Math 137 Lecture Notes

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This is Harvard College's *Math 137*, instructed by Yaim Cooper. The formal name for this class is "Algebraic Geometry"; we will be studying complex varieties.

The permanent URL for this document is `http://web.evanchen.cc/coursework.html`, along with all my other course notes. As usual, Dropbox links expire at the end of the semester.

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§1 February 6, 2015

This is the sixth lecture.

At the end of last class we asked the question.

Question 1.1. What are the maximal ideals of $\mathbb{C}[x_1, \dots, x_n]$?

Here's one of them: $(x_1 - a_1, x_2 - a_2, \dots, x_n - a_n)$, where $a_i \in \mathbb{C}$. We're interest in finding *all* of the maximal ideals, but we won't be able to answer it completely until later.

§1.1 Facts about ideals

Today we'll be discussing the relation between ideals and ring homomorphisms. It's easy to verify that for $f : R \to S$, then $\ker f$ is an ideal, and more generally for that $I \subseteq S$ the ideal $f^{-1}(I)$ is an ideal. (The kernel is the special case where $S = (0)$.)

The reverse is also true: Given $I \subseteq R$ an ideal, we have a natural map $R \to R/I$ with kernel I.

More importantly, here is the so-called "mandatory exercise" from Vakil.

Proposition 1.2

Consider the natural projection $\pi : R \to R/I$. Ideals $J \subseteq R/I$ correspond exactly to ideals $K \subseteq R$ with $K \supseteq I$ through the projection map π.

It's not true in general that ideals map to ideals under a ring homomorphism. For example, consider the inclusion $\mathbb{Z} \hookrightarrow \mathbb{Z}[x]$. The image of the ideal $2\mathbb{Z} \subseteq \mathbb{Z}$ is $2\mathbb{Z} \subseteq \mathbb{Z}[x]$, which is not an ideal since it fails to absorb multiplication by x.

§1.2 $\mathbb{C}$-algebras

Definition 1.3. We say R is a **$\mathbb{C}$-algebra** if it contains $\mathbb{C}$ as a subring. In particular, every $\mathbb{C}$-algebra can be viewed as a $\mathbb{C}$-vector space.

Definition 1.4. Let R be a $\mathbb{C}$-algebra. A **$\mathbb{C}$-subalgebra generated by J** for some set $J \subseteq R$ is the intersection of all algebras containing J. Then R is **finitely generated** if it's generated by a finite set J.

Example 1.5

The algebra $\mathbb{C}[x]$ is finitely generated as a $\mathbb{C}$-algebra, even though it's infinite-dimensional as a $\mathbb{C}$-vector space. Similarly, $\mathbb{C}[x_1, \dots, x_n]$ is a finitely generated $\mathbb{C}$-algebra, but $\mathbb{C}[x_1, x_2, \dots]$ is not finitely generated.

Definition 1.6. A map of $\mathbb{C}$-algebras $\phi : R \to S$ is a ring homomorphism which is also linear over $\mathbb{C}$ (and in particular fixes $\mathbb{C}$).

Example 1.7

The map $\mathbb{C}[x] \to \mathbb{C}[x]$ by $1 \mapsto 1$, $x \mapsto x^2$ is a homomorphism.

Example 1.8

A non-example is $\mathbb{C}[x] \to \mathbb{C}[x]$ by complex conjugation. (Although it is a map of $\mathbb{R}$-algebras!)

§1.3 Hilbert Basis Theorem

In previous lectures we defined $\mathcal{V}(\{F_i\}_{i \in I})$; we didn't require in this case that I be finite or even countable. Actually, we will show that we ever only need finitely many such I's. For this, we require the following.

Definition 1.9. A ring R is **Noetherian** if all its ideals are finitely generated.

On homework we'll show this is equivalent to the **ascending chain condition**: there does *not* exist a strictly ascending chain of ideals

$$I_1 \subsetneq I_2 \subsetneq \dots.$$

Theorem 1.10 (Hilbert Basis Theorem)

If R is Noetherian then $R[x]$ is Noetherian.

By induction, $R[x_1, \dots, x_n]$ is Noetherian. And of course $\mathbb{C}$ is Noetherian (there are only two ideals!). But $\mathbb{C}[x_1, x_2, \dots]$ is not Noetherian, because

$$(x_1) \subsetneq (x_1, x_2) \subsetneq \dots.$$

Proof. Take $I \subseteq R[X]$. We wish to show I is finitely generated, so we're going to start throwing in elements of I. We can't quite do this randomly, but here's what we do: let f_1 be a nonzero polynomial of *minimal degree* in I. Then look at $I - (f_1)$. If it's nonempty, we can pick f_2 in it of minimal degree, and $f_3 \in I - (f_1, f_2)$ and so on.

Let me explain why I'm picking the minimal degree. Suppose somehow that $f_N \notin (f_1, f_2, \dots, f_{N-1})$ for a really big N. Then that means that

$$f_N - a_1 f_1 - a_2 f_2 + \dots - a_{N-1} f_{N-1}$$

can't be in the ideal either, for any $a_1, a_2, \dots \in R[x]$. If we look at the minimal f_N, then that means even with all the firepower of the f_i's, we can't even kill the leading term of f_N. That seems like it shouldn't be possible for big enough N, and this is the motivation for the proof.

Let $a_j \in R$ be the leading coefficient of f_j. We're going to use these to blow up an

$$a_N x^{\deg f_N}$$

for big N. Look at the ascending chain

$$(a_1) \subseteq (a_1, a_2) \subseteq \dots$$

which eventually stabilizes. Hence $a_N \in (a_1, \dots, a_{N_1})$ for some big N, meaning

$$a_N = u_1 a_1 + u_2 a_2 + \dots + u_{N-1} a_{N-1}.$$

Then look at

$$f_N - \sum_{j=1} u_j f_j x^{\deg f_{N+1} - \deg f_j}.$$

(The x^{blah} is just there to shift it so that all our a_i line up). By all our discussion this is a polynomial with degree strictly less than $\deg f_N$, contradiction. $\square$

§1.4 Viewing varieties as ideals

Note that if an affine algebraic variety V is a zero locus of $\{f_i\}_{i \in I}$, then in fact it's a zero locus of the entire ideal $(\{f_i\}_{i \in I})$.

So it's better to think about $\mathcal{V}$(ideal) than $\mathcal{V}$(set), because this way we have the same name for the vanishing set. So we can think of affine algebraic varieties by ideals.

In particular, by the Hilbert Basis Theorem, all the ideals are finitely generated. So in Noetherian situations, all our varieties are finitely generated.

§1.5 Flavors of Ideals

We're going to cover various types of ideals and then later see what these correspond to geometrically.

We already know what a maximal ideal is (ideal maximal under inclusion).

Definition 1.11. An ideal $I \subseteq R$ is **prime** if $ab \in I$ if and only if $a \in I$ or $b \in I$.

Example 1.12 (Prime Ideals)

$(5) \subseteq \mathbb{Z}$ is prime. The ideal $(6) \subseteq \mathbb{Z}$ is not prime. Similarly, $(x) \in \mathbb{C}[x]$ is a radical ideal.

Definition 1.13. $I \subseteq \mathbb{R}$ is **radical** if $I = \sqrt{I}$, where the radical $\sqrt{I}$ of I is defined by

$$\sqrt{I} \stackrel{\text{def}}{=} \{a \in R \mid a^n \in I \text{ for some } n > 0\}$$

Example 1.14

The ideal $(5) \subseteq \mathbb{Z}$ is radical; in general prime ideals are radical. We have $\sqrt{(12)} = (6)$, and in general this corresponds to the number-theoretic notion of a radical (the product of distinct primes). So (12) is not radical, but (6) is radical.

Proposition 1.15

Prime ideals are radical. Any ideal of the form $\sqrt{I}$ is also radical.

Remark 1.16. (0) is prime exactly when R is an integral domain, and (0) is maximal exactly when R is a field.

§2 February 11, 2015

Today we will try to make precise the connection between affine algebraic varieties between ideals of $\mathbb{C}[x_1, \dots, x_n]$ and the affine algebraic varieties. In the best of worlds, we would have a bijection between ideals and affine algebraic varieties. Unfortunately, the answer is not so simple.

§2.1 A Small Remark from Evan o'Dorney (in response to homework)

Note: an isomorphism of algebraic varieties is not merely a structure-preserving bijection; the inverse must be a morphism too. Both the forwards and backwards maps must actually be morphisms. Hence for example the map $\mathbb{A}^1 \to \mathbb{A}^2$ by $t \mapsto (t^2, t^3)$ is not an isomorphism, even though it is a bijection, because the inverse is not a map of affine algebraic varieties: there is no polynomial f such that $f(t^2, t^3)$.

Actually, if you draw the curve $\left\{(t^2, t^3) \mid t \in k\right\}$ you get a curve with a singularity.

§2.2 Obtaining Ideals From Varieties

To understand our hypothetical bijection we first ought to decide how to obtain an ideal from a variety. First, if V is the common zero set of $\{f_i\}_{i \in A}$, then we can consider $I = (f_i)_{i \in A}$ and note that $f \in I \implies f(x) = 0$.

A second thing we could do is define the **annihilator** of V

$$\mathcal{I}(V) = \{f \in \mathbb{C}[x_1, \dots, x_n] \mid f(x) = 0 \; \forall x \in V\}.$$

This begs the question: does I equal $\mathcal{I}(V)$? Obviously $I \subseteq \mathcal{I}(V)$. We'll delay the answer to this until later, and consider the other direction.

§2.3 Obtaining Varieties from Ideal

Given an ideal $I \in \mathbb{C}[x_1, \dots, x_n]$ we can define

$$\mathcal{V}(I) = \{x \in \mathbb{C}^n \mid f(x) = 0 \; \forall f \in I\}.$$

Now we ask: is it true that $\mathcal{I}(\mathcal{V}(I)) = I$ and $\mathcal{V}(\mathcal{I}(V)) = V$? One direction can be done quickly.

Proposition 2.1
$\mathcal{V}(\mathcal{I}(V)) = V$.

Proof. $V \subseteq \mathcal{V}(\mathcal{I}(V))$ is tautological. For the other direction, $x \in \mathcal{V}(\mathcal{I}(V)) \implies f(x) = 0 \; \forall f \in \mathcal{I}(V)$. We defined $V = \mathcal{V}((f_i)_{i \in I})$ so everything is clear. □

Unfortunately, it's not true that $\mathcal{I}(\mathcal{V}(I)) = I$.

§2.4 Maximal Ideals

Note that larger ideals correspond to smaller varieties. So this motivates looking at *maximal* ideals, because those should correspond to the smallest (and hopefully easiest to understand) varieties.

Specifically, in a variety V a point $(a_1, \dots, a_n)$ can be considered as a maximal ideal $(x_1 - a_1, x_2 - a_2, \dots, x_n - a_n)$. So this leads us naturally to the following question: are all maximal ideals of this form? The answer is affirmative.

Theorem 2.2 (Maximal Ideals)
All maximal ideals of $\mathbb{C}[x_1, \dots, x_n]$ are of the form $(x_1 - a_1, \dots, x_n - a_n)$.

Theorem 2.3 (Weak Nullstellensatz)
Let $I \subsetneq \mathbb{C}[x_1, \dots, x_n]$ be a proper ideal. Then the variety $\mathcal{V}(I) \neq \varnothing$.

We'll proof (or at least sketch the proof) of the weak nullstellensatz next lecture. In the meantime, however, we will prove that these two theorems are equivalent. (Both these theorems are true over any algebraically closed field, as usual.)

MI Implies WN. Start with $I \subsetneq R$. Then I is contained inside a maximal ideal $M = (x_1 - a_1, \dots, x_n - a_n)$, and that M has a common vanishing point.

The point of this is just that $\mathcal{V}$ reverses containments. □

WN Implies MI. Let M be a maximal ideal. By WN, there is a point $p = (a_1, \dots, a_n) \in \mathcal{V}(M)$. Hence M is the set of all polynomials vanishing at p. But the set of polynomials vanishing at p is $I = (x - a_1, \dots, x - a_n)$. Hence $M \subseteq I$, and by maximality $M = I$. □

So points in $\mathbb{C}^n$ correspond exactly with maximal ideals!

§2.5 Hilbert's Nullstellensatz

Theorem 2.4 (Hilbert's Nullstellanzatz)
In fact, $\mathcal{I}(\mathcal{V}(I)) = \sqrt{I}$.

Example 2.5
Let $I = (x^{2014})$. Then $\mathcal{I}(\mathcal{V}(I)) = (x)$, which was to be expected (the zeros of x^{2014} are the same as the zeros of x).

As a result, the *actual* bijection is that there is bijection between affine algebraic varieties and radical ideals.

WN Implies Nullstellensatz. You can check easily that

$$\mathcal{I}(\mathcal{V}(I)) \subseteq \sqrt{I}.$$

Hence the tricky part is to check that if $f \in \mathcal{I}(\mathcal{V}(I))$, *id est* $f(x) = 0$ for all $x \in \mathcal{V}(I)$, then $f \in \sqrt{I}$.

Take a set of generators $f_1, \dots, f_m$, in the original ring $\mathbb{C}[x_1, \dots, x_n]$; we may assume it's finite by the Hilbert Basis Theorem.

We're going to do a trick now: consider $S = \mathbb{C}[x_1, \dots, x_n, x_{n+1}]$ instead. Consider the ideal $I' \subseteq S$ in the bigger ring generated by $\{f_1, \dots, f_m\}$ and the polynomial $x_{n+1}f - 1$. The point of the last guy is that its zero locus does not touch our copy $x_{n+1} = 0$ of $\mathbb{A}^n$ nor any point in the "projection" of f through $\mathbb{A}^{n+1}$ (one can think of this as $\mathcal{V}(I)$ in the smaller ring direct multiplied with $\mathbb{C}$). Thus $\mathcal{V}(I') = \varnothing$, and by the weak nullstellensatz we in fact have $I' = \mathbb{C}[x_1, \dots, x_{n+1}]$. So

$$1 = g_1 f_1 + \cdots + g_m f_m + g_{m+1}\left(x_{n+1} f - 1\right).$$

Now the hack: **replace every instance of x_{n+1} by** $\frac{1}{f}$, and then clear all denominators. Thus for some large enough integer N we can get

$$f^N = f^N(g_1 f_1 + \cdots + g_m f_m)$$

which eliminates any fractional powers of f in the right-hand side. It follows that $f^N \in I$. $\square$

This hack has a name: the **Rabinowitsch trick**.

§3 February 13, 2015

Recall that last time we showed WN and MI were equivalent, and that WN $\implies$ Hilbert Nullstellensatz. Now we will prove the weak nullstellensatz, thus completing the correspondence.

§3.1 Prime ideals

Before doing so, let's complete the chart: we know that points of $\mathbb{A}^n$ correspond to the maximal ideals while varieties correspond to radical ideals. Finally, we claim the following.

Proposition 3.1

Prime ideals correspond to the irreducible varieties.

Proof. Let V be irreducible. We wish to show $\mathcal{I}(V)$ is prime.

Take any $f, g \in \mathbb{C}[x_1, \dots, x_n]$ such that $fg \in \mathcal{I}(V)$. We wish to show either $f \in \mathcal{I}(V)$ or $g \in \mathcal{I}(V)$. But we have $V \subseteq \mathcal{V}(f) \cup \mathcal{V}(g)$., so by irreducibility we may assume $V = \mathcal{V}(f)$, whence $f \in \mathcal{I}(V)$ as required.

Conversely, suppose $\mathcal{I}(V)$ is prime; we'll show V is irreducible. Suppose for contradiction that $V = V_1 \cup V_2$ is a nontrivial decomposition. Hence $V_i = V \cap A_i$ for some affine algebraic variety A_i in $\mathbb{A}^n$. Hence $V \subsetneq A_1$ and $V \subsetneq A_2$. Thus $\exists f \in \mathcal{I}(A_1)$, $g \in \mathcal{I}(A_2)$, so that neither f nor g are in $\mathcal{I}(V)$. Now $Fg \in \mathcal{I}(A_1 \cup A_2) \subseteq \mathcal{I}(V)$. $\square$

Inclusion-reversal is pretty dizzying.

§4 February 18, 2015

Today we will show the weak nullestellensatz, thus completing the proof of the full Hilbert nullestellensatz.

§4.1 A Coordinate-Change Lemma

Lemma 4.1

Given $g \in \mathbb{C}[x_1, \dots, x_n]$, there exists a change of coordinates

$$(x_1, \dots, x_n) \to (x_1 + \lambda x_n, \dots, x_{n-1} + \lambda_{n-1} x_n, x_n).$$

such that g is monic in x_n (thought of as a polynomial in $\mathbb{C}[x_1, \dots, x_{n-1}][x_n]$).

For example, the polynomial $g(x_1, x_2) = x_1^3 + x_1 x_2^4 + x_2^5 + x_1 x_2^3$. is monic in x_2 (the leading term is x_2^5), but the polynomial $g(x_1, x_2) = x_1^3 + x_1 x_2^4 + x_1^5 + x_1 x_2^3$. is not monic in x_2 (the leading term is $x_1 x_2^4$), but we can look instead at

$$g(x_1 + \lambda_1 x_2, x_2).$$

The "leading x_2 term" will be $\lambda_1^5 + \lambda_1$ (contributed by the second and fourth). Note that there's no x_i's left: we just use the fact that $\mathbb{C}$ is algebraically closed to exhibit a root of $t^5 + t = 1$. The proof if the general case is the same (the point is that the maximal degree terms are x_i^d).

§4.2 Proof of Weak Nullstellensatz

The proof goes by induction on n.

For the case $n = 1$, $\mathbb{C}[x]$ is a principle ideal domain (just by Euclid or whatever) and so any ideal is of the form $I = (f)$. Assume I is not the zero ideal. As I is a proper ideal, we know f is nonconstant, and so we can find a root at which it vanishes.

Now for the inductive step (when $n > 1$). Take any nonconstant $g \in I$ with degree e. By the lemma, we can assume WLOG that g is monic in x_n. Hence we may write

$$g(x_1, \dots, x_n) = x_n^e + \sum_{k=1}^{e-1} g_k(x_1, \dots, x_{n-1}) x_n^k.$$

(We won't use this g until later, but it's important to emphasize that g doesn't depend at all on anything we define below.)

We construct the following two auxilary ideals. We construct $I' \subsetneq \mathbb{C}[x_1, \dots, x_{n-1}]$ of polynomials in I which don't have any x_n terms (here I' is proper since $1 \notin I'$). It has a vanishing point $(a_1, \dots, a_{n-1})$. Next, define the ideal

$$J = \{ f(a_1, \dots, a_{n-1}, x_n) \mid f \in I \} \subseteq \mathbb{C}[x_n]$$

(you can easily check directly it's an ideal).

Claim 4.2. *J is also proper.*

Note that $a_1, \dots, a_n$ depends only on I.

Proof that J is proper. If not, suppose

$$1 = f(a_1, \dots, a_{n-1}, x_n)$$

for some $f \in I$. Write

$$f = f_0 + f_1 x_n + \cdots + f_d x_n^d$$

with $f_0, \dots, f_d \in \mathbb{C}[x_1, \dots, x_{n-1}]$. we find that $f_1(a_1, \dots, a_{n-1}) = \cdots = f_d(a_1, \dots, a_{n-1}) = 0$ and $f_0(a_1, \dots, a_{n-1}) = 1$.

View f and g as polyonmials in x_n with coefficients in $\mathbb{C}[x_1, \dots, x_{n-1}]$ We consider the **resultant** R of f and g, which lives inside the coefficients $R \in \mathbb{C}[x_1, \dots, x_{n-1}]$. The resultant (in determinant form) has the property that it's a linear combination of f and g, meaning $R \in I$, and hence $R \in I'$. Explicitly, if $d = 3$ and $e = 2$ the resultant is

$$R = \det \begin{pmatrix} f_0 & f_1 & f_2 & f_3 & 0 \\ 0 & f_0 & f_1 & f_2 & f_3 \\ e_0 & e_1 & 1 & 0 & 0 \\ 0 & e_0 & e_1 & 1 & 0 \\ 0 & 0 & e_0 & e_1 & 1 \end{pmatrix}$$

The first e rows are dedicated to f, the next d are dedicated to g, so the resultant is a $(d+e) \times (d+e)$ square matrix. I'll let you guess what the general form is :D (Note that $e_2 = 1$ by construction). ■

Since $J \subsetneq \mathbb{C}[x_n]$ is proper, and hence $J = (h)$ for some nontrivial h. Pick a_n such that $h(a_n) = 0$. Then for all $f \in I$ we have

$$f(a_1, \dots, a_n) = 0$$

as desired (note that this a_i really is independent of f).

§4.3 Correspondence

Recall that we had correspondences

- Points correspond to maximal ideals.
- Irreducible varieties correspond to prime ideals.
- Affine algebraic varieties correspond to radical ideals.

Also, we know somehow that V_1 corresponds to I_1 and $V2$ corresponds to I_2, then $V_1 \cap V_2$ corresponds to $I_1 + I_2$ and $V_1 \cup V_p$ corresponds to $I_1 + I_2$.

This correspondence isn't perfect, though.

Example 4.3

Let $C = \mathcal{V}(y - x^2)$ be the parabola and $L = \mathcal{V}(y)$. Then $C \cap L$ is just the point $(0, 0)$. But $(y - x^2) + (y) = (y, x^2)$, which indeed has vanishing set of just the origin, but that's not the ideal we would normally have named: we normally would call it (y, x).

Somehow the fact that we got (y, x^2) instead of (y, x) is significant. The ideal (y, x^2) *remembers* the fact that we have a "double zero" on the x-axis.

So something corresponds to the non-radical ideals. That "something" is schemes.
To be precise about what's true and isn't true: here are the true statements.

$$\begin{aligned}
\mathcal{V}(I) \cap \mathcal{V}(J) &= \mathcal{V}(I+J) \\
\mathcal{V}(I) \cup \mathcal{V}(J) &= \mathcal{V}(IJ) = \mathcal{V}(I \cap J) \\
\mathcal{I}(V_1 \cap V_2) &= \sqrt{\mathcal{I}(V_1) + \mathcal{I}(V_2)} \\
\mathcal{I}(V_1 \cup V_2) &= \sqrt{\mathcal{I}(V_1) \cup \mathcal{I}(V_2)} = \sqrt{\mathcal{I}(V_1) \cap \mathcal{I}(V_2)} = \mathcal{I}(V_1) \cap \mathcal{I}(V_2).
\end{aligned}$$

§5 February 20, 2015

In life, we like to understand objects by understanding the maps on or to them. To this end, we want to examine the morphisms on a variety V. The notion of a coordinate ring does this.

§5.1 Coordinate Ring

We restrict our attention to algebraic (polynomial) functions on a variety V. For example, a valid function is $(a, b, c) \mapsto a$, which we call "x" Similarly we have a canonical projection y and z, and we can create polynomials by combining them.

Definition 5.1. The **coordinate ring** $\mathbb{C}[V]$ of a variety V is the ring of polynomial functions on V.

This is not merely $\mathbb{C}[x_1, \dots, x_n]$. For example, note that in $\mathbb{A}^3$, the variety $V = \mathcal{V}(x^2 + y^2 - z^2)$ has functions $x^2 + y^2$ and z^2, which are the same. What is naturally true is that

Proposition 5.2

Given a variety $V \subseteq \mathbb{A}^n$, which vanishes on $I = \mathcal{I}(\mathcal{V})$, we have a canonical isomorphism
$$\mathbb{C}[V] \cong \mathbb{C}[x_1, \dots, x_n]/I.$$

Proof. There's a natural surjection
$$\mathbb{C}[x_1, \dots, x_n] \twoheadrightarrow \mathbb{C}[V]$$
with kernel I. □

Thus properties of $\mathcal{I}(V)$ correspond to properties of $\mathbb{C}[V]$.

§5.2 Pullback

Suppose we have a morphism $V \xrightarrow{F} W$. Then we can get a map of the coordinate rings $F^\sharp$ according to
$$\mathbb{C}[W] \xrightarrow{F^\sharp} \mathbb{C}[V]$$
$$g \mapsto g \circ F$$

We can draw the diagram more explicitly:

$$\begin{tikzcd} W \arrow[r, "F"] & \mathbb{C} \\ V \arrow[u, "g"] \arrow[ur, "F^\sharp(g)"'] & \end{tikzcd}$$

This can be thought of as a contravariant functor.

$$\begin{tikzcd} V \arrow[dd, "F"'] & \mathbb{C}[V] \\ {} \arrow[r, dashed] & {} \\ W & \mathbb{C}[W] \arrow[uu, "F^\sharp"'] \end{tikzcd}$$

Example 5.3

Let $V = \mathbb{A}^3 \xrightarrow{F} \mathbb{A}^2 = W$ by $(x, y, z) \mapsto (x^2y, x - z)$. Then the pullback is a map

$$\begin{aligned} \mathbb{C}[W] \cong \mathbb{C}[u, v] &\xrightarrow{F^\sharp} \mathbb{C}[x, y, z] \cong \mathbb{C}[V] \\ u &\mapsto x^2y \\ v &\mapsto x - z \end{aligned}$$

Remark 5.4. The *pullback* of a map generalizes the notion of the *dual map* $T^\vee : W^\vee \to V^\vee$ of a map $T : V \to W$ of finite vector spaces.

§5.3 Which rings are coordinate rings?

We now ask: which rings are the coordinate ring of some ideal? There are some obvious requirements.

- Such rings must be $\mathbb{C}$-algebras, of course.
- Such a ring must be finitely generated.
- Such a ring must be reduced.

It turns out that these conditions are sufficient!

Theorem 5.5

Every finitely generated reduced $\mathbb{C}$-algebra is the coordinate ring of some complex affine variety.

Proof. Let R be this map. Because it's finitely generated by some r_i, there is some map

$$\mathbb{C}[x_1, \dots, x_n] \twoheadrightarrow R$$

via $x_i \mapsto r_i$. Let I be the kernel of this map. Since R is reduced, I is radical. Then $R \cong \mathbb{C}[V]$, where $V = \mathcal{V}(I)$. $\square$

§5.4 The equivalence of algebra and geometry

Now the main theorem is the following.

Theorem 5.6

Let $R \xrightarrow{\sigma} S$ be a map of finitely generated reduced $\mathbb{C}$-algebras. Then there exists unique affine varieties V and W (up to isomorphism) and a map F between them so that the diagram

$$\begin{tikzcd} R \arrow[r, "\sigma"] \arrow[d, leftrightarrow, "\cong"'] & S \arrow[d, leftrightarrow, "\cong"] \\ \mathbb{C}[W] \arrow[r, "F^\sharp"'] & \mathbb{C}[V] \end{tikzcd}$$

commutes.

Rephrasing in terms of category theory,

Theorem 5.7 (The Equivalence of Algebra and Geometry)
The pullback is a contravariant functor which induces an (opposite) *equivalence* of the following two categories:

- The category of affine algebraic varieties.
- The category of finitely generated $\mathbb{C}$-algebras.

§6 February 23, 2015

§6.1 Recap

To summarize the results from last time:

- Every finitely generated reduced $\mathbb{C}$-algebra is isomorphic to the coordinate ring of some affine algebraic variety.
- If $F : V \to W$ is a morphism of affine algebraic varieties, then we have a pullback map
$$F^\sharp : \mathbb{C}[W] \to \mathbb{C}[V]$$
between the coordinate rings.
- If $\sigma : R \to S$ is a homomorphism of finitely generated $\mathbb{C}$-algebras then there exists a unique morphism F (up to isomorphism) of the corresponding affine algebraic varieties such that σ is the pullback of F. This gives an equivalence of categories between affine complex varieties and finitely generated reduced $\mathbb{C}$-algebras.

§6.2 Isomorphism of Affine Algebraic Varieties

For categorical reasons, isomorphisms of rings correspond to isomorphisms of varieties.

Example 6.1 (An Isomorphism)
Consider the isomorphism of affine algebraic varieties

$$\mathbb{A}^1 \xrightarrow{F} \mathcal{V}(y - x^2) \subseteq \mathring{A}^2$$

which maps a line into a parabola (via $t \mapsto (t, t^2)$). It induces a map of coordinate rings

$$\mathbb{C}[t] \xleftarrow{F^\sharp} \mathbb{C}[x, y]/(y - x^2).$$

by sending $t \hookleftarrow x$ and $t^2 \hookleftarrow y$.

Example 6.2 (A Non-Isomorphism)
Consider the isomorphism of affine algebraic varieties

$$\mathbb{A}^1 \xrightarrow{F} \mathcal{V}(y^2 - x^3) \subseteq \mathring{A}^2$$

through the map $t \mapsto (t^2, t^3)$. You can think of this as a curve with a cusp and the line $x = 1$, mapped by projection through the origin. But this is not an isomorphism (there's a "singularity" at the origin).

You can see this reflected in the corresponding map of rings. It is given by

$$\mathbb{C}[t] \leftarrow \mathbb{C}[x, y]/(y^2 - x^3)$$

via

$$t^2 \hookleftarrow x \text{ and } t^3 \hookleftarrow y.$$

This is not an isomorphism because it fails to be surjective; it misses $t \in \mathbb{C}[t]$!

§6.3 (Digression) Spectrum of a Ring

We won't touch much on this (for now), but just briefly...

In our previous examples $\mathbb{C}[x_1, \dots, x_n]$ we can think of the points as just the maximal ideals. More generally, given *any* ring R we can think about a "space" where the points are the maximal ideals of R, and the elements of R as the "functions". Hence the points are

$$\max \operatorname{Spec} R = \{\mathfrak{m} \subseteq R \mid \mathfrak{m} \text{ maximal}\}$$

We can even define the Zariski topology on $\max \operatorname{Spec} R$: a set Z is closed if Z is the set of all maximal ideals containing an ideal I. And given $f \in R$, $\mathfrak{m} \in \max \operatorname{Spec} R$, we can think of $f(\mathfrak{m})$ as $[f] \pmod{\mathfrak{m}}$.

This is unfortunately far from perfect: a morphism $R \to S$ of rings does not necessarily preserve maximal ideals. For example, look at the map $\mathbb{Z} \hookrightarrow \mathbb{Q}$. The zero ideals in $\mathbb{Z}$ are far from maximal, but they are maximal in $\mathbb{Q}$. In particular, the pullback of the maximal ideal $(0) \subseteq \mathbb{Q}$ is the non-maximal ideal $(0) \subseteq \mathbb{Z}$.

This motivates us instead to consider

$$\operatorname{Spec} R = \{\mathfrak{p} \subseteq R \mid \mathfrak{p} \text{ prime}\}$$

and so on.

§6.4 Complex Projective Space

It turns out we can get many more examples of varieties by looking at $\mathbb{CP}^n$ instead of by looking at $\mathbb{C}^n$. This requires me to tell you what $\mathbb{CP}^n$ is. It turns out to be a "compacification" of $\mathbb{C}^n$.

(As usual, in what follows all of our geometric pictures will really be $\mathbb{RP}^n$: we again treat $\mathbb{C}$ as a straight line.)

Here's the definition: $\mathbb{CP}^n$ can be thought of as the set of all lines through the origin in $\mathbb{C}^{n+1}$.

Example 6.3 ($\mathbb{CP}^1$)

Consider the bundle of lines through the origin in $\mathbb{C}^2$, and project it onto the line $x = 1$. Thus $\mathbb{CP}^1$ can be thought of as the complex line, plus a point at infinity.

The unit circle might have been a decent approximation, but it suffered from the issue that every line hits two points on the unit circle.

Example 6.4 ($\mathbb{CP}^1$)

Consider the bundle of lines through the origin in $\mathbb{C}^2$, and project it onto the line $x = 1$. This is an "almost-bijection"; we have a "point at infinity" caused by the line $x = 0$ in $\mathbb{C}^2$.

Thus $\mathbb{CP}^1$ can be thought of as the complex line, plus a point at infinity.

Example 6.5 ($\mathbb{CP}^2$)

In an analogous way, we can project $\mathbb{CP}^2$ onto the plane $x = 1$, which misses only a copy of $\mathbb{CP}^1$ (the lines through the origin contained in the yz-plane).

Hence

$$\mathbb{CP}^2 = \mathbb{C}^2 \cup \mathbb{CP}^1 = \mathbb{C}^2 \cup \mathbb{C} \cup \{\infty\}.$$

In general, we have

$$\mathbb{CP}^n = \mathbb{C}^n \cup \mathbb{CP}^{n-1} = \mathbb{C}^n \bigcup \mathbb{C}^{n-1} \cup \dots \{\infty\}.$$

§6.5 Manifold

In this section we only consider $n = 2$ for simplicity of notation. We can think of the space

$$\mathbb{CP}^2 \overset{\text{def}}{=} \{p = (x_0, x_1, x_2) \mid p \neq 0\} / ((x_0, x_1, x_2) \sim (\lambda x_0, \lambda x_1, \lambda x_2)).$$

So the point is that we consider a *chart* $U_0 \simeq \mathbb{C}^2$, where U_0 is the set of points with nonzero x-coordinate. Similarly, we can define a chart U_1 for the points with nonzero y-coordinate, and so on. These three charts cover $\mathbb{CP}^2$.

Now we want to make $\mathbb{CP}^2$ into a complex manifold. You can check that the *transition functions* are all holomorphic.

§6.6 Projective Varieties

Anyways, the point of all this is that we want to think about projective varieties (those that live in $\mathbb{CP}^n$) in addition to our affine varieties (living in $\mathbb{C}^n$).

In $\mathbb{C}^n$ we could consider polynomial functions. Now we want to consider what functions we might put on $\mathbb{CP}^n$, whose points have homogeneous coordinates

$$[x_0 : x_1 : \dots : x_n].$$

This leads us to the following definition:

Definition 6.6. A polynomial $F \in \mathbb{C}[x_0, \dots, x_n]$ is **homogeneous** if all terms have the same degree.

By taking $m + 1$ of these functions, this gives us a good notion of a function from $\mathbb{CP}^n$ to $\mathbb{CP}^m$. Note also that given a polynomial $f \in \mathbb{C}[x_1, \dots, x_n]$, we can get a homogeneous polynomial $\tilde{f} \in \mathbb{C}[x_0, x_1, \dots, x_n]$ by throwing in x_0's (for example, $x_1^2 + x_2^3 \to x_0 x_1^2 + x_2^3$).

§7 February 25, 2015

Today we want to define projective varieties in a way similar to the way we defined algebraic varieties.

§7.1 Functions on a Projective Variety

Consider $\mathbb{CP}^2 = \{[x_0 : x_1 : x_2]\}$. We wish to consider zero loci in $\mathbb{CP}^2$, just like we did in affine space, and hence obtain a notion of a projective variety.

But this isn't so easy: for example, the function "x_0" is not a well-defined function on points in $\mathbb{CP}^2$ because $[x_0 : x_1 : x_2] = [5x_0 : 5x_1 : 5x_2]$! So we'd love to consider these "pseudo-functions" that still have zero loci. These are just the homogeneous polynomials.

Definition 7.1. A function $F \in \mathbb{C}[x_0, \dots, x_n]$ is called **homogeneous** if all terms have the same degree d. Equivalently,

$$F(\lambda x_0, \dots, \lambda x_n) = \lambda^d F(x_0, \dots, x_n).$$

The homogeneous condition is really necessary. For example, to require "$x_0 - 1 = 0$" makes no sense, since the points $(1 : 1 : 1)$ and $(2015 : 2015 : 2015)$ are the same.

It's trivial to verify that homogeneous polynomials do exactly what we want: hence we can do the following.

Definition 7.2. A **projective algebraic variety** in $\mathbb{CP}^2$ is the common zero locus of an arbitrary collection of homogeneous polynomials in $n+1$ variables.

Example 7.3

Let's try to picture the variety

$$\mathcal{V}(x^2 + y^2 - z^2) \subseteq \mathbb{CP}^2$$

which consists of the points $[x : y : z]$ such that $x^2 + y^2 = z^2$. If we take the U_i charts defined last time, we obtain the following:

- When $x = 1$, we get a hyperbola $\mathcal{V}(1 + y^2 - z^2)$.
- When $y = 1$, we get a hyperbola $\mathcal{V}(1 + x^2 - z^2)$.
- When $z = 1$, we get a circle $\mathcal{V}(x^2 + y^2 - 1)$.

Example 7.4

A second way to do this is picture the double cone

$$\mathcal{V}(x^2 + y^2 - z^2) \subseteq \mathbb{A}^3.$$

This cone has the property that if a point p is on it, then the entire line through the origin and p is in it. So you can think of the corresponding variety in $\mathbb{CP}^2$ as the set of lines which make up the double cone.

§7.2 Projective Analogues of Affine Results

Definition 7.5. Given $V \subseteq \mathbb{CP}^n$ we can consider

$$\mathcal{I}(V) = \{F \in \mathbb{C}[x_1, \dots, x_n] \mid F(p) = 0 \; \forall p \in V\}.$$

This is a homogeneous ideal of V.

Proposition 7.6

$\mathcal{I}(V)$ is radical and finitely generated.

Proof. Check it. $\square$

Theorem 7.7 (Homogeneous Nullestellensatz)

There is a natural bijection between projective varieties in $\mathbb{CP}^n$ and the radical ideas of $\mathbb{C}[x_1, \dots, x_n]$ with homogeneous generators *except* for the ideal $(x_0, \dots, x_n)$.

Unfortunately, because elements $\mathbb{C}[x_0, \dots, x_n]$ cannot be thought of as functions on a projective variety, we can't define the coordinate ring in the same way.

Nonetheless, we still can put the Zariski topology on $\mathbb{CP}^n$; the closed sets will be the projective subvarieties. (We can also take a Euclidean topology via our charts.) Naturally, we induce a topology on every subvariety of $\mathbb{P}^n$ as well.

§7.3 Transforming affine varieties to projective ones

Projective varieties are nice because they are *compact* under the Euclidean topology (and hence in the Zariski topology as well, since the Zariski topology is coarser than the Euclidean topology).

Definition 7.8. Let V be an affine algebraic variety

$$V \subseteq \mathbb{A}^n \subseteq \mathbb{CP}^n.$$

Then the **projective closure** $\overline{V}$ is the closure of V (as a set) in $\mathbb{CP}^n$ (with respect to either the Zariski or Euclidean topology; the answers turn out to be the same).

Note that we're embedding $\mathbb{A}^n$ (which has n coordinates) in $\mathbb{CP}^n$ (which has $n+1$ coordinates).

Example 7.9

Consider the set $V = \mathcal{V}(y - x^2) \subseteq \mathbb{A}^2 \subseteq \mathbb{CP}^2$, where $\mathbb{A}^2$ is thought of as the $z = 1$ chart in $\mathbb{CP}^2$. The lines through the origin passing through points $[t : t^2 : 1]$ the closure contains those lines plus one more: the point $[0 : 1 : 0]$ corresponding to the y-axis.

The closure $\overline{V}$ is more correctly thought of as

$$\overline{V} = \mathcal{V}(zy - x^2) \subseteq \mathbb{CP}^2.$$

That is we *homogenize* each of the polynomials.

In general, given $I \subseteq \mathbb{C}[x_1, \dots, x_n]$ cutting out an affine variety in $\mathbb{A}^n$, we can construct the ideal $\tilde{I} \subseteq \mathbb{C}[x_0, \dots, x_n]$ formed by homogenizing each of the polynomials in I. Then $\mathcal{V}(\tilde{I}) \subseteq \mathbb{CP}^n$ is the closure of $\mathcal{V}(I) \subseteq \mathbb{C}^n$.

Note that you really need to homogenize *all* the polynomial in the ideal; just homogenizing the generators is not sufficient.

Example 7.10 (A Warning: The Twisted Cubic)

Consider the ideal

$$\mathcal{V}(y - x^2, z - xy) \subseteq \mathbb{A}^3$$

which can be parametrized as the twisted cubic

$$TC = \left\{ (t, t^2, t^3) \mid t \in \mathbb{C} \right\}$$

One can check that $I = (y - x^2, z - xy)$ is in fact a radical ideal. But the variety

$$W = \mathcal{V}(wy - x^2, wz - xy) \subseteq \mathbb{CP}^3$$

is not what we want: it still has tuples $(1, t, t^2, t^3)$ but in addition contains the line $w = x = 0$: W contains an entirely new component.

§8 February 27, 2015

Last time we defined the projective space $\mathbb{CP}^n$ and consider projective closures $\overline{V}$ of varieties V.

§8.1 Ideals of Projective Closures: A Cautionary Tale

Theorem 8.1

Let $V \subseteq \mathring{A}^n \subseteq \mathbb{CP}^n$ be an affine algebraic variety. Let $I = \mathcal{I}(V)$, and let $\tilde{I}$ be the homogenization of I. Then $\tilde{I}$ is the ideal which cuts out $\overline{V} \subseteq \mathbb{CP}^n$.

Recall that

- The homogenization $\tilde{f}$ of a polynomial f is done adding on factors of an extra variable x_0 to each term of f so that it becomes homogeneous. For example the homogenization of $x_1^3 + x_2^2 x_3^5$ is $x_0^4 x_1^3 + x_2^2 x_3^5$.
- Recall that the homogenization of an ideal is the homogenization of *every* polynomial in I.

Example 8.2 (Cautionary Example: Twisted Cubic)

Again consider the twisted cubic. In $\mathbb{CP}^3$ we can consider its projective closure (in U_w) as cut out by

$$J = (xz - y^2, wy - x^2, wz - xy).$$

If we fix $w = 1$ then we recover the ideal

$$I = (xz - y^2, y - x^2, z - xy) = (y - x^2, z - xy).$$

But it's not the case that $J = (wy - x^2, wz - xy)$. Indeed, J only contains $w(xz - y^2)$ but not the function $xz - y^2$; as a result, $\mathcal{V}(J) \subseteq \mathbb{CP}^n$ cuts out the twisted cubic *as well as* the line $w = x = 0$.

§8.2 Proof of Theorem

Lemma 8.3 (On Homework)

Let I be an ideal. Then

(a) $\tilde{I}$ is an ideal.

(b) Assume I is radical. Then $\tilde{I}$ is radical.

Now we prove the theorem.

Proof. Set U_0 to be the $x_0 = 1$ plane in which we embed the original V.

First we show $\overline{V} \subseteq \mathcal{V}(\tilde{I})$. Let $G \in \tilde{I}$; we wish to show G vanishes on $\overline{V}$. WLOG G is homogeneous. It suffices to show that $V \subseteq V(G)$ (where $V(G)$ is the vanishing set of G), since $V(G)$ is closed in $\mathbb{CP}^n$ and it will follow that $\overline{v} \subseteq V(G)$. But $V = \overline{V} \cap U_0$, and G restricted to U_0 must vanish along V for tautological reasons.

Let's now prove $V(\tilde{I}) \subseteq \overline{V}$. It's equivalent to show $\tilde{I} \supseteq \mathcal{I}(\overline{V})$ by the nullstellensatz. So we want to show if G vanishes on $\overline{V}$ (in particular, it's homogeneous) then it's in $\tilde{I}$. But if G vanishes on $\overline{V}$ then it certainly vanishes on $\overline{V} \cap U_0 = V$, so $g(x_1, \dots, x_n) = G(1, x_1, \dots, x_n)$; hence $g \in I$. We would be done if $\tilde{g} = G$, but this is not exactly right, since if $G = x_0^3 x_1 + x_0 x_1 x_2^2$ then $\tilde{g} = x_0^2 x_1 + x_1 x_2^2$. What is true that $G = x_0^r \tilde{g}$; but $\tilde{g} \in \tilde{I}$ and multiples of $\tilde{g}$ should be in $\tilde{I}$; hence $G \in \tilde{I}$. $\square$

§8.3 Morphisms of projective varieties

Let $V \subseteq \mathbb{CP}^n$, $W \subseteq \mathbb{CP}^m$ and $V \xrightarrow{F} W$ a map of sets. We say it is a **morphism of projective varieties** if

> for every point $p \in V$, there exists a (Zariski) neighborhood $U_p \ni p$ in V and some homogeneous polynomials $F_0, \dots, F_m \in \mathbb{C}[x_0, \dots, x_n]$ such that the restriction to this neighborhood U_p is given by
>
> $$q \mapsto [F_0(q) : \dots : F_m(q)] \, .$$

For this definition to make any sense, $\deg F_0 = \deg F_1 = \dots = \deg F_m$; otherwise scaling q would mess up the coordinates. (It's okay if some polynomials are zero as long as the others are of the same degree.) Also, the polynomials should never be all zero at the same point. (This can be fixed – if all F_i simultaneously vanish on the points $T \subseteq U_p$ then T is closed so we can look at $U_p \setminus T$ instead.)

This should look familiar if you have experience with manifolds: just like a manifold should "look locally" like Euclidean space, a projective varieties should "look locally" like affine space and a map of projective varieties should "look locally" like an affine map.

§8.4 Examples of projective maps

Example 8.4

Consider the map $\mathbb{CP}^1 \to \mathbb{CP}^2$ by

$$[s : t] \mapsto [s^2 : st : t^2].$$

This is well-defined since it's homogeneous and doesn't hit the "point" $[0 : 0 : 0]$ for any point in $\mathbb{CP}^1$ (since $[0 : 0] \notin \mathbb{CP}^1$).

Note that the points in the range all land in the locus $xz - y^2 = 0$. So in fact we can think of this as a map to a curve C:

$$\mathbb{CP}^1 \to C = \mathcal{V}(y^2 - xz) \subseteq \mathbb{CP}^2.$$

In fact, this is an isomorphism.

In the above example, we're taking $U_p = \mathbb{CP}^1$ for every p.

§9 March 2, 2015

§9.1 Examples of Morphisms

Last time we discussed morphisms of projective varieties. Recall that a map $F : V \to W$ is a morphism is for every point $p \in V$, there's a (Zariski) neighborhood $U_p \ni p$ such that F is locally a polynomial map.

For example, the map $\mathbb{CP}^1 \to \mathbb{CP}^2$ by $[s : t] \mapsto [s^2 : st : t^2]$.

Example 9.1 (A Multi-Chart Example)

Let $C = \mathcal{V}(xz - y^2)$. We have a map $\mathbb{CP}^1 \to C$ by again $[s : t] \mapsto [s^2 : st : t^2]$.

Now we construct a map $C \to \mathbb{CP}^1$ by

$$[x : y : z] \mapsto \begin{cases} [x : y] & x \neq 0 \\ [y : z] & z \neq 0. \end{cases}$$

For this to be a well-defined map, we'd need for a point $x, z \neq 0$ to have $[x : y] = [y : z]$. This obviously not true for $\mathbb{CP}^2$ in general but is true by definition for the points on C. (Note that "$x \neq 0$" and "$y \neq 0$" are Zariski neighborhoods.)

§9.2 Isomorphisms of Projective Varieties

Definition 9.2. A morphism $F : V \to W$ of projective varieties is an **isomorphism** if it there is an inverse morphism $G : W \to V$ (meaning $F \circ G = \mathrm{id}_W$ and $G \circ F = \mathrm{id}_V$).

Example 9.3

We claim the above maps given an isomorphism from $\mathbb{CP}^1$ to C. We have

$$\mathbb{CP}^1 \to C \to \mathbb{CP}^1$$

by

$$[s : t] \mapsto [s^2 : st : t^2] \mapsto \begin{cases} [s^2 : st] & \text{if } s \neq 0 \\ [st : t^2] & \text{if } t \neq 0 \end{cases} = [s : t].$$

The other direction $C \to \mathbb{CP}^1 \to C$ is

$$[x : y : z] \mapsto \begin{cases} [x : y] & \text{if } x \neq 0 \\ [y : z] & \text{if } z \neq 0 \end{cases} \mapsto \begin{cases} [x^2 : xy : y^2] & \text{if } x \neq 0 \\ [y^2 : yz : z^2] & \text{if } x \neq 0 \end{cases} = [x : y : z]$$

using $y^2 = xz$ in both cases.

Recall that in affine varieties, isomorphisms of varieties induce varieties of coordinate rings. However, this is NOT true in projective space. The above isomorphism $\mathbb{CP}^1 \cong C$ is in fact a counterexample.

Indeed, the coordinate ring of $\mathbb{CP}^1$ is $\mathbb{C}[s, t]$ and the coordinate ring of C is $\mathbb{C}[x, y, z]/(xz - y^2)$. But as $\mathbb{C}$-algebras we have

$$\mathbb{C}[s, t] \ncong \mathbb{C}[x, y, z]/(xz - y^2).$$

Indeed, the left-hand side is the coordinate ring of $\mathbb{A}^2$ and the right-hand side is the coordinate ring of the double cone $xz - y^2 = 0$ in $\mathbb{A}^3$. And you wouldn't expect these to be isomorphic!

The point is that coordinate rings correspond to functions on an affine variety. Coordinate rings of projective varieties are best thought of as the functions on the corresponding affine cone.

§9.3 Projective Equivalence

Let's focus on a stricter set of morphisms of projective varieties. Suppose we want to specify a map

$$\mathbb{CP}^1 \to \mathbb{CP}^1$$

by

$$[s : t] \mapsto [F_0(s,t) : F_0(s,t)].$$

If we want this morphism to be *invertible*, then we better have $\deg F_0 = \deg F_1 = 1$. So we can this as

$$[s : t] \mapsto [as + bt : cs + dt]$$

which has a nice matrix representation by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix}.$$

So inverting this map is as easy as inverting a matrix.

It turns out these are the only automorphisms of $\mathbb{CP}^1$. Hence we might think that the automorphism group of $\mathbb{CP}^1$ is $\mathbf{GL}_2(\mathbb{C})$. But in fact, there is some redundancy. For example the maps T and $2015T$ are the same map. So we must mod out by the center of $\mathbf{GL}_2(\mathbb{C})$, a copy of $\mathbb{C}^*$. We call this the **projective general linear group** and put

$$\operatorname{Aut}(\mathbb{CP}^1) = \mathbf{PGL}_2(\mathbb{C}).$$

Remark 9.4. In complex analysis, the corresponds to the fact that the only biholomorphic maps $\mathbb{C} \to \mathbb{C}$ are the Möbius transformations

$$[s : 1] \mapsto \left[\frac{as+b}{cs+d} : 1 \right].$$

See `http://en.wikipedia.org/wiki/M%C3%B6bius_transformation#Projective_matrix_representations`.

In fact, it turns out these are the *only* automorphisms of $\mathbb{CP}^n$ in general; that is,

$$\operatorname{Aut}(\mathbb{CP}^n) = \mathbf{PGL}_n(\mathbb{C}).$$

In particular, the automorphisms of $\mathbb{CP}^n$ can be defined *globally*; we don't need to use charts to describe them.

We can extend this to any map of varieties.

Definition 9.5. Two projective varieties V, W of $\mathbb{CP}^n$ are **projectively equivalent** if there exists an isomorphism of $\mathbb{CP}^n$ sending one to the other.

Example 9.6

The varieties $\mathcal{V}(x)$ and $\mathcal{V}(y)$ inside $\mathbb{CP}^2$ are projectively equivalent under the map

$$[x : y : z] \mapsto [y : x : z].$$

§9.4 Quasi-projective varieties

We want to start thinking about projectively equivalent varieties as the same variety. More specifically, up until now we've been thinking of varieties as sets which *live inside* some ambient space like $\mathbb{A}^n$ or $\mathbb{CP}^n$. We want to start thinking of these varieties as *intrinsic objects*, the same way we think of a torus as just a torus, not "a torus in $\mathbb{R}^3$".

For now, let's define a *quasi-projective variety* is a *locally closed* subset of $\mathbb{CP}^n$ – that means it is the intersection of a closed and open set in the *Zariski topology*

- For example, each projective variety V is the intersection of V (which is closed) and the entire space $\mathbb{CP}^n$ (which is open).
- Each affine variety is also quasi-projective. The variety itself V is closed (when we embed $\mathbb{A}^n \subseteq \mathbb{CP}^n$ in the "$x_0 = 1$" plane). But in $\mathbb{CP}^n$, the plane $x_0 = 1$ really means $x_0 \neq 0$ (by scaling), and so the old U_0 chart is open.

§10 March 4, 2015

Recall that a quasi-projective variety is a locally (Zariski) closed subset of $\mathbb{CP}^n$.

§10.1 Morphisms (and Examples) of Quasi-Projective Varieties

Remark 10.1. Since the Zariski closed subsets are projective varieties, we can think of a quasi-projective variety as follows: **take some projective variety and throw away some points**.

Definition 10.2. Let $V \subseteq \mathbb{CP}^n$, $W \subseteq \mathbb{CP}^m$ be (quasi-projective) varieties. A map $F : V \to W$ is a **morphism of quasi-projective varieties** if for all $p \in V$, there exists homogeneous polynomials $F_0, \dots, F_m$ such that for some neighborhood $U_p \ni p$, the map $U \to \mathbb{CP}^m$ by

$$q \mapsto [F_0(q) : \dots : F_m(q)]$$

is well-defined and agrees with F.

Isomorphism is done in the obvious way.

Note: In what follows, we will begin shortening "quasi-projective variety" to just "variety".

Recall that projective varieties and affine varieties are varieties, as well as open subsets of projective/affine varieties as well.

§10.2 Affine quasi-projective varieties

Example 10.3

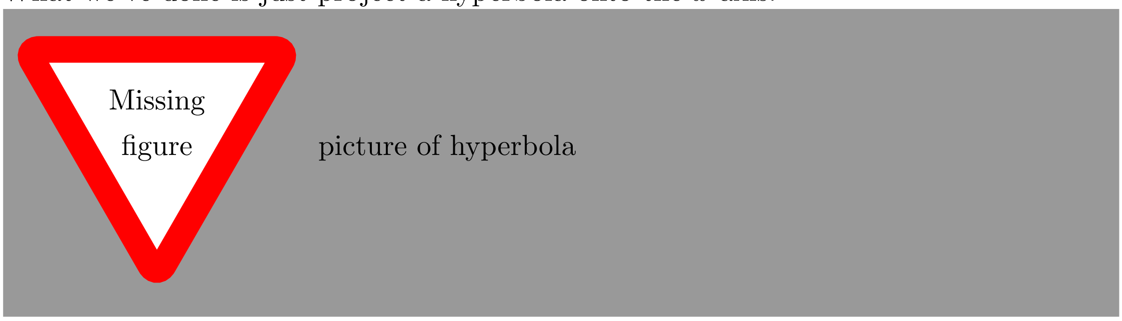
We've already seen $\mathbb{CP}^1$ minus a point, it's $\mathbb{A}^1$. So let's now consider

$$U = \mathbb{CP}^1 \setminus \{0, \infty\} \cong \mathbb{A}^1 \setminus \{0\}$$

is an example of a quasi-projective variety. However, we claim it's isomorphic to the *hyperbola* variety in $\mathbb{A}^2$ given by

$$V = \mathcal{V}(xy - 1).$$

What we've done is just project a hyperbola onto the x-axis:



Here's the actual map. Clearly the map $G : V \to W$ can just be written by $(x, y) \in \mathbb{A}^2 \mapsto (x) \in \mathbb{A}^1$. The inverse map is less obvious because we can't have $\frac{1}{t}$; we would like to write $(t) \mapsto (t, \frac{1}{t})$ but this is of course not valid.

We embed $\overline{V} \subseteq \mathbb{CP}^2$ by $\overline{V} = \mathcal{V}(xy - z^2) \subseteq \mathbb{CP}^2$, and intersect it with the chart $U_z \subseteq \mathbb{CP}^2$, i.e.

$$U_z \cap \overline{V} = V \quad [x : y : 1] \leftrightarrow (x, y).$$

Similarly, we identify $U \subseteq \mathbb{A}^1 \subseteq \mathbb{CP}^1$ by $t \leftrightarrow [t:1]$. So we can then write

$$\mathbb{CP}^1 \xrightarrow{\tilde{F}} \mathbb{CP}^2 \quad [a:b] \mapsto [a^2 : b^2 : ab].$$

On $\mathbb{A}^1 \subseteq \mathbb{CP}^1$, this is just

$$[t:1] \mapsto [t^2 : 1 : t] = \left[t : \frac{1}{t} : 1\right]$$

which is what we wanted. This gives us maps mutually inverse maps F and G, as needed.

Thus, we found that our attempt at an variety is in fact just a previous affine variety. Since we're trying to think about quasi-projective varieties as intrinsic objects, we thus have to say

Definition 10.4. A quasi-projective variety is **affine** if it is isomorphic to some affine algebraic variety in $\mathbb{A}^n$.

Example 10.5

$\mathbb{A}^2 \setminus \{0\}$ is a quasi-projective variety not isomorphic to any projective or affine variety.

Roughly, the idea is that the functions on the space look different than those on projective or affine varieties.

§10.3 Rings of functions

Since we're shortening "quasi-projective variety" to just "variety", we now say "affine variety" to mean "affine quasi-projective variety", and use "(Zariski) closed subset of $\mathbb{A}^n$" for the old meaning. Sorry!

We want to think of *rings of functions* now.

Definition 10.6. For W an affine variety, then the coordinate ring $\mathbb{C}[W]$ is defined to be the coordinate ring of *any* closed subset of $\mathbb{A}^n$ isomorphic to W.

On homework we'll check that this is well-defined.

Example 10.7

Let $U = \mathbb{A}^1 \setminus \{0\}$. Then $\mathbb{C}[U] \cong \mathbb{C}[x,y]/(xy-1)$, which one can also write this as $\mathbb{C}[x, \frac{1}{x}]$. The fact that this coordinate ring cannot be expressed as $\mathbb{C}[t]/I$ for some I reflects the fact that U cannot be isomorphic to an affine variety in one dimension.

§10.4 Complements of hypersurfaces

We now want to show that quasi-projective varieties can be thought of as results from *fusing*[1] *together multiple affine varieties.* hat is, we want a basis of open affine sets for any quasi-projective varieties.

Lemma 10.8

The complement of any hypersurface in a closed subset of $\mathbb{A}^n$ is affine.

[1] I hate glue.

This generalizes the case we did earlier, where we consider the complement of the origin in $\mathbb{A}^1$, giving $U = \mathbb{A}^1 \setminus \{0\}$ which is affine.

Note that a hypersurface is a locus cut out by *one* equation. That's why the lemma doesn't imply $\mathbb{A}^2 \setminus \{0\}$ is affine, because $\{0\}$ is cut out by two equations in $\mathbb{A}^1$.

Proof of Lemma. Suppose W is Zariski-closed in $\mathbb{A}^n$, and

$$V = \mathcal{V}(f) \subseteq \mathbb{A}^n$$

is some hypersurface. Then $W \cap V \subseteq W$ is also Zariski-closed. So the set in consideration is

$$W \setminus (W \cap \mathcal{V}(f))$$

and we want to show it's affine.

We do the same trick by going up a single dimension. Let W is cut out by $F_1, \ldots, F_r$. We will show there are isomorphisms

$$\mathbb{A}^n \supseteq U \longleftrightarrow V \overset{\text{def}}{=} \mathcal{V}(F_1, \ldots, F_r, zf - 1) \subseteq \mathbb{A}^{n+1}.$$

The map $G : V \to U$ should just be projection, while the map $U \to V$ should colloquially be

$$(x_1, \ldots, x_n) \mapsto \left(x_1, \ldots, x_n, \frac{1}{f} \right)$$

and again we have the same trick of passing into projective varieties to make the latter into a polynomial map.

Let's look at the details. Let's recall that

$$U = W \setminus \mathcal{V}(f) = \mathcal{V}(F_1, \ldots, F_r) \setminus \mathcal{V}(f)$$

and

$$V = \mathcal{V}(F_1, \ldots, F_r, zf - 1).$$

First, we want the map

$$G : V \to U \text{ by } (x_1, \ldots, x_n, z) \mapsto (x_1, \ldots, x_n).$$

This indeed maps into V by tautology, and moreover it does not land in $\mathcal{V}(f)$ because of the $zf - 1 = 0$ constraint. Similarly, we may write

$$F : U \to V \text{ by } (x_1, \ldots, x_n) \mapsto \left(x_1, \ldots, x_n, \frac{1}{f(x_1, \ldots, x_n)} \right).$$

Clearly this maps into V (that's how the last term is contrived), The only difficulty now is to show that this can indeed be recast as a polynomial map. The idea is to re-cast this as $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, z)$; the point is that when restricted to V, the map $\frac{1}{f(x_1, \ldots, x_n)} = z$ is indeed a polynomial. $\square$

§11 March 6, 2015

We begin by finishing the proof of the lemma from last time, thus establishing that complements of hypersurfaces in a Zariski closed subset of $\mathbb{A}^n$ are affine (quasi-projective) varieties.

§11.1 Quasi-projective varieties are covered by locally affine sets.

Let U be such a complement (so that U is a quasi-projective variety) and V the corresponding Zariski-closed set. (This is just the situation in the proof of the lemma.) By definition, we have

$$\begin{aligned}\mathbb{C}[U] \overset{\text{def}}{=} \mathbb{C}[v] &= \mathbb{C}[x_1, \dots, x_n, z]/\left(F_1, \dots, F_r, zf - 1\right)\\ &\cong \mathbb{C}[W][z]/(zf-1)\\ &= \mathbb{C}[W][1/f].\end{aligned}$$

Let $V \subseteq \mathbb{CP}^n$ be a quasi-projective variety. We have n charts U_0, U_1, ... of $\mathbb{CP}^n$ which look like affine space, given by $U_i = \{(x_0, \dots, x_n) \mid x_i \neq 0\}$.

Let $V_i = V \cap U_i$. Then V_i is some open set, and so we can write

$$\mathbb{A}^n \supseteq \mathcal{V}(f_1, \dots, f_s) \setminus \mathcal{V}(g_1, \dots, g_t)$$

for some f_i and g_i (closed sets are differences of open sets). Thus V_i is covered by

$$\mathcal{V}(f_1, \dots, f_s) \setminus \mathcal{V}(g_i) \; i = 1, \dots, t.$$

Then because the U_i cover $\mathbb{CP}^n$, we obtain that

> **Every quasi-projective is covered by a bunch of open affine sets in the Zariski topology.**

This is the same way that a manifold is a bunch of copies of Euclidean space. But surprisingly, for quasi-projective varieties the pieces might be in different dimensions.

In fact, you can show that one can get a *basis*, rather than just an open cover.

§11.2 Regular Functions

Now we want to have a notion of a function on a quasi-projective variety.

Definition 11.1. Let U be open and V an affine quasi-projective variety, with $U \subseteq V$. We say $f : U \to \mathbb{C}$ is **regular** at $p \in U$ if there exists $g, h \in \mathbb{C}[V]$ such that $f = g/h$ in some neighborhood of p (in particular $h \neq 0$ on this neighborhood).

We say f is regular on U if it is regular at each of its points.

Definition 11.2. Let $\mathcal{O}_V(U)$ denote the set of all regular functions on U.

Note that if $V \subseteq \mathbb{A}^n$ is a Zariski-closed set and $f, g \in \mathbb{C}[V]$ then f/g is regular on the affine variety $W \overset{\text{def}}{=} V \setminus \mathcal{V}(g)$. In fact, as we saw earlier

$$\mathbb{C}[W] \cong \mathbb{C}[V][\frac{1}{g}] \cong \mathbb{C}[V][z]/(zg-1).$$

So f/g can be identified with the *polynomial* zf on W.

The point is that rational functions should be thought of as polynomials on sufficiently small affine spaces.

Example 11.3 (Examples of Regular Functions)

Here are the examples.

(a) The function $f : \mathbb{A}^2 \to \mathbb{C}$ by $f(x, y) = (x + y)^2 + 2x^5$ is regular.

(b) The function $f : \mathbb{A}^2 \setminus \mathcal{V}(x) \to \mathbb{C}$ by $(x, y) \mapsto y/x$ is a regular function.

Another example is a function $\mathbb{CP}^1 \setminus \{[0 : 1]\} \to \mathbb{C}$ by $[x : y] \mapsto y/x$. This can be thought of as identifying U_0 to $\mathbb{A}^1$.

(c) (Projection from a point in $\mathbb{A}^2$) Pick a point $p \in \mathbb{A}^2$ and a line ℓ not containing p. Then for any $q \neq p$ we can let $f(q)$ be the intersection of the line through p and q with ℓ (unless the lines are not parallel). This gives a function

$$\mathbb{A}^2 \setminus \ell' \to \ell \cong \mathbb{C}$$

where ℓ' is the line through p parallel to ℓ.

Remark 11.4. Note that if $V \subseteq \mathbb{A}^n$ is Zariski-closed, then $\mathbb{C}[V] \subseteq \mathcal{O}_V(V)$. But it turns out this inclusion is equality.

§12 March 9, 2015

§12.1 Basis of Open Affine Sets

Last time we saw that a quasi-projective variety V has a cover by open affine sets. We claim that this gives

Theorem 12.1

The open affine sets form a basis of the Zariski topology of any quasi-projective variety V has a basis of open affine sets.

Proof. It suffices to show that any open cover $\{U_\alpha\}$ of V has a refinement (topology lemma).

Note that each U_α is *itself* a quasi-projective variety, so each U_α has a cover by open affines. This is the desired refinement. $\square$

§12.2 Regular Functions Continued

Recall that for an affine quasi-projective variety V, for an open set $U \subseteq V$ we had the ring of functions $\mathcal{O}_V(U)$.

We now prove the following result.

Theorem 12.2

Let V be an irreducible closed subset of $\mathbb{A}^n$. Then

$$\mathbb{C}[V] = \mathcal{O}_V(V).$$

In other words, if $g : V \to \mathbb{C}$ is regular on V then g is the restriction of some polynomial map in $\mathbb{C}[x_1, \dots, x_n]$.

(This actually holds even if V is reducible.)

Proof. Clearly $\mathcal{O}_V(V) \supseteq \mathbb{C}[V]$, so it suffices to show the reverse inclusion. Let $g : V \to \mathbb{C}$ be regular, meaning for any $p \in V$ there is a neighborhood $U_p \ni p$ such that $g = h_p/k_p$ on U_p (and $k_p(q) \neq 0$ for each $q \in U_p$).

We can replace each U_p with a smaller open affine set. Actually, we can even assume U_p is of the form $V \setminus \mathcal{V}(F_p)$ where F_p is some polynomial.

Since the Zariski topology is compact, we can take a *finite subcover*

$$U_1, \dots, U_t$$

so that each $U_t = V \setminus \mathcal{V}(F_t)$. On each U_i, we have $g = h_i/k_i$, where k_i doesn't vanish on U_i.

But on $U_i \cap U_j$ the representations of g must be equal, *id est*

$$\frac{h_i}{k_i} = \frac{h_j}{k_j}$$

which is enough to imply this equality on all of V, since $U_i \cap U_j$ is dense in V. (Here we've used the fact that V is irreducible.)

Now look at the ideal

$$I = (k_1, k_2, \dots, k_t)\,.$$

Because of the covering, the k_i does not vanish on any point of V. By Hilbert's Nullstellensatz this implies

$$I = \mathbb{C}[x_1, \dots, x_n]/\mathcal{V}(I).$$

Thus we have $\mathbb{C}[x_1, \dots, x_n]/\mathcal{V}(I) \ni 1 = \sum \ell_j k_j$. Over V we then identically have that

$$g = 1 \cdot g = \sum_{j=1}^{t} \ell_j k_j \frac{h_i}{k_i} = \sum \ell_j h_j \in \mathbb{C}[V]. \qquad \square$$

§12.3 Regular functions on quasi-projective varieties

Definition 12.3. Let $U \subseteq V$ be a Zariski-open subset of the quasi-projective variety V. A function $f : U \to \mathbb{C}$ is **regular** at $p \in U$ if there exists an open affine set U' on which f is regular at p.

As before f is regular on U if it's regular at each point of U. The set of such functions we again denote by $\mathcal{O}_V(U)$.

Some remarks.

Remark 12.4. $\mathcal{O}_V(U)$ has the structure of a ring, since it's a set of functions $U \to \mathbb{C}$ and one can check that point-wise sum/product of two functions works. In fact, this ring even has a copy of $\mathbb{C}$ in it (consider constant functions), meaning $\mathcal{O}_V(U)$ is a $\mathbb{C}$-algebra.

Remark 12.5. If $f \in \mathcal{O}_V(U)$, and $W \subseteq U$ is open. Then the restriction of f to W happens to be in $\mathcal{O}_V(W)$. This induces a natural ring homomorphism

$$\mathcal{O}_V(U) \to \mathcal{O}_V(W).$$

Remark 12.6. Suppose f_1, f_2 are regular functions on $U_1 \subseteq V_1$ and $U_2 \subseteq V_2$. If $f_1 = f_2$ on $U_1 \cap U_2$, then there is a unique function f on $U_1 \cup U_2$ which restricts to f_i on U_i, and f is regular.

You can do this for as many sets are you like (even infinitely many).

More generally, a structure like this is called a **sheaf**. You can do this very generally: a variety is a topological space X with a sheaf of functions on it.

One more property (unrelated to sheaves): given a morphism $F : V \to W$ of quasi-projective varieties and an open set $U \subseteq W$, we have a function

$$\mathcal{O}_W(U) \to \mathcal{O}_V(f^{-1}(U))$$

by $f \mapsto f \circ F$.

§12.4 Recasting morphism

Finally, one last definition.

Definition 12.7. A amp $\phi : V \to W$ of quasi-projective varieties is a **morphism** if for all $p \in V$, there exists a neighborhood $U \ni p$ and a neighborhood $U' \ni \phi(p)$ which are both affine such that

- $\phi(U) \subseteq U'$, and
- ϕ restricted to U agrees with an affine map.

This is finally a totally coordinate-free definition. And now we can start giving tons of examples. . .

§13 March 11, 2015

§13.1 Veronese Maps

(Pronounced "veh-roh-NAY-zee".)

The Veronese map is the first example of a "nontrivial" map. Here's the definition.

Definition 13.1. Consider any positive integers n and d. The **dth Veronese mapping of $\mathbb{CP}^n$**, denoted

$$\mathbb{P}^n \xrightarrow{\nu_d} \mathbb{P}^m$$

is given by

$$[x_0 : x_1 : \cdots : x_n] \mapsto \left[x_0^d : x_0^{d-1}x_1 : \dots\right]$$

where the right-hand side consists of *all* the monomials of degree d;[2] hence $m = \binom{d+n}{d} - 1$ (the -1 comes from the fact that $\mathbb{P}^m$ has $m+1$ coordinates.)

Example 13.2

The map $\nu_2 : \mathbb{CP}^1 \to \mathbb{CP}^2$ is given by

$$[s : t] \mapsto [s^2 : st : t^2]$$

and has image given by $\mathcal{V}(xz - y^2)$; in fact this is an isomorphism.

Example 13.3

The map $\nu_3 : \mathbb{CP}^1 \to \mathbb{CP}^3$ is given by

$$[s : t] \mapsto [s^3 : s^2t : st^2 : t^3].$$

This is the "twisted cubic" (or projective closure thereof), and an isomorphism onto its image. Writing $[x : y : z : w]$ for the coordinates of $\mathbb{CP}^3$ the ideal is

$$(xw - yz, z^2 - yw, y^2 - xz).$$

In fact, the Veronese maps are always isomorphisms onto the images! In other words,

Theorem 13.4

A Veronese map $\nu_d : \mathbb{CP}^n \to \mathbb{CP}^m$ is an embedding. In other words, the image of ν_d is a closed subvariety of $\mathbb{CP}^m$ and ν_d gives an isomorphism from $\mathbb{CP}^n$ to $\mathbb{CP}^m$.

Proof. Let W be the image of ν_d in $\mathbb{CP}^m$. Then one can verify that W is cut out by the ideal

$$\left\{z_I - z_J - z_K z_L \mid I, J, K, L \in \{0, 1, \dots\}^{n+1}, I + J = K + L\right\}$$

where if $I = (i_0, i_1, \dots, i_n)$ (of course $i_0 + i_1 + \cdots + i_n = d$), then z_I is the coordinate of $\mathbb{CP}^m$ corresponding to $x_0^{i_0} \dots x_n^{i_n}$.

[2] Stars and bars, anyone?

Now we wish to exhibit an inverse

$$\mathbb{CP}^n \xleftarrow{\Phi_d} W.$$

First, we cover W by the following $n+1$ open affines For $0 \le i \le n$ we consider the open affine $W_i \subseteq W$ where the coordinate indexed by x_i^d is not zero. (This is a much smaller portion of the $m+1$ covers from the standard cover; this is fine because W is images of points in $\mathbb{CP}^n$ and so any point in the image had better have a nonzero monomial.)

Now we define $\mathbb{CP}^n \xleftarrow{\Phi_d} W_i \subseteq W$ on these charts by sending

$$\left[x_i^{d-1}x_0 : x_i^{d-1}x_1 : \dots . : x_i^{d-1}x_n\right] \hookleftarrow \left[x_0^d : \dots : x_n^d\right]$$

which you should think of as just "projection" onto the n coordinates specified; upon reducing this of course equals $[x_0 : x_1 : \dots : x_n]$, but we write the form above to emphasize that this is in fact a map.

This is well defined as a map on W_i since $x_i^{d-1}x_i = x_i^d \neq 0$. Also, one can check that for any U_i and W_j, since it's trivial to check that

$$\left[x_i^{d-1}x_0 : x_i^{d-1}x_1 : \dots . : x_i^{d-1}x_n\right] = \left[x_j^{d-1}x_0 : x_j^{d-1}x_1 : \dots . : x_j^{d-1}x_n\right]$$

provided $x_i, x_j \neq 0$.

Finally, we can see that this map is clearly an inverse by the way we contrived it (suffices to check it for the standard charts U_i of $\mathbb{CP}^n$ and the charts W_i of W). $\square$

§13.2 Ring of regular functions on projective spaces

Definition 13.5. For a quasi-projective variety V, by "ring of regular functions on V" we just mean $\mathcal{O}_V(V)$.

Let V be an affine quasi-projective variety. We already showed last time that $\mathcal{O}_V(V) = \mathbb{C}[V]$ in this case. Today we will want to compute $\mathcal{O}_V(V)$ for V a projective variety. On the homework, we checked that for $V = \mathbb{CP}^n$, we have $\mathcal{O}_V(V)$ consists of only constant functions. We will show that this holds more generally.

Theorem 13.6

Let V be an irreducible projective variety. Then $\mathcal{O}_V(V) = \mathbb{C}$.

Proof. By hypothesis, V is Zariski closed in $\mathbb{CP}^n$. Suppose ϕ is regular on V, so we can take the standard charts U_i of $\mathbb{CP}^n$ and notice that $U_i \cap V$ is Zariski closed in $\mathbb{A}^n$. Hence $\bigcup_{i=0}^n U_i \cap V$ covers V.

Evidently for each i we now have

$$\phi|_{U_I \cap V} \in \mathcal{O}_{U_i \cap V}(U_i \cap V) = \mathbb{C}[U_i \cap V]$$

since $U_i \cap V$ is affine, so the restriction of Φ to $U_i \cap V$ can be thought of as

$$F_i \in \mathbb{C}[x_0, \dots, x_{i-1}, x_{i+1}, \dots x_n.$$

Let $\tilde{F}_i$ is the homogenization of F_i, and consider

$$F_i' \overset{\text{def}}{=} \frac{\tilde{F}_i}{x_i^d}$$

where $d = \deg F_i$. For example, if $F_0 = x_1^2 + x_2^3 x_1$ then $\tilde{F_0} = x_0^2 x_1^2 + x_1 x_2^3$ and $F_0' = \left(\frac{x_1}{x_0}\right)^2 + \frac{x_2^3 x_1}{x_0^4}$. Consequently, F agrees with F_i' on the chart U_i, even under scaling. Hence we have $\Phi = F_0' = F_1' = \dots$.

By adjusting, we may put

$$\phi = \frac{G_0}{x_0^N} = \dots = \frac{G_n}{x_n^N}$$

where G_i is some multiple of F_i' (the point is to make the denominators all the same degree).

We claim that ϕ satisfies a polynomial equation now. Consider any polynomial $g \in \mathbb{C}[V]$, (where $\mathbb{C}[V]$ we mean $\mathbb{C}[x_0, \dots, x_n]$ modded out by the homogeneous ideal generated by $\mathcal{I}(V)$). Assuming $\deg g \ge nN + 2015$, we have $\phi \cdot g \in \mathbb{C}[V]$, because each term in $\phi \cdot g$ must be divisible by some x_i^N. $\square$

§14 March 23, 2015

§14.1 More on Veronese Maps

Note to self: $\nu_2 : \mathbb{CP}^1 \to \mathbb{CP}^2$ and $\nu_3 : \mathbb{CP}^1 \to \mathbb{CP}^3$ are good "example maps". The example ν_2 can be written as

$$[s : t] \mapsto \left[s^2 : st : t^2\right]$$

(or any permutation) which has image $\mathcal{V}(xz - y^2)$. As for ν_3, given by

$$[s : t] \mapsto \left[s^3 : s^2t : st^2 : t^3\right],$$

the image is $\mathcal{V}(yz - xw, y^2 - xz, wy - z^2)$ (here the coordinates in $\mathbb{CP}^3$ are x, y, z, w in that order).

In general, a Veronese map $\nu_d : \mathbb{CP}^n \to \mathbb{CP}^m$ has some image cut out by polynomials $z_I z_J = z_K z_L$ and is an embedding (i.e. $\mathbb{CP}^n$ is isomorphic to its image in $\mathbb{CP}^m$).

Definition 14.1. In the case $n = 1$, the image of $\mathbb{CP}^1 \xrightarrow{\nu_d} \mathbb{CP}^d$ is called the **rational normal curve** of degree d.

In the special case above the set of vanishing guys can be thought of as the determinants of 2×2 minors of the matrix

$$\begin{pmatrix} z_{0,d} & z_{1,d-1} & \cdots & z_{d-1,1} \\ z_{1,d-1} & z_{2,d-2} & \cdots & z_{d,0} \end{pmatrix}.$$

Varieties cut out by such minors are called **determinental varieties**.

§14.2 Functions

We saw that if V is an irreducible projective variety, then $\mathcal{O}_V(V) = \mathbb{C}$.

From this we get the following corollary.

Corollary 14.2

If V is an irreducible projective variety which is isomorphic to an affine variety, then V consists of a single point.

Proof. One one hand, $\mathcal{O}_V(V) = \mathbb{C}$. On the other hand, $\mathcal{O}_V(V) = \mathbb{C}[V]$. From this we deduce the conclusion. □

§15 March 27, 2015

§15.1 Review of Projective Closures

Last time we talked about the *projective closure* of a variety. Given $V \subseteq \mathbb{A}^n$, we embedded in $\tilde{V} \subseteq \mathbb{CP}^n$ by embedding V in $\mathbb{CP}^n$ in a chart U_0 and taking the closure of the image. We proved the theorem that if $V = \mathcal{V}(I)$ then $\tilde{V} = \mathcal{V}(\tilde{I})$, where $\tilde{I}$ is the homogenization of I.

Proposition 15.1

$\tilde{V} \cap U_0$ is still canonically isomorphic to V.

Certainly $V \subseteq \tilde{V} \cap U_0$. Now suppose $p \in \tilde{V} \cap U_0$ is a projective point, meaning that for every $f \in I$ we have

$$\tilde{f}(1, x_1, \dots, x_n) = 0.$$

De-homogenizing, this implies $f(x_1, \dots, x_n) = 0$ for each f, as desired.

Here's another question: how do we compute $\tilde{I}$? Let $I = (f_1, \dots, f_r)$ and $J = (\tilde{f}_1, \dots, \tilde{f}_n)$. We've already seen that $\tilde{I} \neq J$ in general. However, here's a situation where it works.

Theorem 15.2

Let $I = (f_1, \dots, f_r)$ and $J = (\tilde{f}_1, \dots, \tilde{f}_n)$. Assume $V = \mathcal{V}(I)$ is nonempty. If $\mathcal{V}(J)$ is irreducible, then it equals $\tilde{V}$.

This is a sort of analog to the homework problem where we showed that if an affine variety V is irreducible, so is the projective closure $\tilde{V}$.

The canonical example is that if $I = (y - x^2, z - x^3)$ (twisted cubic), its homogenization contains a line at infinity not visible in U_0. This motivates our proof.

Proof. Let $W = \mathcal{V}(J)$. We will show that if W is not $\tilde{V}$, then it is reducible.

We claim that

$$W = \tilde{V} \cup (W \setminus U_0).$$

This is not difficult; moreover since U_0 is Zariski open, $W \setminus U_0 = W \cap (\mathbb{CP}^n \setminus U_0)$, so this is in fact a variety. By irreducibility of W, either

$$W = \tilde{V}$$

and we are done, or

$$W = W \setminus U_0 \implies W \cap U_0 = \varnothing$$

which implies V is empty. □

Finally, let's recall the following homework problem.

Proposition 15.3

If V and W are isomorphic affine varieties, then $\tilde{V} \cong \tilde{W}$.

Our hope is that a map $f : V \to W$ can be extended to a map $\tilde{f} : \tilde{V} \to \tilde{W}$. Unfortunately, this is not true. Take a map

$$\mathcal{V}(y - x^3) \to \mathbb{A}^1 \quad (x, x^3) \mapsto x.$$

We would have to extend this to a map

$$\mathcal{V}(yz^2 - x^3) \mapsto \mathbb{CP}^1 \quad [x : y : z] \mapsto [x : z].$$

(You can check that $(yz^2 - x^3)$ is prime.) Unfortunately, on the chart $y = 1$ this attempts to be an isomorphism $\mathcal{V}(z^2 - x^3) \to \mathbb{CP}^1$ which is not defined at $(0 : 1 : 0)$. This is the counterexample from the very beginning of the course.

In fact, even the x-projection $\mathbb{A}^2 \to \mathbb{A}^1$ works. This just shows in general that projective closure is not functorial.

§15.2 Enumerative Geometry

We work now exclusively in projective space.

Let's start with the simplest question.

Question 15.4. How many lines in the plane $\mathbb{CP}^2$ are there through two points?

As "we learned in kindergarten", the answer is 1. Let's work out the details briefly, and then generalize to harder questions from there. A line in $\mathbb{CP}^2$ corresponds to a vanishing set

$$\mathcal{V}(\alpha_0 x_0 + \alpha_1 x_1 + \alpha_2 x_2) \subseteq \mathbb{CP}^2$$

for some constants α_0, α_1, α_2. Meanwhile we can consider two points

$$\begin{aligned} p_1 &= [a_0 : a_1 : a_2] \\ p_2 &= [b_0 : b_1 : b_2] \end{aligned}$$

So we just want

$$\begin{aligned} 0 &= \alpha_0 a_0 + \alpha_1 a_1 + \alpha_2 a_2 \\ 0 &= \alpha_0 b_0 + \alpha_1 b_1 + \alpha_2 b_2. \end{aligned}$$

This is just a standard linear algebra problem. You get up to scaling exactly one solution except in some degenerate cases; this works as long as $p_1 \neq p_2$.

Indeed, the space of all lines in $\mathbb{CP}^2$ as above is *itself* a space: it corresponds exactly to a point in $\mathbb{CP}^2$. That is, to the line $\alpha_0 x_0 + \alpha_1 x_1 + \alpha_2 x_2$ we can associate the point $[\alpha_0 : \alpha_1 : \alpha_2]$. Then the condition "passes through p_1" corresponds to a line. It's all duality, you see...

We now move to considering conics. A conic is given by

$$\mathcal{V}(\alpha_0 x_0^2 + \alpha_1 x_0 x_1 + \alpha_2 x_0 x_2 + \alpha_3 x_1^2 + \alpha_4 x_1 x_2 + \alpha_5 x_2^2).$$

So the parameter space is $\mathbb{CP}^5$. Thus in exactly the same way, five general points determine a conic.

§16 March 30, 2015

§16.1 Glimpses of Enumerative Geometry

Last time we showed that there is one conic through five points in general position.

You can ask marginally more interesting questions, like

> How many degenerate conics pass through four points p_1, p_2, p_3, p_4?

The set of degenerate conics D is a codimension one space in $\mathbb{CP}^5$. So instead of

$$\bigcap_{i=1}^{5} V_{p_i}$$

being a point like last time, we have

$$D \cap \bigcap_{i=1}^{4} V_{p_i}$$

which is D intersect a line. For dimension reasons we certainly expect it to be finite, but it's not at all clear what the cardinality should be.

Unfortunately, the particular problem for conics is trivial just because a degenerate conic consists of two lines.

Proposition 16.1

There are exactly three degenerate conics through four points in general position.

Proof. A degenerate conic is two lines, so we're just pairing off four points into two lines. There are $\frac{1}{2}\binom{4}{2} = 3$ ways to do this. $\square$

But very hard problems if we go to higher dimension curves: given a degree d curve in $\mathbb{CP}^2$, we can consider the number of singular curves passing through $\binom{d+2}{2} - 1$ points in general position, or even the number of curves with k singular points. The varieties in the parameter space of these singular d-curves is called a **Severi variety**.

The first nontrivial such question is the following:

> How many singular cubics in $\mathbb{CP}^2$ pass through eight points in general position?

The general problem is very, very hard.

In general, given a k-dimensional Zariski closed set, we define the **degree** of V is the number of points in the intersection of V with an $(N-k)$-dimensional linear space. So all the above can be rephrased as finding the degree of the Severi varieties.

Remark 16.2. It turns out that that the image of $\nu_d : \mathbb{CP}^n \to \mathbb{CP}^m$ is in fact degree d. For example, the twisted cubic (the image of $\nu_3 : \mathbb{CP}^1 \to \mathbb{CP}^3$) has degree 3 (since a plane tends to hit it at three points) and the image of the parabola is (predictably) 2.

§16.2 Segre Map

This is another classical example of a projective embedding. The **Segre map** (pronounced SEG-ray) is a map of sets

$$s_{n,m} : \mathbb{CP}^n \times \mathbb{CP}^m \hookrightarrow \mathbb{CP}^N$$

by

$$[x_0, \dots, x_n], [y_0, \dots, y_m] \mapsto [x_0y_0 : x_0y_1 : \dots x_ny_m].$$

Here $N = (n+1)(m+1) - 1$. Again, we'll let z_{ij} denote the coordinate x_iy_j.

(This is a *map of sets* because we haven't put a variety structure on $\mathbb{CP}^n \times \mathbb{CP}^m$. We will later.)

Let's focus specifically on the $n = m = 1$ case. It's cut out by

$$\mathbb{CP}^1 \times \mathbb{CP}^1 \hookrightarrow \mathbb{CP}^3$$

by

$$[u : v], [s, t] \mapsto [us : ut : vs : vt].$$

Here are some properties of this map.

- We're going to find out in a moment that this is an embedding, so we expect the image $S \simeq \mathbb{CP}^1 \to \mathbb{CP}^1$ to be a two-dimensional surface.

  It is in fact true that a codimension one affine variety is in fact cut out by single equation, *id est* is of the form $\mathcal{V}(f)$. And you can see the equation pretty readily:

  $$S = \mathcal{V}(z_{00}z_{11} - z_{01}z_{10}).$$

- If we consider a line in the domain, then its image is a plane. For example, if we fix a c we get

  $$[c : 1], [s : t] \mapsto [cs : ct : s : t]. = \left[c\frac{s}{t} : c : \frac{s}{t} : 1\right].$$

  As Evan o'Dorney remarks: the image is cut out by the equations $z_{00} = cz_{10}$ and $z_{01} = cz_{11}$; hence the image is the intersection of two planes, *id est* a line. Thus *the image of a line is actually a line.*

  This should be surprising, being isomorphic to $\mathbb{CP}^1$ is *way* different than actually being a straight line!

  Even trickier, if we let C vary we find that S *is covered by lines.* And similarly, S is covered by lines going in the other direction.

- Finally, if you interpret $\mathbb{CP}^1$ as a circle (ignoring the fact that $\mathbb{C}$ has dimension one...) then $\mathbb{CP}^1 \times \mathbb{CP}^1$ looks like a torus.

§17 April 1, 2015

§17.1 Segre maps continued

Denote by $\Sigma_{m,n}$ the image of the Segre map

$$\mathbb{CP}^n \to \mathbb{CP}^m \xrightarrow{s_{n,m}} \mathbb{CP}^{(m+1)(n+1)-1}.$$

Last time we saw that $\Sigma_{1,1}$ is a quadric hypersurface in $\mathbb{CP}^3$. We now show (in the notation of last time) the following proposition.

Proposition 17.1
We have

$$\Sigma_{1,1} = \mathcal{V}(z_{00}z_{11} - z_{01}z_{10}).$$

as subsets of $\mathbb{CP}^3$.

Proof. Easy but annoying. Just an algebra exercise to show that $ab = cd$ if and only if $a = wx$, $b = yz$, $c = xy$, $d = wz$ for some w, x, y, z. (For the nontrivial direction you casework on coordinates being zero.) $\square$

You can picture (the real part of) $\Sigma_{1,1}$ as follows:

> Take two metal circle rings, hold one directly above the other, and tie strings between corresponding points on the circle. Twist a little.

§17.2 Topology of $\mathbb{CP}^m \times \mathbb{CP}^n$

In this way, we have a map of sets

$$\mathbb{CP}^m \times \mathbb{CP}^n \twoheadrightarrow \Sigma_{m,n} \subseteq \mathbb{CP}^{(m+1)(n+1)-1}.$$

Thus

Definition 17.2. We endow $\mathbb{CP}^n \times \mathbb{CP}^m$ with the structure of a projective variety through the Segre map $\Sigma_{n,m}$.

Remark 17.3. A second way to define this is to consider functions on $\mathbb{CP}^1 \times \mathbb{CP}^1$ as **bihomogeneous polynomials**: given pairs $[u : v]$ and $[x : y]$ of points we wish to obtain polynomials which are homogeneous in each pair, say

$$u^2y + 2uvx - uvy.$$

Remark 17.4. Asking what the Zariski topology of a space is equivalent to asking for the closed sets, *id est* asking for its functions.

§17.3 Projection

We can project from $\Sigma_{1,1}$ onto its coordinates as follows.

$$\begin{tikzcd}
 & {[a_{00} : a_{01} : a_{10} : a_{11}]} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
{[a_{00} : a_{10}] = [a_{01} : a_{11}]} & & {[a_{10} : a_{11}] = [a_{00} : a_{01}]}
\end{tikzcd}$$

according to which one doesn't give a bunch of zeros.

§17.4 General Equations Cutting Out $\Sigma_{n,m}$

So this concludes our examination of $\Sigma_{1,1}$ in great detail. The general case is exactly the same: consider

$$\mathbb{CP}^n \times \mathbb{CP}^m \xrightarrow{s_{n,m}} \mathbb{CP}^N.$$

Theorem 17.5

As a set, $\Sigma_{n,m} = \mathcal{V}(I)$, where I is the ideal generated by the 2×2 minors of the matrix

$$\begin{pmatrix} z_{00} & \dots & z_{0n} \\ \vdots & \ddots & \vdots \\ z_{m0} & \dots & z_{mn} \end{pmatrix}.$$

Proof. Essentially the same as the proof for $\Sigma_{1,1}$.

Let's write out the details for once. Obviously $\mathcal{V}(I) \subseteq \Sigma_{m,n}$. Take $a \in \mathcal{V}(I)$. Let $a_{pq} \neq 0$ be a nonzero entry in a point

$$a = \begin{pmatrix} a_{00} & \dots & a_{0q} & \dots & a_{0m} \\ a_{p0} & \dots & a_{pq} & \dots & a_{qm} \\ a_{n0} & \dots & a_{nq} & \dots & a_{nm} \end{pmatrix} \in \mathcal{V}(I)$$

and WLOG $a_{pq} = 1$. (Here we're writing the points in matrix form for convenience). Now let $P \subseteq \mathbb{CP}^n$ be the point corresponding to the column containing a_{pq} and $Q \subseteq \mathbb{CP}^m$ the point corresponding to the row containing a_{pq}. We claim $\Sigma_{m,n}(P, Q) = a$. Compute

$$\Sigma_{m,n}(P, Q) = \begin{pmatrix} a_{0q} \\ \vdots \\ a_{nq} \end{pmatrix} \cdot \begin{pmatrix} a_{p0} & \dots & a_{pm} \end{pmatrix}.$$

Hence the (i, j)th coordinate is

$$a_{iq}a_{pj} \overset{\mathcal{V}(I)}{=} a_{ij}a_{pq} = a_{ij}$$

Thus $\Sigma_{m,n}(P, Q) = a$ as desired. $\square$

§18 April 3, 2015

We now know that $\Sigma_{n,m}$ is a projective variety.

Finally, we verify that

Proposition 18.1

As sets, $S_{m,n}$ is a bijection between $\mathbb{CP}^n \times \mathbb{CP}^m$.

Proof. All we have to do is check that it's injective, which is obvious. □

§18.1 Grassmanians

Definition 18.2. The **Grassmanian** $\operatorname{Gr}(k,n)$ as a set is the set of k-planes in $\mathbb{C}^n$ through the origin.. (Variety structure to be given later.)

Example 18.3 (Trivial Grassmanians)

$\operatorname{Gr}(n,n)$ has exactly one element; there is only one n-plane in $\mathbb{C}^n$ at all! Similarly, $\operatorname{Gr}(0,n)$ consists of the points passing through the origin of $\mathbb{C}^n$; hence $\operatorname{Gr}(0,n)$ is also isomorphic to a point.

It is true that

Proposition 18.4

$\operatorname{Gr}(k,n) \simeq \operatorname{Gr}(n-k,n)$.

Proof. Orthogonal complements. We won't do much with this in this class, since orthogonal thing aren't too geometric. □

Example 18.5 ($k = 1$ gives Projective Space)

We have

$$\operatorname{Gr}(1,n) \cong \mathbb{CP}^{n-1}$$

because it consists of the lines in $\mathbb{C}^n$.

Let's move on to more Grassmanians. Since $\operatorname{Gr}(2,3) \simeq \operatorname{Gr}(1,3)$, the next simplest thing we can think of is $\operatorname{Gr}(2,4)$.

Remark 18.6. (Evan o'Dorney): Because we're in the complex numbers, the line orthogonal to a plane can in fact lie in the plane. "You can think about that."

Example 18.7

Consider $\operatorname{Gr}(2,4)$. It is the space of 2-planes in $\mathbb{CP}^4$. We can mod out by scaling against $\mathbb{CP}^4$, since the planes are "nice".

So in addition the description of planes in $\mathbb{CP}^4$, we can imagine $\operatorname{Gr}(2,4)$ as the set of lines in $\mathbb{CP}^3$; when thought of this way we write

$$\mathbb{G}(1,3).$$

In general, for $n, k \geq 1$ we can put

$$\mathrm{Gr}(n,k) \simeq \mathbb{G}(n-1, k-1)$$

by the same interpretation above.

§19 April 6, 2015

Today we want to construct $\mathrm{Gr}(k, n)$ and consider it as an algebraic variety.

§19.1 Representation of Grassmanians as Matrices

Interpret $\mathrm{Gr}(2, 4)$ as the set of 2-planes. Any basis can represent a 2-plane, so we can think of elements of $\mathrm{Gr}(2, 4)$ as pairs of linearly independent vectors in $\mathbb{C}^4$. Thus we can write it as

$$\begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{pmatrix}.$$

Of course, each plane has many names. So we want to mod out by $\mathbf{GL}_k(\mathbb{C})$, the possible automorphisms of the plane: thus we can think of

$$\mathrm{Gr}(2, 4) \cong \{2 \times 4 \text{ matrices of full rank}\} / \text{action of } \mathbf{GL}_k(\mathbb{C}).$$

Thus in general, we may set

$$\mathrm{Gr}(k, n) = \{k \times n \text{ matrices of full rank}\} / \mathbf{GL}_k(\mathbb{C}).$$

§19.2 Dimension of the Grassmanian

Proposition 19.1
The dimension of $\mathrm{Gr}(k, n)$ is $k(n - k)$.

Proof. Since the set of full rank $k \times n$ matrices has dimension kn, while $\dim \mathbf{GL}_k(\mathbb{C})$ has dimension k^2, the dimension is

$$kn - k^2 = k(n - k).$$

□

Note that this matches the duality $\mathrm{Gr}(n - k, n) \cong \mathrm{Gr}(k, n)$.

§19.3 Embedding Grassmanian into Projective Space

We can embed a Grassmanian

$$\mathrm{Gr}(n, k) \hookrightarrow \mathbb{CP}^{\binom{n}{k}-1}$$

as follows

$$\begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{k1} & \dots & a_{kn} \end{pmatrix} \mapsto [\dots, \Delta_{i_1, \dots, i_k}, \dots].$$

We need to check this is well-defined on the quotient; that is for $T \in \mathbf{GL}_k(\mathbb{C})$ and M a $k \times n$, we want M and TM to get sent to the same point. But in fact the T just multiplies all the coordinates by $\det T$ for linear algebra reasons.

This map is called the **Plücker embedding**. It happens to be injective, though we won't prove it.

§19.4 Grassmanian is a Variety

Now we want to show that the image is a variety. We *could* write out the equations which cut it out, but this is very complicated. Instead, it turns out we can invoke the following result.

Theorem 19.2 (Chow's Theorem)
Every compact complex submanifold of $\mathbb{CP}^n$ is a projective variety. Moreover, meromorphic functions between such submanifolds are in fact morphisms of varieties.

We won't show this is compact either... but the point is that the right picture is that it's a compact submanifold.

Moduli spaces (beyond the scope of this class) provide a third way of not having to write out polynomials.

§20 April 8, 2015

§20.1 Grassmanian is a Complex Manifold

We have our usual setup

$$\mathrm{Gr}(k,n) \hookrightarrow \mathbb{CP}^n.$$

We'll demonstrate that $\mathrm{Gr}(k,n)$ has a chart U by

$$\Delta_{i_1,\dots,i_k} \neq 0.$$

Suppose $\Lambda \in \mathrm{Gr}(k,n)$ satisfies this property, meaning its first k columns of the associated matrix M form an invertible matrix G. Then

$$G^{-1}M = \begin{pmatrix} 1 & \dots & 0 & a'_{1,k+1} & \dots & a_{1,n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 1 & a'_{k,k+1} & \dots & a_{k,n} \end{pmatrix}.$$

Hence every element of U has a canonical representative given by the last $n-k$ columns. Thus $U \subseteq \mathbb{CP}^N$ is isomorphic to $\mathbb{C}^{k(n-k)}$.

§20.2 Degree of a Variety

Definition 20.1. Let $V \subseteq \mathbb{CP}^n$ be a projective subvariety. Then the **degree** of V is the maximum (finite) number of intersect points with a linear subspace (of the correct dimension).

The parenthetical remark is just that we expect a dimension k subvariety of $\mathbb{CP}^n$ to intersect a dimension $n-k$ subvariety in a nonzero but finitely many points.

Adult algebraic geometers use multiplicity counting rather than "maximum finite number".

Theorem 20.2

Let F be an irreducible homogeneous polynomial of degree d. Then $\mathcal{V}(F) \subseteq \mathbb{CP}^n$ has degree d.

Proof. Straightforward. Polynomial bash to reduce it to the fact that a general degree d polynomial in one variable has exactly d roots. $\square$

Another remark is made about schemes keeping track of multiplicity.

§20.3 Degree is not preserved under isomorphism!

Astonishingly, degree is not preserved! Just consider

$$\nu_2 : \mathbb{CP}^1 \to \mathbb{CP}^2$$

which maps a line to a "parabola". Now $\mathbb{CP}^1$ has degree one while $\mathbb{CP}^2$ has degree two!

More generally,

$$\nu_d : \mathbb{CP}^1 \to \mathbb{CP}^d$$

maps degree one varieties to degree d varieties. (Proof next time.)

We do have a little preservation.

Proposition 20.3

Degrees of varieties are invariant under automorphisms of $\mathbb{CP}^n$.

Proof. Linear subspaces get sent to linear subspaces under automorphisms of $\mathbb{CP}^n$ (since these are just linear transformations by definition). $\square$

§21 April 10, 2015

Last time we defined the degree, which is *not* intrinsic to the variety and rather reflects how a variety sits inside $\mathbb{CP}^n$. In particular, we saw that if $V \subseteq \mathbb{CP}^n$ is an (irreducible reduced) hypersurface, then $\deg V$ is the degree of the associated polynomial.

§21.1 Degree of the Veronese Map

We will now show that $\nu_3 : \mathbb{CP}^1 \to \mathbb{CP}^3$, by

$$[s : t] \mapsto [s^3 : s^2t : st^2 : t^3].$$

has degree 3. In other words, we want to count the number of points of intersection of $C \cap H$, where C is the twisted cubic (image of ν_3) and H is a (dimension two) hyperplane.

Label the coordinates of $\mathbb{CP}^3$ by $[x : y : z : w]$. Let's take a very convenient plane H by $x = 0$ to begin with. Solving, $x = s^3 = 0 \implies s = 0$ and hence there's only a single point $[0 : 0 : 0 : 1]$. Hence $\deg V \ge 1$. But as you might guess, we can probably get more than that. (In real algebraic geometry, we fortunately have multiplicity and $[0 : 0 : 0 : 1]$ would somehow get counted three times).

We might try $y = 0$, but there are only points $[1 : 0 : 0 : 0]$ and $[0 : 0 : 0 : 1]$.

OK, now let's try $x = w$. Then we have $s^3 = t^3$ and hence s, t cannot be zero. So the possible values of $s : t$ are the three cube roots of unity, which gives us three points of intersection.

Show rigorously that ≤ 3 points happen is possible, but not much fun; again multiplicity takes care of everything if we had it. For our purposes we just note that a hyperplane amounts to

$$As^3 + Bs^2t + Cst^2 + Dt^3 = 0$$

and for any choice of constants A, B, C, D not all zero there are at most three possible values of $s : t$. (One has a sense that we are *pulling back* the plane in $\mathbb{CP}^n$ into $\mathbb{CP}^1$, where we can bring to bear our single-variable polynomials.)

In any case the obvious generalization holds with the same proof.

Theorem 21.1 (Degree of Veronese Maps)
The image of the Veronese map $\nu_d : \mathbb{CP}^1 \to \mathbb{CP}^d$ has degree d.

§21.2 Complete Intersections and Degrees

It's worth nothing that this is a counterexample to a hopeful conjecture: the image C of ν_3 is cut out by three *quadrics* (degree two surfaces), but $2^3 \neq 3$.

Remark 21.2. If we take just two of the quadrics Q_1, Q_2, cutting out C and then in fact the intersection $Q_1 \cap Q_2$ is C plus a line, which has degree $3 + 1 = 4$. This works about as expected ($2^2 = 4$). But something weird happens when we intersect with Q_3: we have $Q_1 \cap Q_2 \cap Q_3 = C$. So intersection of Q_3 just threw away the extra line. In particular, *intersecting with Q_3 doesn't even change the dimension.*

In fact, C is weird in the sense that it is cut out by *three* equations cut out a dimension one surface in $\mathbb{CP}^3$. This is in some sense the reason why 2^3 does not correctly compute the degree of V.

For nicer situations (like $Q_1 \cap Q_2$ above).

Remark 21.3. V is called a **complete intersection** if the codimension of V equals the minimal number of generators in the ideal $\mathcal{I}(V)$.

This gives us our generalization of $\mathcal{V}(F)$.

Theorem 21.4

If V is a complete intersection and $F_1, \dots, F_r$ are a minimal set of generators for $\mathcal{I}(V)$, then

$$\deg V = \deg F_1 \dots \deg F_r.$$

Proof omitted, but this is just to give some intuition for why these things behave the way that they do. We also won't actually *use* the theorem either, because we still don't actually have a rigorous handle on dimension either.

The Veronese curve C is really the first example of a surface which is not a complete intersection.

§21.3 Curves in Projective Plane

Higher dimensions are difficult, but let us do one thing in $\mathbb{CP}^2$.

Theorem 21.5 (Bezout)

If C and D are (distinct irreducible) curves in $\mathbb{CP}^2$ of degree c and d, then $C \cap D$ has at most cd points of intersection, and exactly cd points with multiplicity.

The "irreducible" condition is to avoid infinity by accident: for example, if $C = \mathcal{V}(y^2 - x^2)$ and $D = \mathcal{V}(y - x)$.

Higher dimension analogs in fact hold.

Remark 21.6. After "dimension" and "degree", we can think of something called a **Hilbert function**, which is a polynomial encoding both dimension and degree. (This again depends on embeddings.)

§21.4 How do you tell apart curves?

In some sense, there are two types of ways to study curves: curves as embedded objects, and curves as isomorphism of classes.

If we have two curves, we might first start by looking at the so-called genus to try and distinguish them. (Genus makes sense here: one-dimensional complex curves are two-dimensional surfaces.) If that didn't work, maybe we would count singularities. But if it turns out, say, they're both smooth, then it's *really hard*.

One possible way is using the so-called line bundles. Another way is perhaps using maps.

Remark 21.7. Genus is an example of a *moduli space*, the analog of a parameter space for isomorphism classes.

For example, there is only one more isomorphism class for genus zero. For genus one, we get a moduli space of $\mathbb{CP}^1$, which can be handled in some ways. For example, elliptic curves have a computable *j-invariant* which in fact determines the isomorphism class of elliptic curves completely. But in genus two and higher, the situation is just very complicated. . .

§22 April 13, 2015

§22.1 Tangent Space of Affine Spaces

Let $V = \mathcal{V}(f_1, \dots, f_n)$ be a variety and $p \in V$ a point. We want to consider the **tangent space** to V at the point p, which is the union of all the lines which intersect V with multiplicity greater than one. This can be defined in one variable, as follows. Without loss of generality, call p the origin. Since a line through the origin can be parametrized as

$$\ell = \left\{ (tx_1, \dots, tx_n) \mid t \in \mathbb{A}^1 \right\}$$

we can substitute this into each of the F_i, and look at $F_i(tx_1, \dots, tx_n)$ for each i. Then it's tangent if t is a double root for *every* i.

Example 22.1
At the origin, the parabola $\mathcal{V}(y - x^2)$ has tangent space $\mathcal{V}(y)$, the x-axis.

The main theorem is the following.

Theorem 22.2
Let $V = \mathcal{V}(F_1, \dots, F_n)$ and p a point in it. Denote

$$G_i(\vec{x}) = (dF_i)_p(\vec{x} - p)$$

for each i, where (dF_i) is the total derivative. Then $T_pV = \mathcal{V}(G_1, \dots, G_n)$; in particular T_pV is linear.

Moreover, T_pV does not depend on the choice of generators.

Remark 22.3. This is especially easy when $p = 0$, since $(dF_i)_p$ is just the linear part of dF_i.

Here we want to regard dF_i as a formal operation, since the F_i are all polynomials.

Proof. Just use the formal properties of the derivatives to deduce the first part.

Now we just want to show that this doesn't depend on the choice of generators If $\tilde{F}_1$, ..., $\tilde{F}_m$ is another set of generators, then we can put

$$\tilde{F}_i = \sum_{j=1}^{n} H_{ij} F_j$$

for $i = 1 \dots, m$. Taking the total derivative at $p = 0$, and using the product rule gives

$$(d\tilde{F}_i)_p = \sum_{j=1}^{n} H_{ij}(p)(dF_j)_p + F_j(p)(dH_{ij})_p = \sum_{j=1}^{n} H_{ij}(p)(dF_j)_p.$$

Thus $\mathcal{V}(\{\tilde{F}_i\}) \supset \mathcal{V}(\{F_i\})$; reverse inclusion follows by symmetry. $\square$

Example 22.4

Let us find the tangent space to the parabola $\mathcal{V}(y - x^2)$ at a point (a, a^2). Set $u = x - a$ and $v = y - a^2$, so the parabola becomes

$$y - x^2 = (v + a^2) - (u + a)^2 = v - u^2 - 2au.$$

Taking the linear part to get

$$\mathcal{V}(v - 2au) = \mathcal{V}(y - a^2 - 2a(x - a)) = \mathcal{V}(y - 2ax + a^2).$$

§22.2 Tangent Spaces in Projective Space

Let $V \subseteq \mathbb{CP}^n$ be a projective variety. One could take an affine chart, compute the tangent space of that chart, and then take the closure.

Alternatively, take the affine cone $\tilde{V} \in \mathbb{A}^{n+1}$. Take the tangent space of p of $\tilde{V}$. It is some plane passing through the origin, and thus gives rise to a tangent space in $\mathbb{CP}^n$ by reversing the cone operation.

§22.3 Tangent Spaces Determine Singular Points

It turns out to be true that

Theorem 22.5

If V is a variety, then

$$\dim T_p V \geq \dim V.$$

Thus for $p \in V$, if $\dim T_p V = \dim V$ then we say p is **smooth**; otherwise p is **singular**. Thus intuitively the dimension of the tangent space *usually* equals $\dim V$ and the singular points are the "special" points.

§23 April 22, 2015

We're doing an abridged version of Chapter 8 in Smith. Highlights include Riemann-Roch.

§23.1 Vector Bundles and Line Bundles

Definition 23.1. A **vector bundle** of rank n consists of the following.

1. Varieties E (the **total space**) and X (the **base space**),
2. A projection map $\pi : E \to X$, and
3. An open cover U_i of X such that
$$\pi^{-1}(U_i) \simeq U_i \times \mathbb{C}^n.$$

It is required that the diagram

$$\begin{tikzcd} \pi^{-1}(U_i) \arrow[rr, leftrightarrow, "\phi_i"] \arrow[dr, "\pi"'] & & U_i \times \mathbb{C}^n \arrow[dl, "\mathfrak{p}"] \\ & U_i & \end{tikzcd}$$

commutes; the ϕ_i is called a **local trivialization**. Moreover, we require that the isomorphisms are linearly compatible, in the sense that on $U_i \cap U_j$, the map

$$(U_i \cap U_j) \times \mathbb{C}^n \xrightarrow{\phi_j \circ \phi_i^{-1}} (U_i \cap U_j) \times \mathbb{C}^n$$

by

$$(x, v) \mapsto (x, \phi_j \circ \phi_i^{-1}(v))$$

is a linear map of $\mathbb{C}^n$ for each fixed x.

We call it a **line bundle** if it has rank 1.

Line bundles are useful since (a) we have tangent bundles to curves, and (b) they describe maps to projective space (?). So you can understand curves well if you understand the line bundles well.

By abuse of notation, we will abbreviate the vector bundle to $E \xrightarrow{\pi} X$ or E.

Picture: you have a variety X, and a copy of $\mathbb{C}^n$ floating above each point of $p \in X$. We denote this copy by E_p, and treat it as a vector space.

§23.2 Sections

Definition 23.2. If we have a diagram

$$E \xrightarrow{\pi} X \xrightarrow{\sigma} E$$

which composes to the identity, then σ is called a **global section**.

Picture: σ takes each point $p \in X$ to a point $\sigma(p) \in E_p$.

Example 23.3

The **zero section** is the section sending each $p \in X$ to the origin of E_p.

Local versions:

Definition 23.4. Let $E \xrightarrow{\pi} X$ is a vector bundle and $U \subseteq X$ is open, then a **section of E over U** is a morphism σ such that

$$\pi^{-1}(U) \xrightarrow{\pi} U \xrightarrow{\sigma} \pi^{-1}(U)$$

is the identity. We denote this by $\mathcal{E}(U)$.

Picture: σ takes each point $p \in U$ to a point $\sigma(p) \in E_p$.

This gives us a sheaf $\mathcal{E}$. . . and the set of global sections is exactly $\mathcal{E}(X)$. This is also denoted $H^0(X, \mathcal{E})$ or $\Gamma(X, \mathcal{E})$.

Observe that $H^0(X, \mathcal{E})$ is an $\mathcal{O}_X(X)$ module (more generally I guess, $\mathcal{E}(U)$ is $\mathcal{O}_X(U)$ module). Given two sections σ_1 and σ_2, the sum is $\sigma_1 + \sigma_2$. And given a regular function $f \in \mathcal{O}_X(U)$, we can take $f \cdot \sigma$ to literally be $f(p) \cdot \sigma(p)$.

§23.3 The Trivial Bundle

Example 23.5 (Trivial Bundle)

Given any integer n, we literally take $E = X \times \mathbb{C}^n$

$$X \times \mathbb{C}^n \xrightarrow{\pi} X$$

and the projection is $(p, v) \mapsto p$. This is called the **trivial bundle**.

For this we just have to take the open cover $\{X\}$. There's no compatibility condition to check, since there's only one isomorphism.

When $n = 1$, we get the trivial line bundle

$$E = X \times \mathbb{C} \xrightarrow{\pi} X.$$

Note that a section $\sigma : U \to U \times \mathbb{C}$ amounts to specifying $p \mapsto (p, \sigma(p))$. In other words, a *section of the trivial line bundle amounts to a regular function*. Therefore it follows that

$$\mathcal{E} \simeq \mathcal{O}_X$$

quite literally. In fact,

$$\mathcal{E}(U) \simeq \mathcal{O}_X(U).$$

Thus, even the trivial bundle is important!

§23.4 The Tautological Bundle

The **tautological line bundle** B is the line bundle on $\mathbb{CP}^n$ defined as follows. We consider $\mathbb{CP}^n$ as the set of lines through the origin of $\mathbb{A}^{n+1}$. Then for each $p \in \mathbb{CP}^n$, we want B_p to be the line corresponding to p.

More specifically, set

$$B = \{(x, \ell) \mid x \in \ell\} \subseteq \mathbb{C}^{n+1} \times \mathbb{CP}^n$$

and you can check this is an algebraic variety. Then we get a projection map onto the first coordinate

$$B \xrightarrow{\pi} \mathbb{CP}^n.$$

(And you can do this with Grassmanians.)

Let's work out the details of this as a cover with open sets. For simplicity, work with $n = 1$. Take the usual cover of $\mathbb{CP}^1$ with affine charts U_0, U_1. So we want to specify the isomorphism

$$\pi^{-1}(U_0) \to \mathbb{C} \times U_0.$$

Here U_0 is the set of lines in $\mathbb{C}^2$ through the origin other than the y-axis. So $\pi^{-1}(U_0)$ consists of pairs (p, ℓ), where $p \in \ell$ and ℓ is not the y-axis. Then the projection is just

$$(p, \ell) \mapsto (x\text{-coord of } p, \ell)\,.$$

§23.5 Tangent Bundle

There is an obvious map $T_X \to X$ by mapping $T_pX \mapsto p$. In particular, if X is a curve, then T_X is a line bundle. (Even in higher dimensional space! Tangent spaces of curves in $\mathbb{CP}^{100}$ still are lines. In fact, the tangent space is intrinsic.)

Given $E \to X$ a vector bundle, we can construct a dual bundle $E^\vee \to X$, or $\Lambda^r E \to X$, say. We often consider $T_X^\vee$ the **contangent bundle** (of rank n), and we can consider the canonical line bundle $\Lambda^n T_X^\vee$.

Note that if X was a curve, $n = 1$ and we're already good.

§24 April 24, 2015

Last time we defined vector bundles and focused on line bundles. In particular, we had the *trivial line bundle* and the *tautological line bundle.*

§24.1 More on the Tautological Bundle

Recall that the tautological bundle is defined as

$$B = \{(X, \ell) \mid x \in \ell\} \subseteq \mathbb{C}^{n+1} \times \mathbb{CP}^n.$$

Then a global section $\sigma : \mathbb{CP}^n \to B$ amounts to picking a point on each line of $\mathbb{CP}^1$. In particular, we can interpret σ as a map $\mathbb{CP}^n \to \mathbb{C}^{n+1}$, and we can think of it as

$$(\sigma_0(p), \dots, \sigma_n(p))$$

where $\sigma_0, \dots, \sigma_n : \mathbb{CP}^n \to \mathbb{C}$. But that means each σ_i is constant! (Projective space has a bland ring of functions. . .) Thus σ is constant, i.e. there is a fixed $p \in \mathbb{C}^{n+1}$ such that

$$\sigma(\ell) = (p, \ell) \quad \forall \ell.$$

For this to be in B, we require $p = 0$. Thus the only global section σ is the zero section.

§24.2 Dual Bundle

We now take the dual bundle to B, called H, the **hyperplane bundle**. We define it as

$$H = \{(f, \ell) \mid f \text{ linear functional on } \ell\} \subseteq (\mathbb{C}^{n+1})^\vee \times \mathbb{CP}^n.$$

(The fact you can do this embedding is not obvious.) Taking $e_0, \dots, e_n$ as the $\mathbb{C}$-basis for $\mathbb{C}^{n+1}$, we have that $(\mathbb{C}^{n+1})^\vee$ has basis $e_i^\vee$. Thus the vector space of global sections of H is an $n+1$ dimensional space. We denote the sheaf of sections of H by

$$\mathcal{O}_{\mathbb{CP}^n}(1).$$

§24.3 Why Line Bundles?

Suppose we want to map $X \to \mathbb{CP}^1$. Take a line bundle $L \xrightarrow{\pi} X$ and pick two sections $s_0, s_1 \in \Gamma(X, L)$. Then one map we could consider is

$$X \to \mathbb{CP}^1 \quad \text{by} \quad p \mapsto [s_0(p) : s_1(p)].$$

To check this is well-defined, we need that it's independent of the choice of trivialization (the isomorphism), and the s_0 and s_1 don't both vanish at the same point. For the second point, one has to just pick s_0 and s_1 carefully.

The first point works in general. Specifically, a trivialization around $U \ni p$ is an isomorphism

$$U \xrightarrow{s_i} \pi^{-1}(U) \simeq U \times \mathbb{C}$$

by

$$p \xrightarrow{s_i} s_i(p) \xrightarrow{\phi} (p, \tilde{s}_i(p)).$$

So the map "should" be written as $p \mapsto [\tilde{s}_0(p) : \tilde{s}_1(p)]$, and we need that this is independent of the choice of ϕ and U.

This follows from the *compatibility condition* on the definition of a line bundle. Specifically, suppose we have U', ϕ'; then on $U \cap U'$ we have that

$$\phi \circ \phi^{-1} = \lambda \text{ linear}.$$

Thus $\tilde{s}_i(x) = \lambda(x) \cdot \tilde{s}'_i(x)$.

Example 24.1

Let $X = \mathbb{CP}^n$, and take the bundle $H \to \mathbb{CP}^n$. Let's pick $s_0 = e_0^\vee$ and $s_1 = e_1^\vee$. This generates a map

$$p \mapsto \left[e_0^\vee(p) : e_1^\vee(p)\right]$$

that is

$$[x_0 : \cdots : x_n] \mapsto [x_0 : x_1] .$$

Unfortunately, this breaks on the locus $x_0 = x_1 = 0$.

Example 24.2

If we do the same thing with $\mathbb{CP}^n \to \mathbb{CP}^n$ and the $n+1$ sections $e_0^\vee, \dots, e_n^\vee$ then we get the identity map.

Example 24.3

If we do the same thing with $\mathbb{CP}^n \to \mathbb{CP}^{n-1}$ and the n sections $e_0^\vee, \dots, e_{n-1}^\vee$ then we get the map

$$[x_0 : x_1 : \cdots : x_n] \mapsto [x_0 : \cdots : x_{n-1}] .$$

So this is defined on $U_0 \cup \cdots \cup U_{n-1}$ which is $\mathbb{CP}^n$ minus the single point $[0 : \cdots : 0 : 1]$. So this is projection from a point.

So we're seeing that as we add more and more sections, the maps becomes "more and more defined". Also, observe that $n = 1$ linearly independent global sections is nothing more than an automorphism of $\mathbb{CP}^n$.

§24.4 Maps $\simeq$ Line Bundles

Moral: if we have X and a line bundle with some nonzero sections, we can get a map to $\mathbb{CP}^n$ by picking $n+1$ sections. It turns out this goes the other way, too! Given a map

$$X \xrightarrow{f} \mathbb{CP}^n$$

we take the line bundle H on $\mathbb{CP}^n$ (the tautological bundle has no nonzero sections, so it's unlikely to do any good) and we define the pullback f^*H. This gives

$$\begin{tikzcd} f^*H \arrow[d] & H \arrow[d] \\ X \arrow[r, "f"] & \mathbb{CP}^n \end{tikzcd}$$

Then f is the map

$$p \mapsto [s_0(p) : \cdots : s_n(p)]$$

where $s_i = f^* e_i^\vee$.

§25 April 27, 2015

Today we're going to do the Riemann-Roch Theorem.

§25.1 Divisors and Degree

Let $L \to X$ be a line bundle, and $\mathcal{L}$ the sheaf of sections of L. Let $s \neq 0$ be a global section. Then s intersects the zero section at some subset of X, and is a **divisor** of $\mathcal{L}$.

Example 25.1

Let C be an irreducible curve and $L \to C$ a line bundle. (In what follows, all varieties are irreducible.) Then any divisor associated to $\mathcal{L}$ is a finite set of points, since it's the intersection of two (dimension one) curves.

In this case we define the **degree of L** to be the number of points in D for any divisor D associated to $\mathcal{L}$ (counted with multiplicity). This doesn't depend on the choice of D (which we won't prove).

In fact, this also coincides with the degree we already defined as follows. Since we have a line bundle, we can extract a map from C to projective space; in that case, the degree corresponds to the image of C in this projective space with a hyperplane (represented by the section), which literally is (when counted with multiplicity) the degree.

§25.2 Tensor Product of Line Bundles

One last operation (again we're thinking $\mathcal{L}$ as the sheaf of sections rather than the total space).

Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two line bundles, and define a sheaf $\mathcal{L}_1 \otimes \mathcal{L}_2$ as follows:

$$(\mathcal{L}_1 \otimes \mathcal{L}_2)(U) \overset{\text{def}}{=} \mathcal{L}_1(U) \otimes \mathcal{L}_2(U).$$

(This is not really a tensor product, but let's sweep that down...).

§25.3 The Riemann-Roch Theorem

Theorem 25.2 (Riemann-Roch Theorem)

Let C be an (irreducible) curve and $L \to C$ a line bundle with sheaf of sections $\mathcal{L}$. Then

$$\dim\left(H^0(C, \mathcal{L})\right) - \dim\left(H^0(C, \mathcal{L}^\vee \otimes \omega)\right) = \deg \mathcal{L} + (1 - g)$$

where g is the genus of C.

We're interested in $\dim(H^0(C, \mathcal{L}))$, since that will tell us the maps from C to projective space. The second term $\dim\left(H^0(C, \mathcal{L}^\vee \otimes \omega)\right)$ is a "correction" term; often we hope its zero. The term $\deg \mathcal{L}$ is in practice easy to compute.

The genus is a standard property of curves (analogous to dimension). Recall that we're over $\mathbb{C}$, so "curves" aren't actually one-dimensional and it makes sense to to talk about the genus.

§25.4 Example: Bundles Over H

Let H be the hyperplane bundle over $\mathbb{CP}^1$, so that the global sections of the resulting sheaf $\mathcal{H}$ correspond to the linear functionals. Hence $\mathcal{H} \otimes \mathcal{H}$ has global sections which are bilinear functions.

Observe that $\dim H^0(\mathbb{CP}^1, \mathcal{H}) = 2$, corresponding to the linear functionals. Observe that $\dim H^0(\mathbb{CP}^1, \mathcal{H} \otimes \mathcal{H}) = 3$, corresponding to the quadratic functionals on $\mathbb{CP}^1$. More generally,

$$\dim H^0\left(\mathbb{CP}^1, \mathcal{H}^{\otimes n}\right) = n + 1$$

with basis x_0^n, $x_0^{n-1}x_1$, ..., x_1^n.

Now, our motivation for studying line bundles in the first place was that line bundles induced maps into projective space. In particular, we expect that if we pick $n+1$ linearly independent sections from $\mathcal{H}^{\otimes n}$, we should get some map from $\mathbb{CP}^1$ into $\mathbb{CP}^n$. It doesn't really matter which one we pick since they're all related by an automorphism of $\mathbb{CP}^n$, so we may as well pick $x_0^n, \dots, x_1^n$ and hence in this way we recover

$$\mathbb{CP}^1 \xrightarrow{\nu_n} \mathbb{CP}^n$$

the Veronese map of degree n. In particular, $\mathcal{H}^{\otimes n}$ has degree n.

In fact, it turns out that $\mathcal{B}$, $\mathcal{H}$ and their tensors are *all* the line bundles over $\mathbb{CP}^1$. ($\mathcal{B}$ and $\mathcal{H}$ are duals, so they eat each other. In fact they're usually denoted $\mathcal{O}(-1)$ and $\mathcal{O}(1)$.)

You might see vaguely where this is going...we want $\mathcal{B}^{\otimes -n}$ to have degree $-n$. And somehow you want negative degree guys to have no global sections.

§25.5 Canonical Bundle

The **canonical bundle** ω is the nth exterior power of the cotangent bundle. For curves, $n = 1$ and so this is just the cotangent bundle.

§26 April 29, 2015

Today we'll try to use Riemann-Roch to classify conics.

First, let's start by making a list of the line bundles we know.

- On $\mathbb{CP}^n$ we have the tautological bundle, the hyperplane bundle, and powers of these.
- On an arbitrary X, we have a canonical bundle.

That's not a lot, so let's give more examples of line bundles.

§26.1 Divisors, and their associated sheaves

Let C be a smooth, irreducible projective curve over $\mathbb{C}$. We want to construct sheaves of sections of line bundles.

Definition 26.1. A **divisor** D on C is a finite formal sum

$$D = \sum m_i p_i$$

where $p_i \in C$, $m_i \in \mathbb{Z}$. Its **degree** is $\sum m_i$.

Definition 26.2. Define the sheaf $\mathcal{O}(D)$ by setting

$$\mathcal{O}(D)(U) = \{\text{rational functions } f = h/g \mid \text{poles of } f \text{ at } p_i \text{ have order} \le m_i.\}.$$

Example 26.3

$\mathcal{O}(0)$ is just the sheaf of regular functions. Thus $\mathcal{O}$ is a generalization of "regular functions" in which we allow some poles but place restrictions on how bad those poles can be.

This has the following relation to line bundles: let L be a line bundle and $\mathcal{L}$ the sheaf of sections. Take a nonzero section $s \in H^0(C, \mathcal{L})$ and let D be the divisor of zeroes of s. This lets us construct $\mathcal{O}(D)$. It turns out that

$$\mathcal{O}(D) \simeq \mathcal{L}$$

and in particular $\mathcal{O}(D)$ doesn't depend on D, up to isomorphism! (This is not too surprising, since the degree of each divisor equals to the degree of $\mathcal{L}$.)

We can allow m_i to be negative as well; if $m_i < 0$ then the requirement means "f has a zero of multiplicity $\ge m_i$".

Definition 26.4. Abbreviate $\mathcal{O}(0)$ to just $\mathcal{O}$.

Next, some observations:

- $H^0(C, \mathcal{O}) = \mathbb{C}$, since the regular functions on a curve are just the constant functions. In particular, $\dim H^0(C, \mathcal{O}) = \mathbb{C}$.
- If D is a divisor of degree $D < 0$, then

  $$\dim H^0(C, \mathcal{O}(D)) = 0.$$

  This follows from a result on compact Riemann surface: complex functions here cannot have more zeros than poles.

- If $\deg D > 0$, then
$$\deg D = \deg \mathcal{O}(D).$$
Let's see this in the case that $m_i > 0$ for all i. To see this, pick sections $s_0, \dots, s_n \in H^0(C, \mathcal{O}(D))$; this gives a map from C to projective space via
$$p \mapsto [s_0(p) : \dots : s_n(p)].$$
Let's count the intersections of this image with of C with the hyperplane. If we just conveniently pick the hyperplane to be $x_n = 0$, then we get exactly $\deg D$ intersections (corresponding to zeros of s_n); thus $\deg D = \deg \mathcal{O}(D)$.

§26.2 Genus

We black box the following result:

Theorem 26.5

We have $\dim H^0(C, \omega) = g$, where g is the genus of C and the left-hand side is the global sections of the canonical bundle.

Theorem 26.6 (Riemann-Roch Restated)

Let C be an irreducible smooth projective curve, and let K be any divisor of the global section of ω. Then we have
$$\dim H^0(C, \mathcal{O}(D)) - \dim H^0(C, \mathcal{O}(K - D)) = \deg D + 1 - g.$$

The previous theorem is the special case $D = 0$.

§26.3 Applications

Theorem 26.7

Let K be the canonical divisor of a curve C. We have $\deg K = 2g - 2$.

Proof. First, we apply Riemann-Roch to the canonical bundle ω on C. Thus
$$\dim H^0(C, \mathcal{O}(K)) - \dim H^0(C\mathcal{O}(K - K)) = \deg k + 1 - g.$$
Thus
$$g - 1 = \deg K + 1 - g$$
hence
$$\deg K = 2g - 2. \qquad \square$$

Next, we try to classify all curves for genus zero.

Theorem 26.8

The only curve of genus zero is $\mathbb{CP}^1$, the Riemann sphere.

Proof. Clearly we have $\mathbb{CP}^1$ the Riemann sphere. Are there others?

Suppose C is such a curve, and let pick $p \in C$; consider $\mathcal{O}(p)$. By the Riemann Roch Theorem we have

$$\dim H^0(C, \mathcal{O}(p)) - \dim H^0(C, \mathcal{O}(K - P)) = 1 + 1 - 0.$$

Also, by our previous result we have $\deg K = -2$ Hence $\mathcal{O}(K - P)$ has degree -3, and as we saw already negative degrees give zero. Hence the above term vanishes and

$$\dim H^0(C, \mathcal{O}(p)) = 2.$$

The constant functions contribute one to the degree, but the dimension of two so there ought to be a nonconstant global section of $\mathcal{O}(p)$ somewhere. Thus, choosing two linearly independent sections of $H^0(C, \mathcal{O}(p))$ gives a map

$$m : C \to \mathbb{CP}^1 \quad \text{by} \quad x \mapsto [s_0(x) : s_1(x)].$$

Let's pick $s_0 = 1$ and s_1 to b some nonconstant section. Thus our map is

$$x \mapsto [1 : s_1(x)].$$

In particular, $p \mapsto [1 : s_1(p)] = [0 : 1]$ since s_1 has a pole at p. But for other points $x \neq p$ the image of x is not $[0 : 1]$.

In fact, this map is *injective* onto $\mathbb{CP}^1$ (equivalently, s_1 is injective). Thus m is an isomorphism, as desired. □

Theorem 26.9

The only curves of genus one are ???

Remark 26.10. It's instructive to try to apply Riemann Roch on the divisor $D = p$ as before. This gives

$$\dim H^0(C, \mathcal{O}(p)) - \dim H^0(C, \mathcal{O}(K - p)) = 1 + 1 - 1 = 1.$$

As $\deg K = 0$, $\deg K - p = -1$ and thus the second term vanishes. Hence we have

$$\dim H^0(C, \mathcal{O}(p)) = 1.$$

Since we know about constant sections, it follows these are the *only* sections of $\mathcal{O}(p)$. This lets us map to $\mathbb{CP}^0\dots$ which we could do already.

Proof. Use Riemann-Roch on $D = np$. This gives

$$\dim H^0(C, \mathcal{O}(np)) - \dim H^0(C, \mathcal{O}(K - np)) = n + 1 - 1 = n.$$

As $\deg(K - np) < 0$, we get in general that

$$\dim H^0(C, \mathcal{O}(np)) = n.$$

These sheaves are not unrelated; we have

$$H^0(C, \mathcal{O}) \subseteq H^0(C, \mathcal{O}(p)) \subseteq H^0(C, \mathcal{O}(2p)) \subseteq \dots.$$

Let's pick basis elements inductively. At $H^0(C, \mathcal{O}(p))$ we have a basis element 1. At $H^0(C, \mathcal{O}(2p))$ extend it to basis $1, x$. At $H^0(C, \mathcal{O}(3p))$ extend it to basis $1, x, y$.

Now we use the NUMBER SIX. Consider

$$H^0(C, \mathcal{O}(6p)).$$

We can exhibit seven elements

$$\left\{1, x^3, y^2, xy, x^2, x, y\right\}$$

which are in here by pole counting (for example, since x has pole of order 2 it follows that x^3 has pole of order 6). Moreover, since x^3 and y^2 both have a pole of order six, they appear with nonzero coefficient, and we can write

$$Ay^2 + Bxy + Cy = Ex^3 + Fx^2 + Gx + H$$

and using a suitable change of variables we can make this into

$$y'^2 = x'(x' - 1)(x' - c)$$

for some constant c.

Finally, we have a map $C \to \mathbb{CP}^2$. Consider the map $C \to \mathbb{CP}^2$ given by $\mathcal{O}(3p)$, using the sections 1, x', y', that is

$$q \mapsto \left[1 : x'(q) : y'(q)\right].$$

But since x', y' satisfy the above relation, they all live in ELLIPTIC CURVES. That is, any genus one curve can be embedded as a cubic in $\mathbb{CP}^2$. $\square$