This is a (quick) English translation of the complex numbers note I wrote for Taiwan IMO 2014 training. Incidentally I was also working on an airplane.

1 The Complex Plane

Let \( \mathbb{C} \) and \( \mathbb{R} \) denote the set of complex and real numbers, respectively.

Each \( z \in \mathbb{C} \) can be expressed as

\[
z = a + bi = r (\cos \theta + i \sin \theta) = re^{i\theta}
\]

where \( a, b, r, \theta \in \mathbb{R} \) and \( 0 \leq \theta < 2\pi \). We write \( |z| = r = \sqrt{a^2 + b^2} \) and \( \arg z = \theta \).

More importantly, each \( z \) is associated with a conjugate \( \overline{z} = a - bi \). It satisfies the properties

\[
\begin{align*}
\overline{w \pm \overline{z}} &= \overline{w} \pm \overline{z} \\
\overline{w \cdot \overline{z}} &= \overline{w} \cdot \overline{z} \\
\overline{w/z} &= \overline{w}/\overline{z} \\
|z|^2 &= z \cdot \overline{z}
\end{align*}
\]

Note that \( z \in \mathbb{R} \iff z = \overline{z} \) and \( z \in i\mathbb{R} \iff z + \overline{z} = 0 \).

We represent every point in the plane by a complex number. In particular, we’ll use a capital letter (like \( Z \)) to denote the point associated to a complex number (like \( z \)).
Complex numbers add in the same way as vectors. The multiplication is more interesting: for each \( z_1, z_2 \in \mathbb{C} \) we have

\[
|z_1z_2| = |z_1||z_2| \quad \text{and} \quad \arg z_1z_2 = \arg z_1 + \arg z_2.
\]

This multiplication lets us capture a geometric structure. For example, for any points \( Z \) and \( W \) we can express rotation of \( Z \) at \( W \) by \( 90^\circ \) as

\[
z \mapsto i(z - w) + w.
\]

![Figure 2: \( z \mapsto i(z - w) + w \).](image)

### 2 Elementary Propositions

First, some fundamental formulas:

**Proposition 1.** Let \( A, B, C, D \) be pairwise distinct points. Then \( AB \perp CD \) if and only if

\[
\frac{d - c}{b - a} + \frac{(d - c)}{(b - a)} = 0.
\]

*Proof.* It’s equivalent to \( \frac{d - c}{b - a} \in i\mathbb{R} \iff \arg \left( \frac{d - c}{b - a} \right) \equiv \pm 90^\circ \iff AB \perp CD. \) \( \square \)

**Proposition 2.** Let \( A, B, C \) be pairwise distinct points. Then \( A, B, C \) are collinear if and only if \( \frac{c - a}{c - b} \in \mathbb{R} \); i.e.

\[
\frac{c - a}{c - b} = \frac{(c - a)}{(c - b)}.
\]

*Proof.* Similar to the previous one. \( \square \)

**Proposition 3.** Let \( A, B, C, D \) be pairwise distinct points. Then \( A, B, C, D \) are concyclic if and only if

\[
\frac{c - a}{c - b} : \frac{d - a}{d - b} \in \mathbb{R}.
\]

*Proof.* It’s not hard to see that \( \arg \left( \frac{c - a}{c - b} \right) = \angle ACB \) and \( \arg \left( \frac{d - a}{d - b} \right) = \angle ADB \). (Here angles are directed). \( \square \)
Now, let’s state a more commonly used formula.

**Lemma 4** (Reflection About a Segment). Let $W$ be the reflection of $Z$ across $AB$. Then

$$w = \frac{(a - b)\overline{z} + \overline{a}b - a\overline{b}}{b - a}.$$

Of course, it then follows that the foot from $Z$ to $\overline{AB}$ is exactly $\frac{1}{2}(w + z)$.

**Proof.** According to Figure 4 we obtain

$$\frac{w - a}{b - a} = \frac{(z - a)}{(b - a)} = \frac{\overline{z} - \overline{a}}{b - \overline{b}}.$$

From this we derive $w = \frac{(a - b)\overline{z} + \overline{a}b - a\overline{b}}{b - a}.$ \qed

Here are two more formulas.

**Theorem 5** (Complex Shoelace). Let $A, B, C$ be points. Then $\triangle ABC$ has signed area

$$\frac{i}{4} \begin{vmatrix} a & \overline{a} & 1 \\ b & \overline{b} & 1 \\ c & \overline{c} & 1 \end{vmatrix}.$$

In particular, $A, B, C$ are collinear if and only if this determinant vanishes.
Proof. Cartesian coordinates.

Often, Theorem 5 is easier to use than Proposition 2.
Actually, we can even write down the formula for an arbitrary intersection of lines.

**Proposition 6.** Let $A, B, C, D$ be points. Then lines $AB$ and $CD$ intersect at

$$\frac{(ab - \bar{a}b)(c - d) - (a - b)(\bar{c}d - cd)}{(\bar{a} - b)(c - d) - (a - b)(\bar{c} - \bar{d})}.$$

But unless $d = 0$ or $a, b, c, d$ are on the unit circle, this formula is often too messy to use.

3 The Unit Circle, and Triangle Centers

On the complex plane, the **unit circle** is of critical importance. Indeed if $|z| = 1$ we have

$$\overline{z} = \frac{1}{z}.$$  

Using the above, we can derive the following lemmas.

**Lemma 7.** If $|a| = |b| = 1$ and $z \in \mathbb{C}$, then the reflection of $Z$ across $AB$ is $a + b - ab\overline{z}$, and the foot from $Z$ to $AB$ is

$$\frac{1}{2}(z + a + b - ab\overline{z}).$$

**Lemma 8.** If $A, B, C, D$ lie on the unit circle then the intersection of $AB$ and $CD$ is given by

$$\frac{ab(c + d) - cd(a + b)}{ab - cd}.$$

These are much easier to work with than the corresponding formulas in general. We can also obtain the triangle centers immediately:

**Theorem 9.** Let $ABC$ be a triangle center, and assume that the circumcircle of $ABC$ coincides with the unit circle of the complex plane. Then the circumcenter, centroid, and orthocenter of $ABC$ are given by $0, \frac{1}{3}(a + b + c), a + b + c$, respectively.

Observe that the Euler line follows from this.

**Proof.** The results for the circumcenter and centroid are immediate. Let $h = a + b + c$. By symmetry it suffices to prove $AH \perp BC$. We may set

$$z = \frac{h - a}{b - c} = \frac{b + c}{b - c}.$$

Then

$$\overline{z} = \left(\frac{b + c}{b - c}\right) = \frac{\overline{b} + \overline{c}}{\overline{b} - \overline{c}} = \frac{\frac{1}{b} + \frac{1}{c}}{\frac{1}{b} - \frac{1}{c}} = \frac{c + b}{c - b} = -z$$

so $z \in \mathbb{R}$ as desired.

We can actually even get the formula for the incenter.

**Theorem 10.** Let triangle $ABC$ have incenter $I$ and circumcircle $\Gamma$. Lines $AI, BI, CI$ meet $\Gamma$ again at $D, E, F$. If $\Gamma$ is the unit circle of the complex plane then there exists $x, y, z \in \mathbb{C}$ satisfying

$$a = x^2, b = y^2, c = z^2$$

and $d = -yz, e = -zx, f = -xy$.

Note that $|x| = |y| = |z| = 1$. Moreover, the incenter $I$ is given by $-(xy + yz + zx)$.

**Proof.** Show that $I$ is the orthocenter of $\triangle DEF$. 


4 Some Other Lemmas

**Lemma 11.** Let $A, B$ be on the unit circle and select $P$ so that $\overline{PA}, \overline{PB}$ are tangents. Then

$$p = \frac{2ab}{a+b}.$$

*Proof.* Let $M$ be the midpoint of $AB$ and set $O = 0$. One can show $OM \cdot OP = 1$ and that $O, M, P$ are collinear; the result follows from this. □

![Figure 5: Two tangents. $p = \frac{2ab}{a+b}$](image)

**Lemma 12.** For any $x, y, z$, the circumcenter of $\triangle XYZ$ is given by

$$\frac{|x\bar{x} - 1|}{z - \bar{z}} \div \frac{|y\bar{y} - 1|}{z - \bar{z}} \div \frac{|z\bar{z} - 1|}{z - \bar{z}}.$$

This formula is often easier to apply if we shift $z$ to the point 0 first, then shift back afterwards.

5 Examples

**Example 13** (MOP 2006). Let $H$ be the orthocenter of triangle $ABC$. Let $D, E, F$ lie on the circumcircle of $ABC$ such that $\overline{AD} \parallel \overline{BE} \parallel \overline{CF}$. Let $S, T, U$ respectively denote the reflections of $D, E, F$ across $BC, CA, AB$. Prove that points $S, T, U, H$ are concyclic.

*Proof.* Let $(ABC)$ be the unit circle and $h = a + b + c$. WLOG, $\overline{AD}, \overline{BE}, \overline{CF}$ are perpendicular to the real axis (rotate appropriately); thus $d = \pi$ and so on. Thus $s = b + c - bc\overline{d} = b + c - abc$ and so on; we now have

$$\frac{s-t}{s-u} = \frac{b-a}{c-a} \quad \text{and} \quad \frac{h-t}{h-u} = \frac{b+abc}{c+abc}.$$ 

Compute

$$\frac{s-t}{s-u} \cdot \frac{h-t}{h-u} = \frac{(b-a)(c+abc)}{(c-a)(b+abc)} = \frac{(\frac{1}{b} - \frac{1}{c})(\frac{1}{c} + \frac{1}{abc})}{(\frac{1}{c} - \frac{1}{a})(\frac{1}{b} + \frac{1}{abc})} \Rightarrow \frac{s-t}{s-u} \cdot \frac{h-t}{h-u} \in \mathbb{R}$$

as desired. □
Example 14 (Taiwan TST 2014). In $\triangle ABC$ with incenter $I$, the incircle is tangent to $CA$, $AB$ at $E$, $F$. The reflections of $E$, $F$ across $I$ are $G$, $H$. Let $Q$ be the intersection of $GH$ and $BC$, and let $M$ be the midpoint of $BC$. Prove that $IQ$ and $IM$ are perpendicular.

Solution. Let $D$ be the foot from $I$ to $BC$, and set $(DEF)$ as the unit circle. (This lets us exploit the results of Section 3.) Thus $|d| = |e| = |f| = 1$, and moreover $g = -e$, $h = -f$. Let $x = d = \frac{1}{d}$ and define $y$, $z$ similarly. Then

$$b = \frac{2}{d + f}.$$

Similarly, $c = \frac{2}{x+y}$, so

$$m = \frac{1}{2}(b + c) = \frac{1}{x + y} + \frac{1}{x + z}.$$ $$= \frac{2x + y + z}{(x + y)(x + z)}.$$

Next, we have $Q = DD \cap GH$, which implies

$$q = \frac{dd(g + h) - gh(d + d)}{d^2 - gh} = \frac{1}{x^2} \left( -\frac{1}{y} - \frac{1}{z} \right) - \frac{1}{yz} \frac{\frac{2}{x} - \frac{1}{yz}}{\frac{1}{x^2} - \frac{1}{yz}} = \frac{2x + y + z}{x^2 - yz}. $$

so

$$m/q = \frac{x^2 - yz}{(x + y)(x + z)}.$$

Now,

$$\frac{m}{q} = \frac{1}{x^2} - \frac{1}{yz} = \frac{yz - x^2}{(x + y)(x + z)} = -m/q.$$

thus $m/q \in i\mathbb{R}$, as desired. 

Example 15 (USAMO 2012). Let $P$ be a point in the plane of $\triangle ABC$, and $\gamma$ a line through $P$. Let $A'$, $B'$, $C'$ be the points where the reflections of lines $PA$, $PB$, $PC$ with respect to $\gamma$ intersect lines $BC$, $AC$, $AB$ respectively. Prove that $A'$, $B'$, $C'$ are collinear.

Solution. Let $p = 0$ and set $\gamma$ as the real line. Then $A'$ is the intersection of $bc$ and $p\bar{a}$. So, using Proposition 6 we get

$$a' = \frac{\bar{a}(bc - \bar{b}c)}{(b - c)\bar{a} - (b - c)a}.$$
Note that
\[ a' = \frac{a(bc - bc)}{(b - c)a - (b - c)a}. \]

Thus by Theorem 5, it suffices to prove

\[
0 = \begin{vmatrix}
\frac{a(bc - bc)}{(b - c)a - (b - c)a} & \frac{a(bc - bc)}{(b - c)a - (b - c)a} & 1 \\
\frac{b(\bar{c}a - c\bar{a})}{(c - a)b - (c - a)b} & \frac{b(\bar{c}a - c\bar{a})}{(c - a)b - (c - a)b} & 1 \\
\frac{c(\bar{a}b - a\bar{b})}{(a - b)c - (a - b)c} & \frac{c(\bar{a}b - a\bar{b})}{(a - b)c - (a - b)c} & 1
\end{vmatrix}.
\]

This is equivalent to

\[
0 = \begin{vmatrix}
\frac{a(b - \bar{c})}{b(\bar{c}a - c\bar{a})} & \frac{b(\bar{c}a - c\bar{a})}{b(\bar{c}a - c\bar{a})} & \frac{c(\bar{a}b - a\bar{b})}{c(\bar{a}b - a\bar{b})} & \frac{a(b - \bar{c})}{(b - c)a} - (b - c)a \\
\frac{\bar{b}(\bar{c}a - c\bar{a})}{\bar{b}(\bar{c}a - c\bar{a})} & \frac{b(\bar{c}a - c\bar{a})}{b(\bar{c}a - c\bar{a})} & \frac{c(\bar{a}b - a\bar{b})}{c(\bar{a}b - a\bar{b})} & \frac{\bar{b}(\bar{c}a - c\bar{a})}{(\bar{c}a - c\bar{a}) - (b - a)c} \\
\frac{\bar{c}(\bar{a}b - a\bar{b})}{\bar{c}(\bar{a}b - a\bar{b})} & \frac{c(\bar{a}b - a\bar{b})}{c(\bar{a}b - a\bar{b})} & \frac{\bar{a} - \bar{b}}{\bar{a} - \bar{b}} & \frac{\bar{c}(\bar{a}b - a\bar{b})}{(\bar{c}a - c\bar{a})} (\bar{a} - \bar{a})
\end{vmatrix}.
\]

Evaluating the determinant gives

\[
\sum_{\text{cyc}} ((b - \bar{c})\bar{a} - (b - c)a) \cdot \begin{vmatrix} b & \bar{b} \\ c & \bar{c} \end{vmatrix} \cdot (\bar{c}a - c\bar{a}) (\bar{a} - \bar{a}) = 0.
\]

or, noting the determinant is \( bc - \bar{b}c \) and factoring it out,

\[
(bc - \bar{b}c)(\bar{c}a - c\bar{a})(\bar{a} - \bar{a}) \sum_{\text{cyc}} (ab - ac + \bar{c}a - \bar{b}a) = 0.
\]

\[\Box\]

**Example 16** (Taiwan TST Quiz 2014). Let \( I \) and \( O \) be the incenter and circumcenter of \( ABC \). A line \( \ell \) is drawn parallel to \( BC \) and tangent to the incircle of \( ABC \). Let \( X, Y \) be on \( \ell \) so that \( I, O, X \) are collinear and \( \angle XIY = 90^\circ \). Show that \( A, X, O, Y \) are concyclic.

**Solution.** Let \( X' \) and \( Y' \) respectively denote the reflections of \( X \) and \( Y \) across \( I \). Note that \( X, Y \) lie on \( BC \). Also, let \( P, Q \) be the intersections of \( T \) with the circumcircle.

Of course, \( (ABC) \) is the unit circle. Let \( j \) be the complex number corresponding to \( I \) (to avoid confusion with \( i = \sqrt{-1} \)). Thus,

\[
x' = \frac{(bc - \bar{c}c) (j - 0) - (\bar{j}0 - j\bar{0}) (b - c)}{(b - \bar{c}) (j - 0) - (b - c) (\bar{j} - \bar{0})} = \frac{j \cdot c^2 - b^2}{bc} - \frac{(b - c)\bar{j}}{j + bc\bar{j}}.
\]

We seek \( y' \) now. Consider the quadratic equation in \( z \) given by

\[
\frac{z - j}{j} + \frac{1 - j}{j} = 0 \iff z^2 - 2jz + j/\bar{j} = 0.
\]
Its zeros in $z$ are $p$ and $q$, which implies that $p + q = 2j$ and $pq = j/j$ (by Vieta!). From this we can compute

$$y' = \frac{pq(b + c) - bc(p + q)}{pq - bc} = \frac{j(b + c) - 2bcj}{j - bcj} = \frac{j(b + c) - 2bcj}{j - bcj}.$$

which gives

$$x = 2j - x' = \frac{j(2j - b - c + 2bcj)}{j + bcj} \quad \text{and} \quad y = 2j - y' = \frac{j(2j - b - c)}{j - bcj}.$$

From this we can obtain

$$y - x = \frac{(2j - b - c)(j + bcj) - (2j - b - c + 2bcj)(j - bcj)}{(j - bcj)(j + bcj)(j + bcj)}$$

$$= j \cdot \frac{2b \cdot c \cdot j(2j - b - c) - 2b \cdot c \cdot j(j - bcj)}{(j - bcj)(j + bcj)(j + bcj)}$$

$$= j \cdot \frac{2b \cdot c \cdot j(j - b - c + bcj)}{(j - bcj)(j + bcj)}$$

$$X = \frac{y - x}{x} = \frac{2b \cdot c \cdot j(j - b - c + bcj)}{(j - bcj)(2j - b - c + 2bcj)}$$

$$A = \frac{y - a}{a} = \frac{j(2j - b - c - a) + abcj}{a(j - bcj)}$$

We need to prove $X/A = X/Y$. Now set $a = x^2$, $b = y^2$, $c = z^2$, $j = -(xy + yz + zx)$, $j = \frac{x + y + z}{xyz}$ (this is a different $x, y$ than the points $X$ and $Y$.) So, the above rewrites as

$$X = \frac{2xz^2(x + y + z)(y^2)(x + y + z) + y^2 + z^2 + xy + yz + zx + 2y(x + y + z)}{(y + z)(x^2 - xy + yz + z) + 2y(x + y + z) + 2y(x + y + z)}$$

$$= \frac{2yz(x + y + z)(2xyz + \sum xy^2)}{(y + z)(x^2 - yz)(x(y + z)(2x + y + z) + 2yz(x + y + z))}$$
$$= \frac{2yz(x + y + z)(x + y)(x + z)}{(x^2 - yz)((x^2 + yz)(y + z) + (xy + yz + zx)(x + y + z))}$$

and

$$A = \frac{(xy + yz + zx)(x + y + z)^2 - xy(z)(x + y + z)}{x^2((-xy + yz + zx) + \frac{yz}{x}(x + y + z))} = \frac{(x + y + z)(x + y)(y + z)(z + x)}{x(yz - x^2)(y + z)}$$

thus

$$X/A = \frac{-2xyz}{(x^2 + yz)(y + z) + (x + y + z)(xy + yz + zx)}$$

$$= \frac{-2}{x^2 + \frac{1}{y}y + \frac{1}{z}z + \frac{1}{z}y + \frac{1}{y}z + \frac{1}{x}z + \frac{1}{x}y + \frac{1}{y}x + \frac{1}{z}x} = X/A.$$

\[ \square \]

6 Practice Problems

1. Let $ABCD$ be cyclic. Let $H_A, H_B, H_C, H_D$ denote the orthocenters of $BCD, CDA, DAB, ABC$. Show that $AH_A, BH_B, CH_C, DH_D$ are concurrent.

2. (China TST 2011) Let $\Gamma$ be the circumcircle of a triangle $ABC$. Assume $\overline{AA'}, \overline{BB'}, \overline{CC'}$ are diameters of $\Gamma$. Let $P$ be a point inside $ABC$ and let $D, E, F$ be the feet from $P$ to $BC, CA, AB$. Let $X$ be the reflection of $A'$ across $D$; define $Y$ and $Z$ similarly. Prove that $\triangle XYZ \sim \triangle ABC$.

3. In circumscribed quadrilateral $ABCD$ with incircle $\omega$, prove that the midpoint of $AC$ and the midpoint of $BD$ are collinear with the center of $\omega$.

4. (Simson Line) Let $ABC$ be a triangle and $P$ a point on its circumcircle.

(a) Let $D, E, F$ be the feet from $P$ to $BC, CA, AB$. Show that $D, E, F$ are collinear.

(b) Moreover, prove that the line through these points bisects $PH$, where $H$ is the orthocenter of $ABC$.

5. (PUMaC Finals) Let $\gamma$ and $I$ be the incircle and incenter of triangle $ABC$. Let $D, E, F$ be the tangency points of $\gamma$ to $BC, CA, AB$ and let $D'$ be the reflection of $D$ about $I$. Assume $EF$ intersects the tangents to $\gamma$ at $D$ and $D'$ at points $P$ and $Q$. Show that $\angle DAD' + \angle PIQ = 180^\circ$.

6. (Schiffler Point) Let triangle $ABC$ have incenter $I$. Prove that the Euler lines of $\triangle AIB, \triangle BIC, \triangle CIA, \triangle ABC$ are concurrent.

7. (USA TST 2014) Let $ABCD$ be a cyclic quadrilateral and let $E, F, G, H$ be the midpoints of $AB, BC, CD, DA$. Call $W, X, Y, Z$ the orthocenters of $AHE, BEF, CFG, DHG$. Prove that $ABCD$ and $WXYZ$ have the same area.

8. (Iran 2004) Let $O$ be the circumcenter of $ABC$. A line $\ell$ through $O$ cuts $\overline{AB}$ and $\overline{AC}$ at points $X$ and $Y$. Let $M$ and $N$ be the midpoints of $\overline{BY}, \overline{CX}$. Show that $\angle MON = \angle BAC$.

9. (APMO 2010) Let $ABC$ be an acute triangle, where $AB > BC$ and $AC > BC$. Denote by $O$ and $H$ the circumcenter and orthocenter. The circumcircle of $AHB$ intersects $AB$ again at $M$; the circumcircle of $AHC$ intersects $AC$ again at $N$. Prove that the circumcenter of triangle $MNH$ lies on line $OH$. 

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10. (Iran 2013) Let $ABC$ be acute, and $M$ the midpoint of minor arc $\overarc{BC}$. Let $N$ be on the circumcircle of $ABC$ such that $\overline{AN} \perp \overline{BC}$, and let $K$, $L$ lie on $AB$, $AC$ so that $\overline{OK} \parallel \overline{MB}$, $\overline{OL} \parallel \overline{MC}$. (Here $O$ is the circumcenter of $ABC$). Prove that $NK = NL$.

11. (MOP 2006) Cyclic quadrilateral $ABCD$ has circumcenter $O$. Let $P$ be a point in the plane and let $O_1$, $O_2$, $O_3$, $O_4$ be the circumcenters of $PAB$, $PBC$, $PCD$, $PDA$. Show that the midpoints of $O_1O_3$, $O_2O_4$, $OP$ are concurrent.