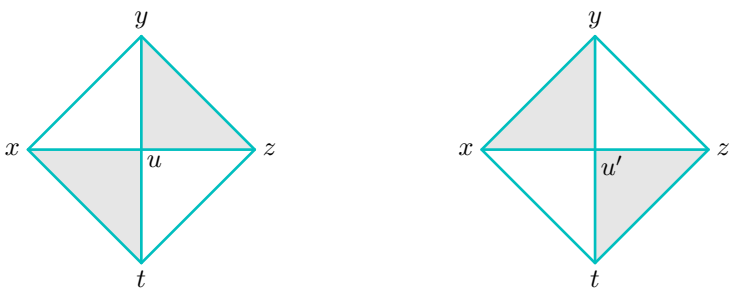
As usual, we consider such configurations up to certain moves. The following moves are permitted:

- Given a unit square formed by two triangles of the same color, we can change the diagonal.

- Suppose four triangles meet at a vertex u (and thus the colors alternate). Then we switch the colors of all four triangles while replacing the variable u with

$$u' = \frac{xz + yt}{u}$$

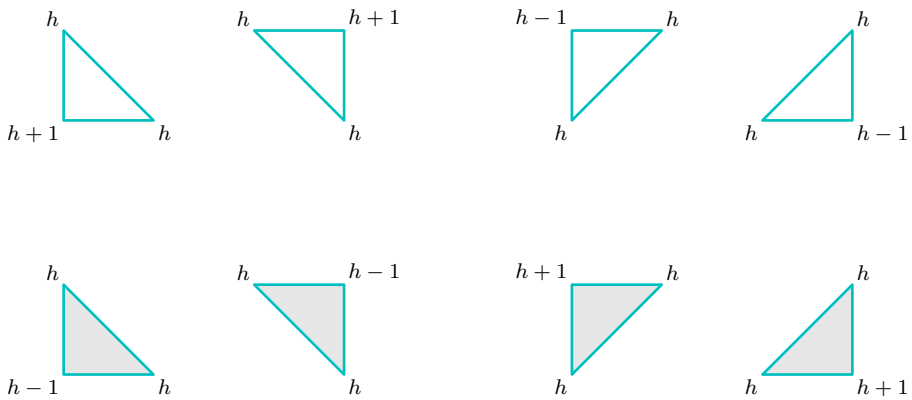
given by the Ptolemy relation.



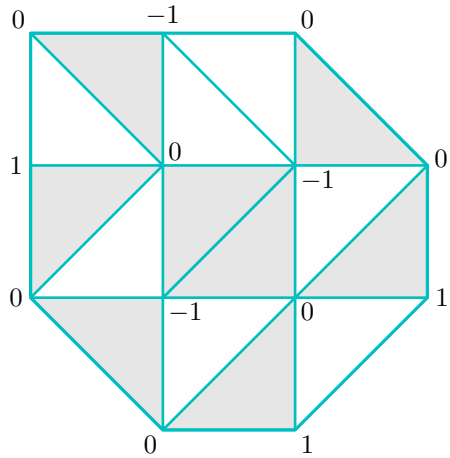
Given a chessboard triangulation, we can then define a **height function**

$$h \colon R \cap \mathbb{Z}^2 \to \mathbb{Z}$$

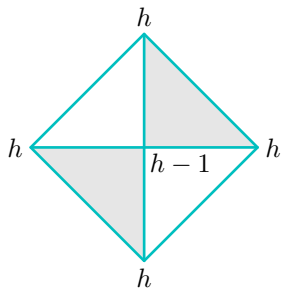
up to constant shifts, by dictating that its values along any triangle fall into the following shapes:

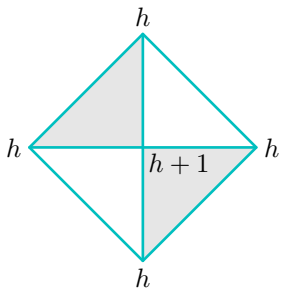


Here is an example.



You can visualize this with the height function h actually representing the height of the point off the paper. Then, e.g. the color-swap move corresponds to taking a half-octahedron and replacing it with the mirror image.





§9.2 Octahedron recurrence

We continue to push the 3D picture by embedding our tiled region into 3D according to height.

Define the lattice

$$L = \left\{(x, y, z) \in \mathbb{Z}^3 \mid x + y + z \equiv 0 \pmod 2\right\}.$$

As a $\mathbb{Z}$-module this is isomorphic to $\mathbb{Z}^3$ with basis given by $(0, 1, 1)$, $(1, 0, 1)$, $(1, 1, 0)$. It has natural layers given by

$$L_h \stackrel{\text{def}}{=} \{(x, y, z) \in L \mid z = h\}.$$

Definition 9.2. A **little octahedron** is an octahedron $ABCDEF$ such that $ABCD$ is a square in L_h of side length $\sqrt{2}$, E is in L_{h+1}, F is in L_{h-1}.

Then, the Ptolemy relation amounts to specifying $f \colon L \to \mathbb{R}_{>0}$ such that

$$f(E)f(F) = f(A)f(C) + f(B)f(D).$$

what is this?

Now fix a constant $k \in \mathbb{Z}$. We define

$$\begin{aligned} \widetilde{L} &\stackrel{\text{def}}{=} \left\{(a,b,c,d) \in \mathbb{Z}^4 \mid a+b+c+d=k\right\} \\ &\subset V \stackrel{\text{def}}{=} \left\{(a,b,c,d) \in \mathbb{R}^4 \mid a+b+c+d=k\right\} \simeq \mathbb{R}^3. \end{aligned}$$

There is a natural correspondence

$$\widetilde{L} \longleftrightarrow L \quad \text{by} \quad \begin{cases} x & \mapsto \ a+b \\ y & \mapsto \ b+c \\ z & \mapsto \ c+a. \end{cases}$$

§9.3 Hyperplane arrangements

§10 September 28, 2018

§11 October 1, 2018

§11.1 Tropical calculus

We continue our discussion of tropical calculus by completing the table of analogies below.

Rational calculus	Tropical calculus
$x \cdot y$	$x + y$
x/y	$x - y$
$x + y$	$\max(x, y)$
1	0
2	0
$\frac{1}{\frac{1}{x}+\frac{1}{y}} = \frac{xy}{x+y}$	$\min(x, y) = -\max(-x, -y)$

It turns out that most of our previous results will have tropical variants. For example, we had the octahedron recurrence

$$E \cdot E' = A \cdot C + B \cdot D.$$

which becomes the tropical octahedron recurrence

$$E + E' = \max(A + C, B + D).$$

§11.2 RSK

One can formulate the RSK correspondence using the following recursions and picture.

$$\begin{aligned} \mu'_{k+1} + \mu_k &= \max(\lambda_{k+1}, \nu_{k+1}) + \min(\lambda_k, \nu_k) \\ \mu'_1 &= \max(\lambda_1, \nu_1) + x. \end{aligned}$$

The translation of this (birational toggle) gives

$$\begin{aligned} \mu'_{k+1} &= \frac{(\lambda_{k+1} + \nu_{k+1}) \frac{\lambda_k \nu_k}{\lambda_k + \nu_k}}{\mu_k} \\ \mu'_1 &= (\lambda_1 + \nu_1) \cdot x. \end{aligned}$$

We are going to now change variables to get the octahedron recurrence out of this. Inspired by Green's theorem, we make a change of variables

$$\begin{aligned} \ell_k &= \lambda_1 + \cdots + \lambda_k \\ m_k &= \mu_1 + \cdots + \mu_k \\ n_k &= \nu_1 + \cdots + \nu_k. \end{aligned}$$

This gives

$$\begin{aligned} m'_{k+1} + m_k &= \max(\lambda_{k+1}, \nu_{k+1}) + \min(\lambda_k, \nu_k) \\ &\quad + \max(\lambda_k, \nu_k) + \min(\lambda_{k-1}, \nu_{k-1}) \\ &\quad + \cdots + \max(\lambda_1, \nu_1) + x \\ &= \max(\lambda_{k+1}, \nu_{k+1}) \\ &\quad + (\lambda_k + \nu_k) + (\lambda_{k-1} + \nu_{k-1}) + \dots \\ &\quad + (\lambda_1 + \nu_1) + x \\ &= \max(\lambda_{k+1}, \nu_{k+1}) + \ell_k + n_k + x \\ &= \max(\ell_{k+1} + m_k, \ell_k + m_{k+1}) + x. \end{aligned}$$

In summary we have

$$m'_{k+1} + m_k = \max(\ell_{k+1} + n_k, \ell_k + n_{k+1}) + x$$

which is almost the same as the octahedron recurrence, but with a $+x$ term.

$$\begin{tikzcd} & n_{k+1} \arrow[d, dash] & \\ \ell_{k+1} \arrow[r, dash] & m_k \arrow[r, dash] \arrow[d, dash] & n_k \\ & \ell_k & \end{tikzcd}$$

Taking the geometric lift (i.e. detropicalizing) gives

$$m'_{+1} \cdot m_k = (\ell_{k+1} \cdot n_k + \ell_k \cdot n_{k+1}) \cdot x$$

§11.3 A general triply indexed recursion

We now consider the following more general setup. We have an input matrix $X = (x_{ij})$ and we will output a triply indexed matrix $Y = (y_{ijk})$ such that

$$y_{ij(k+1)} y_{ijk} = (y_{i,j-1,k+1} \cdot y_{i,j+1,k} + y_{i-1,j,k+1} \cdot y_{i+1,j,k}) \cdot x_{ij}.$$

Define a re-scaled version

$$\widetilde{y}_{ijk} = \frac{y_{ijk}}{\prod_{\substack{i' \le i+k-1 \\ j' \le j+k-1}} x_{i'j'}}$$

so that we have the recursion instead

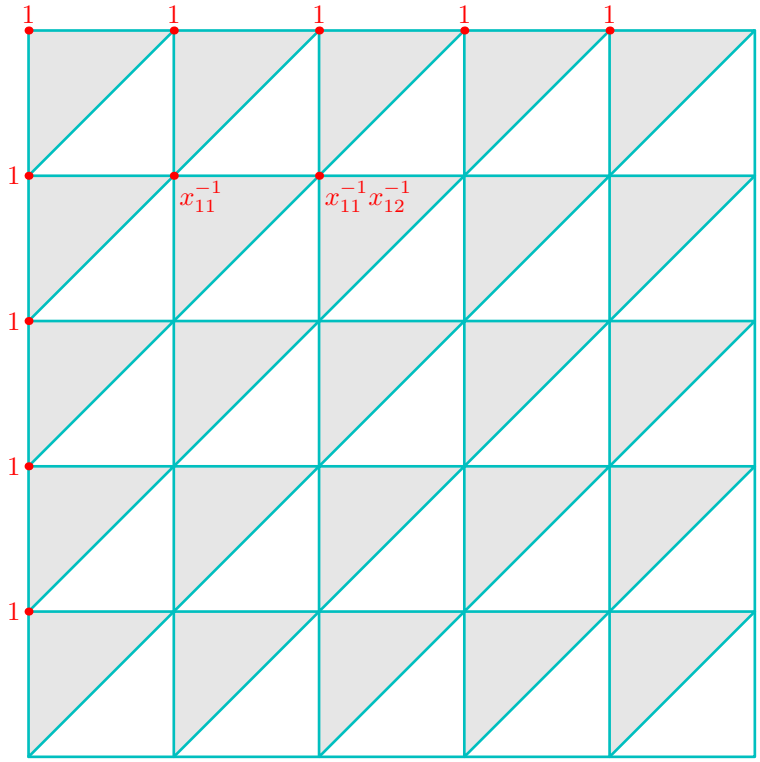
$$\widetilde{y}_{ij(k+1)} \widetilde{y}_{ijk} = \widetilde{y}_{i,j-1,k+1} \cdot \widetilde{y}_{i,j+1,k} + \widetilde{y}_{i-1,j,k+1} \cdot \widetilde{y}_{i+1,j,k}.$$

which now has the data of the X matrix captured in the initial values

$$\widetilde{y}_{ij0} = \frac{1}{\prod_{\substack{i' \le i-1 \\ j' \le j-1}} x_{i'j'}}$$

instead of having it in the recursion.

One can imagine the $k = 0$ values as forming an "input chessboard".



We allow ourselves to apply an octahedron move for a vertex v (including in the first row or column) if there are no diagonal edges.

§12 October 12, 2018

§12.1 General situation

For $G = \mathrm{GL}_n$, we may let

- B denote the upper triangular matrices (the Borel subgroup of G),
- B_- denote the lower triangular matrices (the opposite Borel subgroup of G),
- $G^{>0}$ the subgroup of totally positive matrices,
- $G^{\geq 0}$ the subgroup of totally nonnegative matrices,
- W the Weyl group (which for us is S_n).

We pick this notation since in general one can do the same construction with G a connected, reductive algebraic group over an algebraically closed field, B a Borel subgroup of G, and W a Weyl group of G corresponding to a maximal torus of B.

There exist two **Bruhat decompositions** of G by

$$G = \bigsqcup_{u \in W} BuB = \bigsqcup_{v \in W} B_- v B_-.$$

Exercise 12.1. Prove this.

This means we can define the **double Bruhat cells**

$$G_{u,v} = (BuB) \cap (B_- v B_-)$$

as well as their positive parts $G_{u,v}^{>0} = G_{u,v} \cap G^{\geq 0}$.

Theorem 12.2 (Double Bruhat decomposition of $G^{\geq 0}$)
We have

$$G^{\geq 0} = \bigsqcup_{u,v \in W} G_{u,v}^{>0}.$$

Moreover, each piece $G_{u,v}^{>0}$ is homeomorphic to an "open ball" $\mathbb{R}_{>0}^{\ell(u)+\ell(v)+n}$.

Finally, if we order the cells by containment, the resulting partial order is the product of two copies of the *strong Bruhat order*.

Example 12.3

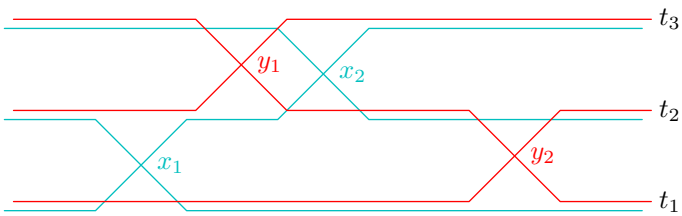
- $G^{>0} = G_{w_0,w_0}^{>0}$ is the top cell.
- $T^{>0} = G_{\mathrm{id},\mathrm{id}}^{>0}$ consisting of diagonal matrices with positive entries is the bottom cell.

§12.2 Lusztig's parametrization

Define $E_i(x) = F_i(x)$ to be the matrix with 1's on the diagonal, and x in the $(i+1)$st column, e.g.

$$E_2(x) = F_2(x) = \begin{bmatrix} 1 & & & & \\ & 1 & x & & \\ & & 1 & & \\ & & & 1 & \\ & & & & \ddots \end{bmatrix}$$

To parametrize this, we take a shuffle of any two reduced decompositions $u = s_{i_1} \dots s_{i_\ell}$ and $v = s_{j_1} \dots s_{j_r}$. Rather than writing the definition, here is an example with $n = 3$, $u = s_1 s_2$ in blue, $v = s_2 s_1$ in red:



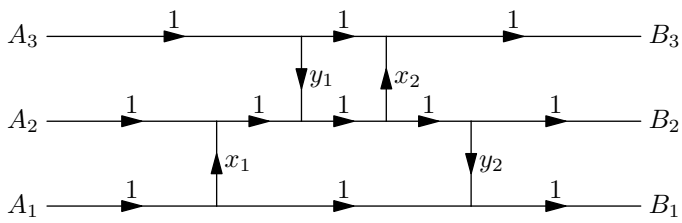
then

$$A = \operatorname{diag}(t_1, t_2, t_3) F_1(y_2) E_2(x_2) F_2(y_1) E_1(x_1).$$

Theorem 12.4

Any $A \in G_{u,v}^{>0}$ can be written uniquely in this form.

One can draw this without color in the following way:



(The choice of weight 1 doesn't matter much, since one can rescale any junction (x, y, z) to $(x/t, y/t, z/t)$ for $t > 0$.)

The resulting matrix A is totally nonnegative.

This follows from a general lemma:

Lemma 12.5 (Linström Lemma, aka Gessel-Viennot method)

Let Γ be any acyclic directed planar graph drawn on the plane such that all sources $A_1, \dots, A_n$ are on the left and $B_1, \dots, B_n$ are on the right of a curve which encloses Γ. Denote the edge weights by x_e. Then define a matrix $A = (a_{ij})$ by

$$a_{ij} = \sum_{p \colon A_i \to B_j} \prod_{e \in P} x_e.$$

Then

$$\det A = \sum_{(P_1, \dots, P_n)} \prod_i \prod_{e \in P_i} x_e$$

where the product is across tuples $P_i \colon A_i \to B_i$ of *non-crossing* paths (not even common vertices).

Involution proof. We have

$$\det(A) = \sum_{w \in S_n} \operatorname{sign}(w) \sum_{\substack{P_1 \colon A_1 \to B_{w(1)} \\ P_2 \colon A_2 \to B_{w(2)} \\ \cdots}} \prod_i \prod_{e \in P_i} x_e$$

Thus we seek a sign-reserving, weight-preserving involution of path tuples $(P_1, \dots, P_n)$ with at least one crossing; this way, only the terms with no crossing will remain.

The hardest part is to define the "first crossing"; we do so as follows.

- Let i be the minimal index such that there exists a crossing on P_i.
- Let C be the first crossing on P_i.
- Let j be the minimal index $j > i$ such that P_j passes through the vertex C.

Then, swap the tails of P_i and P_j at C. This map

$$(P_1, \dots, P_n) \mapsto (P_1 \dots, P_i', \dots, P_j', \dots, P_n).$$

gives the desired involution and flips the sign. Indeed, since we changed w by a transposition, so the change is by sign, and the multiset of edges is the same so the weight is the same. Finally, the (careful) way we chose "first crossing" ensures that the point C remains as the first crossing. □

Note that $\det(A)$ is a nonnegative expression. In particular, if $x_e > 0$ for every edge e, then every minor of A is necessarily nonnegative, by applying the lemma to minors of A. Thus:

Corollary 12.6

The matrix A from the wiring diagram earlier is totally nonnegative.

In fact, there is a following inverse Lindström lemma:

Proposition 12.7

Every totally nonnegative matrix can be achieved as the corresponding matrix for a wiring diagram as above.

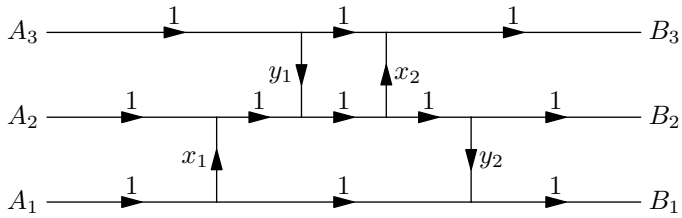
§13 October 15, 2018

Last time we introduced double wiring diagrams; today we will use these to give two parametrizations of the double Bruhat cells $G_{u,v}^{>0}$.

§13.1 Lustzig's parametrization continued

Recall last time we took $n = 3$, $u = s_1 s_2$, $v = s_2 s_1$, and obtained a decompostion

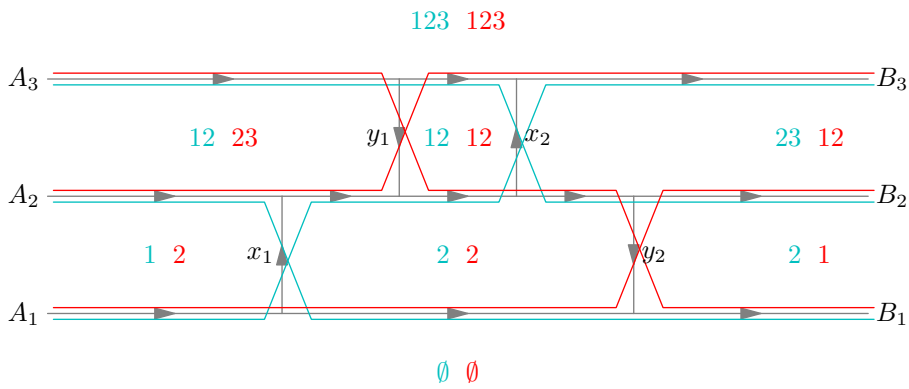
$$A = \operatorname{diag}(t_1, t_2, t_3) F_1(y_2) E_2(x_2) F_2(y_1) E_1(x_1).$$



This gives us Lusztig variables x_1, x_2, y_1, y_2, t_1, t_2, t_3, hence we have a seven-dimensional cell, that is

$$\dim G_{u,v} = \ell(u) + \ell(v) + n = 7.$$

Now, let's reintroduce the colored strands from before. First consider the blue strands; above the blue strand starting from i, we write a blue i in every cell which lies above the blue strand starting from i. Similarly we write a red i in every cell which lies above the red strand ending at B_i.



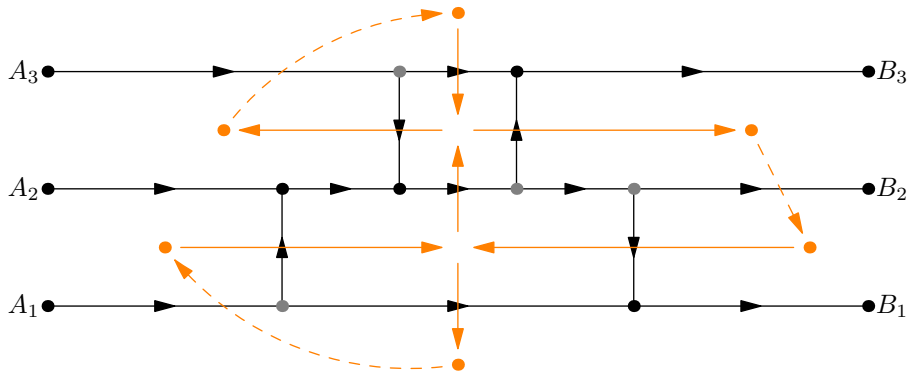
This will give us four **chamber minors** $\Delta_{1,2}$, $\Delta_{2.2}$, $\Delta_{2.1}$, $\Delta_{12,23}$, $\Delta_{12,12}$, $\Delta_{23,12}$, $\Delta_{123,123}$.

Indeed, this gives us another system of variables; in particular, the quantities

$$\begin{aligned} \#\text{chamber minors} &= \#\text{regions} - 1 \\ \#\text{edge variables} &= \#\text{edges} - \#\text{vertices} \end{aligned}$$

should be equal, wich holds by Euler. We now wish to give a parametrization of x_i, y_i, t_i in terms of the chamber minors. (Relevant keywords: chamber ansatz, twist map.)

If we mark all the A_i and B_i in black, then we can draw the usual quiver (as before); I will have a few extra edges (dashed) induced by the black

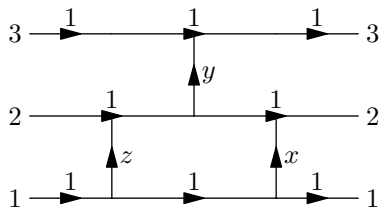


§13.2 Lusztig's tranfsormation

Notice that the braid relation $s_1s_2s_1 \leftrightarrow s_2s_1s_2$ corresponds to the identity

$$E_1(x)E_2(y)E_1(z) = E_2(x')E_1(y')E_2(z').$$

Picture:



If we do the calculation, this amounts to

$$\begin{bmatrix} 1 & x+z & yz \\ & 1 & y \\ & & 1 \end{bmatrix} = \begin{bmatrix} 1 & y' & x'y' \\ & 1 & x'+z' \\ & & 1 \end{bmatrix}$$

and solving this gives

$$\begin{aligned} x' &= \frac{yz}{x+z} \\ y' &= x+z \\ z' &= \frac{xy}{x+z}. \end{aligned}$$

As for the Lewis Carroll transformation, if we set

$$E_1(x)F_1(y) = F_1(x')E_1(y') \begin{bmatrix} t_1 & 0 \\ 0 & t_2 \end{bmatrix}$$

solving this now gives

$$\begin{aligned} x' &= \frac{y}{1+xy} \\ y' &= x(1+xy) \\ t_1 &= (1+xy)^{-1} \\ t_2 &= 1+xy. \end{aligned}$$

§14 October 24, 2018: Protean chromatic polynomial (lecture by Bruce Sagan)

Seems the slides are online, so I'll just link them: http://users.math.msu.edu/users/sagan/Slides/pcp.pdf