Barycentric Coordinates for the Impatient

Max Schindler* Evan Chen†

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I suppose it is tempting, if the only tool you have is a hammer, to treat everything as if it were a nail.

1 Introduction

Stop practicing your bash. It gives you an unfair advantage.
–dragon96

This is intended to be an abridged version of the more thorough "Barycentric Coordinates in Olympiad Geometry". It contains three things: formula sheet, examples, and problems.

Most of the space, as one can see, is consumed by the example problems; after all, this is a bash technique, and any real bash which is less than two pages long is considered very successful.

Enjoy!

1.1 Standard Formulas

Throughout this paper, $\triangle ABC$ is a triangle with vertices in counterclockwise order. The lengths will be abbreviated $a = BC$, $b = CA$, $c = AB$. These correspond with points in the vector plane $\vec{A}$, $\vec{B}$, $\vec{C}$.

For arbitrary points P, Q, R, $[PQR]$ will denote the signed area of $\triangle PQR$.[1]

Definition. Each point in the plane is assigned an ordered triple of real numbers $P = (x, y, z)$ such that

$$\vec{P} = x\vec{A} + y\vec{B} + z\vec{C} \quad \text{and} \quad x + y + z = 1$$

Theorem 1 (Line)**.** *The equation of a line is $ux + vy + wz = 0$ where u, v, w are reals. (These u, v and w are unique up to scaling.)*

*Mewto55555, Missouri. I can be contacted at igoroogenflagenstein@gmail.com.
†v_Enhance, SFBA. I can be reached at chen.evan6@gmail.com.

[1] For ABC counterclockwise, this is positive when P, Q and R are in counterclockwise order, and negative otherwise. When ABC is labeled clockwise the convention is reversed; that is, $[PQR]$ is positive if and only if it is oriented in the same way as ABC. In this article, ABC will always be labeled counterclockwise.

Coordinates of Special Points

From this point on, the point $(kx : ky : kz)$ will refer to the point (x, y, z) for $k \neq 0$. In fact, the equations for the line and circle are still valid; hence, when one is simply intersecting lines and circles, it is permissible to use these un-homogenized forms in place of their normal forms. Again, the coordinates here are not homogenized!

Point	Coordinates	Sketch of Proof
Centroid	$G = (1 : 1 : 1)$	Trivial
Incenter	$I = (a : b : c)$	Angle bisector theorem
Symmedian point	$K = (a^2 : b^2 : c^2)$	Similar to above
Excenter	$I_a = (-a : b : c)$, etc.	Similar to above
Orthocenter	$H = (\tan A : \tan B : \tan C)$	Use area definition
Circumcenter	$O = (\sin 2A : \sin 2B : \sin 2C)$	Use area definition

If absolutely necessary, it is sometimes useful to convert the trigonometric forms of H and O into expressions entirely in terms of the side lengths (cf. [3, 5]) by

$$O = (a^2(b^2 + c^2 - a^2) : b^2(c^2 + a^2 - b^2) : c^2(a^2 + b^2 - c^2))$$

and

$$H = ((a^2 + b^2 - c^2)(c^2 + a^2 - b^2) : (b^2 + c^2 - a^2)(a^2 + b^2 - c^2) : (c^2 + a^2 - b^2)(b^2 + c^2 - a^2))$$

Definition. The *displacement vector* of two (normalized) points $P = (p_1, p_2, p_3)$ and $Q = (q_1, q_2, q_3)$ is denoted by $\overrightarrow{PQ}$ and is equal to $(p_1 - q_1, p_2 - q_2, p_3 - q_3)$.

A Note on Scaling Displacement Vectors

In EFFT, one can write a displacement vector (x, y, z) as $(kx : ky : kz)$, and the theorem will still be true. This is also true for Strong EFFT, but NOT for the distance formula.

Theorem 4 (Evan's Favorite Forgotten Trick)**.** *Consider displacement vectors* $\overrightarrow{MN} = (x_1, y_1, z_1)$ *and* $\overrightarrow{PQ} = (x_2, y_2, z_2)$*. Then* $MN \perp PQ$ *if and only if*

$$0 = a^2(z_1y_2 + y_1z_2) + b^2(x_1z_2 + z_1x_2) + c^2(y_1x_2 + x_1y_2)$$

Corollary 5. *Consider a displacement vector* $\overrightarrow{PQ} = (x_1, y_1, z_1)$*. Then* $PQ \perp BC$ *if and only if*

$$0 = a^2(z_1 - y_1) + x_1(c^2 - b^2)$$

Corollary 6. *The perpendicular bisector of* BC *has equation*

$$0 = a^2(z - y) + x(c^2 - b^2)$$

Theorem 7 (Distance Formula)**.** *Consider a displacement vector* $\overrightarrow{PQ} = (x, y, z)$*. Then*

$$|PQ|^2 = -a^2yz - b^2zx - c^2xy$$

Theorem 8. *The general equation of a circle is*

$$-a^2yz - b^2zx - c^2xy + (ux + vy + wz)(x + y + z) = 0$$

for reals u, v, w*.*

Corollary 9. *The circumcircle has equation*

$$a^2yz + b^2zx + c^2xy = 0$$

Theorem 10 (Area Formula). *The area of a triangle with vertices $P = (x_1, y_1, z_1)$, $Q = (x_2, y_2, z_2)$ and $R = (x_3, y_3, z_3)$ is*

$$[PQR] = [ABC] \cdot \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

Corollary 11 (First Collinearity Criteria). *The points $P = (x_1 : y_1 : z_1)$, $Q = (x_2 : y_2 : z_2)$ and $R = (x_3 : y_3 : z_3)$ are collinear if and only if*

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = 0$$

Corollary 12 (Line Through 2 Points). *The equation of a line through the points $P = (x_1 : y_1 : z_1)$ and $Q = (x_2 : y_2 : z_2)$ is*

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x & y & z \end{vmatrix} = 0$$

Corollary 13 (Second Collinearity Criteria). *The points $P = (x_1, y_1, z_1)$, $Q = (x_2, y_2, z_2)$ and $R = (x_3, y_3, z_3)$, are collinear if and only if*

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

Cyclic variations hold.

Theorem 14 (Strong EFFT). *Suppose M, N, P and Q are points with*

$$\overrightarrow{MN} = x_1\overrightarrow{AO} + y_1\overrightarrow{BO} + z_1\overrightarrow{CO}$$
$$\overrightarrow{PQ} = x_2\overrightarrow{AO} + y_2\overrightarrow{BO} + z_2\overrightarrow{CO}$$

If either $x_1 + y_1 + z_1 = 0$ or $x_2 + y_2 + z_2 = 0$, then $MN \perp PQ$ if and only if

$$0 = a^2(z_1y_2 + y_1z_2) + b^2(x_1z_2 + z_1x_2) + c^2(y_1x_2 + x_1y_2)$$

Corollary 15. *The equation for the tangent to the circumcircle at A is $b^2z + c^2y = 0$.*

Definition (Conway's Notation). Let S be twice the area of the triangle. Define $S_\theta = S \cot \theta$, and define the shorthand notation $S_{\theta\phi} = S_\theta S_\phi$.

Fact. We have $S_A = \frac{-a^2+b^2+c^2}{2} = bc \cos A$ and its cyclic variations. (We also have $S_\omega = \frac{a^2+b^2+c^2}{2}$, where ω is the Brocard angle. This follows from $\cot \omega = \cot A + \cot B + \cot C$.)

Fact. We have the identities

$$S_B + S_C = a^2$$

and

$$S_{AB} + S_{BC} + S_{CA} = S^2$$

Fact. $O = (a^2S_A : b^2S_B : c^2S_C)$ and $H = (S_{BC} : S_{CA} : S_{AB}) = \left(\frac{1}{S_A} : \frac{1}{S_B} : \frac{1}{S_C}\right)$.

Theorem 16 (Conway's Formula). *Given a point P with counter-clockwise directed angles $\measuredangle PBC = \theta$ and $\measuredangle BCP = \phi$, we have $P = (-a^2 : S_C + S_\phi : S_B + S_\theta)$.*

Lemma 17 (Parallelogram Lemma). *The points $ABCD$ form a parallelogram iff $A + C = B + D$ (here the points are normalized), where addition is done component-wise.*

Lemma 18 (Concurrence Lemma). *The three lines $u_ix + v_iy + w_iz = 0$, for $i = 1, 2, 3$ are concurrent if and only if*

$$\begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{vmatrix} = 0$$

1.2 More Obscure Formulas

Here's some miscellaneous formulas and the like. These were not included in the main text.

From [7],

Theorem 19 (Leibniz Theorem)**.** *Let Q be a point with homogeneous barycentric coordinates $(u : v : w)$ with respect to $\triangle ABC$. For any point P on the plane ABC the following relation holds:*

$$uPA^2 + vPB^2 + wPC^2 = (u + v + w)PQ^2 + uQA^2 + vQB^2 + wQC^2$$

1.2.1 Other Special Points

Point	Coordinates
Gregonne Point [3]	$\mathrm{Ge} = ((s-b)(s-c) : (s-c)(s-a) : (s-a)(s-b))$
Nagel Point [3]	$\mathrm{Na} = (s-a : s-b : s-c)$
Isogonal Conjugate [1]	$P^* = \left(\frac{a^2}{x} : \frac{b^2}{y} : \frac{c^2}{z}\right)$
Isotomic Conjugate [1]	$P^t = \left(\frac{1}{x} : \frac{1}{y} : \frac{1}{z}\right)$
Feuerbach Point [8]	$F = \big((b+c-a)(b-c)^2 : (c+a-b)(c-a)^2 : (a+b-c)(a-b)^2\big)$
Nine-point Center	$N = (a\cos(B-C) : b\cos(C-A) : c\cos(A-B))$

1.2.2 Special Lines and Circles

Nine-point Circle	$-a^2yz - b^2xz - c^2xy + \frac{1}{2}(x+y+z)(S_Ax + S_By + S_Cz) = 0$
Incircle	$-a^2yz - b^2zx - c^2xy + (x+y+z)\big((s-a)^2x + (s-b)^2y + (s-c)^2z\big) = 0$
A-excircle [9]	$-a^2yz - b^2zx - c^2xy + (x+y+z)\big(s^2x + (s-c)^2y + (s-b)^2z\big) = 0$
Euler Line [9]	$S_A(S_B - S_C)x + S_B(S_C - S_A)y + S_C(S_A - S_B)z = 0$

2 Example Problems

> Graders received some elegant solutions, some not-so-elegant solutions, and some so not elegant solutions.
> – Delong Meng

Let us now demonstrate the power of the strategy on a few... victims.

2.1 USAMO 2001/2

We begin with a USAMO problem which literally (figuratively?) screams for an analytic solution.

2.1.1 Problem and Solution

Problem (USAMO 2001/2)**.** Let ABC be a triangle and let ω be its incircle. Denote by D_1 and E_1 the points where ω is tangent to sides BC and AC, respectively. Denote by D_2 and E_2 the points on sides BC and AC, respectively, such that $CD_2 = BD_1$ and $CE_2 = AE_1$, and denote by P the point of intersection of segments AD_2 and BE_2. Circle ω intersects segment AD_2 at two points, the closer of which to the vertex A is denoted by Q. Prove that $AQ = D_2P$.

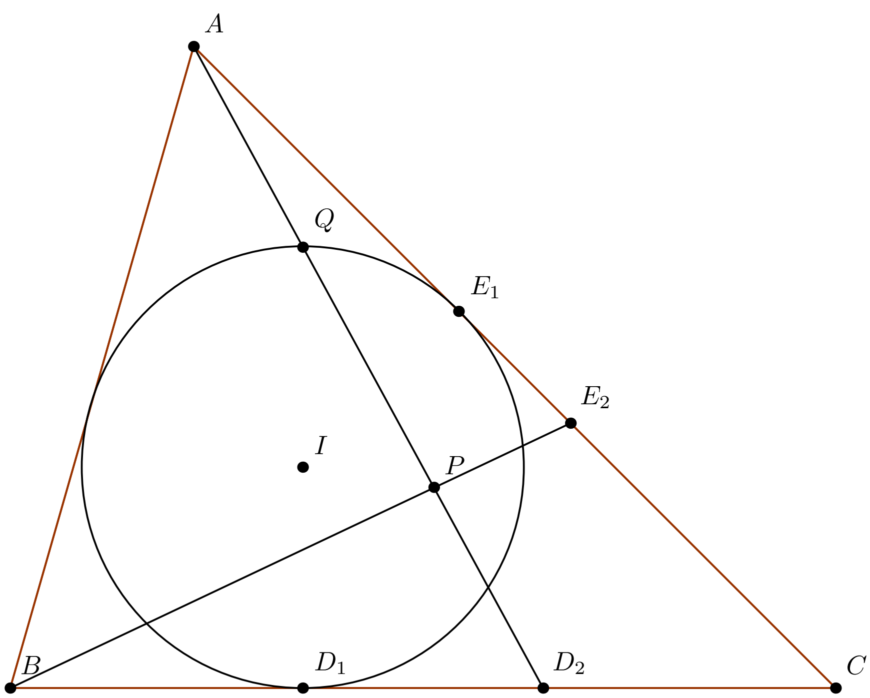

Figure 1: USAMO 2001/2

Solution. (Evan Chen) So we use barycentric coordinates.

It's obvious that un-normalized, $D_1 = (0 : s-c : s-b) \Rightarrow D_2 = (0 : s-b : s-c)$, so we get a normalized $D_2 = \left(0, \frac{s-b}{a}, \frac{s-c}{a}\right)$. Similarly, $E_2 = \left(\frac{s-a}{b}, 0, \frac{s-c}{b}\right)$.

Now we obtain the points $P = \left(\frac{s-a}{s}, \frac{s-b}{s}, \frac{s-c}{s}\right)$ by intersecting the lines $AD_2 : (s-c)y = (s-b)z$ and $BE_2 : (s-c)x = (s-a)z$.

Let Q' be such that $AQ' = PD_2$. It's obvious that $Q'_y + P_y = A_y + D_{2y}$, so we find that $Q'_y = \frac{s-b}{a} - \frac{s-b}{s} = \frac{(s-a)(s-b)}{sa}$. Also, since it lies on the line AD_2, we get that $Q'_z = \frac{s-c}{s-b} \cdot Q'_y = \frac{(s-a)(s-c)}{sa}$. Hence,

$$Q'_x = 1 - \frac{((s-b)+(s-c))(s-a)}{sa} = \frac{sa - a(s-a)}{sa} = \frac{a}{s}$$

Hence,

$$Q' = \left(\frac{a}{s}, \frac{(s-a)(s-b)}{sa}, \frac{(s-a)(s-c)}{sa}\right)$$

Let $I = \left(\frac{a}{2s}, \frac{b}{2s}, \frac{c}{2s}\right)$. We claim that, in fact, I is the midpoint of $Q'D_1$. Indeed,

$$\begin{aligned}
\frac{1}{2}\left(0 + \frac{a}{s}\right) &= \frac{a}{2s} \\
\frac{1}{2}\left(\frac{(s-a)(s-b)}{sa} + \frac{s-c}{a}\right) &= \frac{(s-a)(s-b) + s(s-c)}{2sa} \\
&= \frac{ab}{2sa} \\
&= \frac{b}{2s} \\
\frac{1}{2}\left(\frac{(s-a)(s-c)}{sa} + \frac{s-b}{a}\right) &= \frac{c}{2s}
\end{aligned}$$

Implying that Q lies on the circle; in particular, diametrically opposite from D_1, so it is the closer of the two points. Hence, $Q = Q'$, so we're done.

□

2.1.2 Commentary

The unusual points D_2, and E_2 make this problem ripe for barycentric coordinates, since they can be written as ratios on the appropriate sides. The point Q' is also easy to construct because of lengths.

Now it turns out that it's well known that AD_2 passes through the point directly above D_1, so it is merely a matter of length chasing to construct Q' and subsequently use midpoints.

2.2 USAMO 2008/2

We follow this with another problem, made possible by EFFT (theorem 4). As we shall see, EFFT is particularly useful for constructing perpendiculars to the sides of the triangle.

2.2.1 Problem and Solution

Problem (USAMO 2008/2)**.** Let ABC be an acute, scalene triangle, and let M, N, and P be the midpoints of $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$, respectively. Let the perpendicular bisectors of $\overline{AB}$ and $\overline{AC}$ intersect ray AM in points D and E respectively, and let lines BD and CE intersect in point F, inside of triangle ABC. Prove that points A, N, F, and P all lie on one circle.

Solution. (Evan Chen) Set $A = (1,0,0)$, $B = (0,1,0)$ and $C = (0,0,1)$. Evidently $P = (\frac{1}{2}, \frac{1}{2}, 0)$, etc. Check that the equation of the line AM is $y = z$.

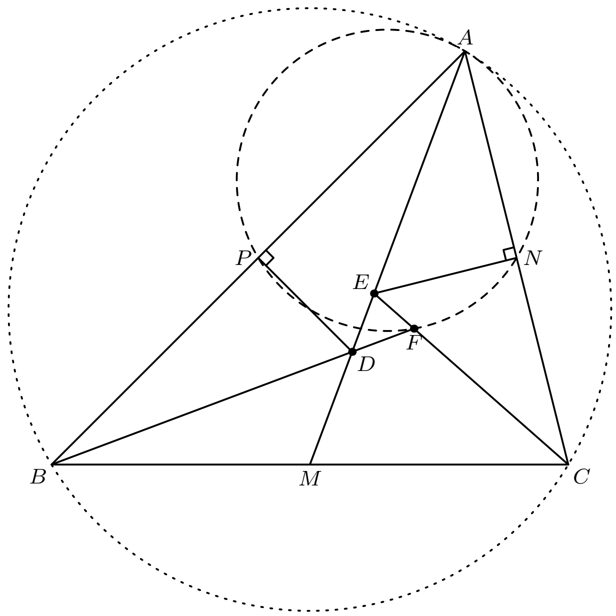

Figure 2: USAMO 2008/2

We will now compute the coordinates of the point D. Write $D = (1-2t, t, t)$ as it lies on the line AM. Applying EFFT to $DP \perp AB$,

$$\begin{aligned}\left(\left(t-\frac{1}{2}\right)-\left(\frac{1}{2}-2t\right)\right)\left(-c^2/2\right)+t\left(a^2-b^2/2\right)=0\\ \Rightarrow (3t-1)(-c^2)+t(a^2-b^2)=0\\ \Rightarrow (a^2-b^2-3c^2)t=-c^2\\ \Rightarrow t=\frac{c^2}{3c^2+b^2-a^2}\end{aligned}$$

Call this value j. Then $D = (1-2j, j, j)$. Similarly, $E = (1-2k, k, k)$ where $k = \frac{b^2}{3b^2+c^2-a^2}$.

Now the line BD has equation $\frac{z}{x} = \frac{j}{1-2j}$, whilst the line CE has equation $\frac{y}{x} = \frac{k}{1-2k}$. So if $F = (p, q, r)$, then $p+q+r = 1$, $\frac{r}{p} = \frac{j}{1-2j}$, and $\frac{q}{p} = \frac{k}{1-2k}$.

In fact, $\frac{r}{p} = \frac{c^2}{(3c^2+b^2-a^2)-2c^2} = \frac{c^2}{c^2+b^2-a^2}$, and $\frac{q}{p} = \frac{b^2}{b^2+c^2-a^2}$. Evidently $\frac{1}{p} = 1 + \frac{r}{p} + \frac{q}{p} = 1 + \frac{b^2+c^2}{c^2+b^2-a^2} = \frac{2S+a^2}{S} = 2 + \frac{a^2}{S}$. Let $S = c^2+b^2-a^2$ for convenience.

Dilate F to $F' = 2F - A = (2p-1, 2q, 2r)$. It suffices to show this lies on the circumcircle of triangle ABC (since this will send P and N to B and C, respectively), which has equation $a^2yz + b^2zx + c^2xy = 0$. Hence it suffices to show that

$$\begin{aligned}0 = a^2(2q)(2r) + b^2(2r)(2p-1) + c^2(2p-1)(2q)\\ \Leftrightarrow 0 = 4a^2\frac{q}{p}\frac{r}{p} + (2+\frac{1}{p})(2b^2\frac{r}{p} + 2c^2\frac{q}{p})\\ \Leftrightarrow -(2-(2+\frac{a^2}{S}))(2b^2\frac{c^2}{S} + 2c^2\frac{b^2}{S}) = 4a^2\frac{b^2c^2}{S^2}\\ \Leftrightarrow a^2(2b^2c^2 + 2c^2b^2) = 4a^2b^2c^2\\ \Leftrightarrow 4a^2b^2c^2 = 4a^2b^2c^2\end{aligned}$$

which is true. □

2.2.2 Commentary

EFFT provides a method for constructing the perpendicular bisectors (in actuality this is just corollary 6), from which it is not hard at all to intersect lines. The final conclusion involves a circle, so we dilate the points such that this circle is simply the circumcircle, which has a very simple form.

2.3 ISL 2001 G1

We now present an application of Conway's Formula to an ISL G1.

2.3.1 Problem and Solution

Problem (ISL 2001/G1)**.** Let A_1 be the center of the square inscribed in acute triangle ABC with two vertices of the square on side BC. Thus one of the two remaining vertices of the square is on side AB and the other is on AC. Points B_1, C_1 are defined in a similar way for inscribed squares with two vertices on sides AC and AB, respectively. Prove that lines AA_1, BB_1, CC_1 are concurrent.

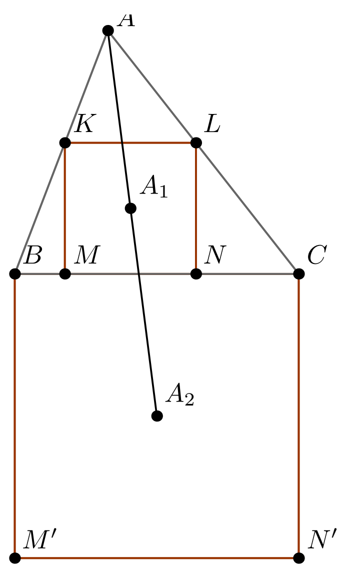

Figure 3: ISL 2001 G1

Solution. (Max Schindler) There is an obvious homothety centered on A that maps the inscribed square with center A_1 to one with center A_2, which is constructed externally off side BC. B_2 and C_2 can be defined similarly. It then suffices to show that AA_2, BB_2, CC_2 concur.

By Conway's formula, $A_2 = (-a^2 : S_C + S\cot 45 : S_B + S\cot 45) = (-a^2 : S_C + S : S_B + S)$. It can be very easily verified that AA_2, and similarly BB_2 and CC_2, all go through the point $((S_B+S)(S_C+S) : (S_A+S)(S_C+S) : (S_B+S)(S_A+S))$ and are thus concurrent (this point is called the Outer Vecten Point). □

2.3.2 Commentary

Homothety (similar to that in problem 2.2) allowed us to create a convenient expression for S by Conway's Formula, at which point Conway's Formula cleared the problem.

2.4 2012 WOOT PO4/7

We now exhibit yet another problem in which perpendicular bisectors can be constructed via EFFT, which are then used to determine a circumcenter. The method was used by v_Enhance during the actual test and received 7/.7. See [6].

2.4.1 Problem and Solution

Problem (2012 WOOT PO4/7)**.** Let ω be the circumcircle of acute triangle ABC. The tangents to ω at B and C intersect at P, and AP and BC intersect at D. Points E, F are on AC and AB, respectively, such that $DE \parallel BA$ and $DF \parallel CA$.

(a) Prove that points F, B, C, and E are concyclic.

(b) Let A_1 denote the circumcenter of cyclic quadrilateral $FBCE$. Points B_1 and C_1 are defined similarly. Prove that AA_1, BB_1, and CC_1 are concurrent.

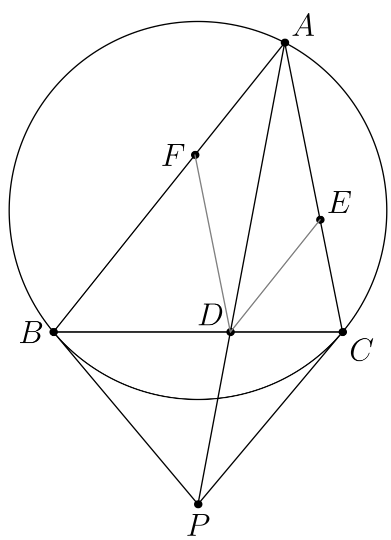

Figure 4: WOOT Practice Olympiad 4, Problem 7

Solution. (Evan Chen) Let $A = (1, 0, 0)$, etc. Since the symmedian point has $K = (a^2 : b^2 : c^2)$, we find that $D = \left(0, \frac{b^2}{b^2+c^2}, \frac{c^2}{b^2+c^2}\right)$. Then, it's obvious that $E = \left(\frac{b^2}{b^2+c^2}, 0, \frac{c^2}{b^2+c^2}\right)$ and $F = \left(\frac{c^2}{b^2+c^2}, \frac{b^2}{b^2+c^2}, 0\right)$. In that case, notice that since the general form of a circle is $a^2yz + b^2zx + c^2xy + (ux + vy + wz)(x + y + z) = 0$, the circle passing through

$$\omega : a^2yz + b^2zx + c^2xy - \frac{b^2c^2}{b^2+c^2} \cdot x(x + y + z) = 0$$

passes through BFEC, so it's cyclic.

Suppose $A_1 = (x_0, y_0, z_0)$. Let $M_1 = \frac{B+F}{2} = \left(\frac{1}{2}\frac{c^2}{b^2+c^2}, \frac{1}{2} + \frac{1}{2}\frac{b^2}{b^2+c^2}, 0\right)$. Then $A_1M_1 \perp AB$, applying EFFT and rearranging yields

$$x_0 - y_0 + \frac{a^2 - b^2}{c^2} z_0 = \frac{c^2}{b^2+c^2}$$

Doing a similar thing, we find that A_1 is given by

$$\begin{aligned}
x_0 + y_0 + z_0 &= 1 \\
x_0 - y_0 + \frac{a^2 - b^2}{c^2} z_0 &= \frac{c^2}{b^2 + c^2} \\
x_0 + \frac{a^2 - c^2}{b^2} y_0 - z_0 &= \frac{b^2}{b^2 + c^2}
\end{aligned}$$

Then by Cramer's rule, we can bash and get $\frac{y_0}{z_0} = \frac{a^2+b^2-c^2}{a^2-b^2+c^2}$. Then it's easy to get the desired conclusion by Ceva. □

2.4.2 Commentary

Once we realize that AD is a symmedian, the symmedian point immediately gives us a nice form for D, and hence E and F by similar triangles. The condition that $FBCE$ is cyclic is made easier since two of the points are vertices of the triangle, making both v and w vanish; hence the end computation is no longer hard.

For the second part, EFFT allows us to use perpendicular bisectors to construct the circumcenter; we can then compute the ratio that the cevian AA_1 divides BC into. Ceva's Theorem guarantees that these will cancel cyclically, so we just compute the proper determinants, knowing that they will cancel cyclically at the end.

The part about Ceva is worth remembering: any time you are trying to prove three cevians are concurrent, barycentric coordinates may provide an easy way to compute the ratios in Ceva's Theorem.

2.5 ISL 2005 G5

We now present a solution to an ISL problem involving Strong EFFT (theorem 1.1). This problem was proposed by Romania, and is relatively difficult to approach synthetically.

2.5.1 Problem and Solution

Problem (ISL 2005/G5)**.** Let $\triangle ABC$ be an acute-angled triangle with $AB \neq AC$. Let H be the orthocenter of triangle ABC, and let M be the midpoint of the side BC. Let D be a point on the side AB and E a point on the side AC such that $AE = AD$ and the points D, H, E are on the same line. Prove that the line HM is perpendicular to the common chord of the circumscribed circles of triangle $\triangle ABC$ and triangle $\triangle ADE$.

Solution. (Max Schindler) We use barycentric co-ordinates.

Let the length $AD = AE = \ell$.

Then, $D = (c - \ell : \ell : 0)$, $E = (b - \ell : 0 : \ell)$, and, using Conway's Notation, $H = \left(\frac{1}{S_A} : \frac{1}{S_B} : \frac{1}{S_C}\right)$.

Since they are collinear,

$$\begin{aligned}
\det \begin{pmatrix} c-\ell & \ell & 0 \\ b-\ell & 0 & \ell \\ \frac{1}{S_A} & \frac{1}{S_B} & \frac{1}{S_C} \end{pmatrix} &= 0 \\
\Rightarrow \frac{-\ell(c-\ell)}{S_B} - \ell\left(\frac{b-\ell}{S_C} - \frac{\ell}{S_A}\right) &= 0 \\
\Rightarrow \frac{cS_C - bS_B}{S_B S_C} &= \ell\left[\frac{1}{S_A} + \frac{1}{S_B} + \frac{1}{S_C}\right]
\end{aligned}$$

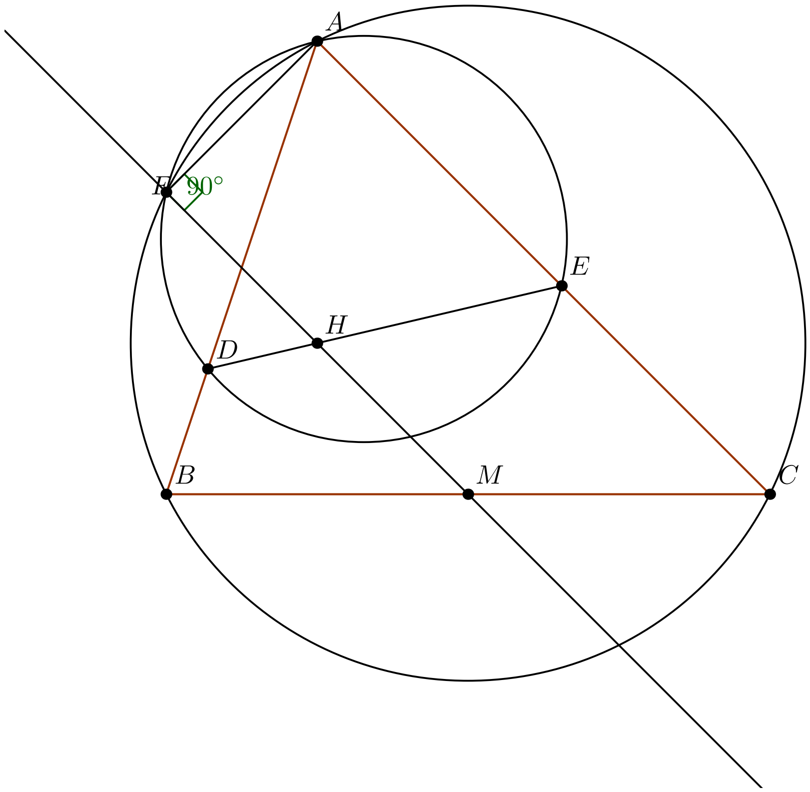

Figure 5: ISL 2005 G5

Thus, $\ell = \frac{S_A(cS_C - bS_B)}{S_BS_C + S_AS_C + S_AS_B}$.

Substituting in $S_A = \frac{b^2+c^2-a^2}{2}$ and similar, remembering that $S_BS_C + S_AS_C + S_AS_B = S^2$ gives:

$$\ell = \frac{(b^2+c^2-a^2)[c(a^2+b^2-c^2)+b(a^2-b^2+c^2)]}{(a+b+c)(a+b-c)(b+c-a)(c+b-a)}$$

It can easily be seen that $c = -b$ is a root of the bracketed part of the numerator; factoring it out leaves $a^2 + 2bc - b^2 - c^2 = a^2 - (b-c)^2 = (a-b+c)(a+b-c)$.

Thus,

$$\ell = \frac{(b^2+c^2-a^2)(b+c)(a-b+c)(a+b-c)}{(a+b+c)(a+b-c)(b+c-a)(c+b-a)} \Rightarrow \boxed{\ell = \frac{(b^2+c^2-a^2)(b+c)}{(a+b+c)(b+c-a)}}$$

Meanwhile, the circumcircle of ADE has equation given by $-a^2yz - b^2zx - c^2xy + (x+y+z)(ux+vy+wz) = 0$, as usual, so upon substituting the points A, D and E we find that

$$\begin{aligned} u &= 0 \\ -c^2(c-\ell)(\ell) + c\left((c-\ell)u + \ell v\right) &= 0 \\ -b^2(b-\ell)(\ell) + b\left((b-\ell)u + \ell w\right) &= 0 \end{aligned}$$

Conveniently enough, the first equation immediately gives $u = 0$, and we can easily solve for v and w to get $v = c(c-\ell)$ and $w = b(b-\ell)$.

Now we need the common chord of the two circles

$$\begin{aligned} -a^2yz - b^2zx - c^2xy &= 0 \\ -a^2yz - b^2zx - c^2xy + (x+y+z)(c(c-\ell)y + b(b-\ell)z) &= 0 \end{aligned}$$

But this is easy: subtract the two equations. We find that the common chord has equation

$$c(c-\ell)y + b(b-\ell)z = 0$$

Now, we have everything we need to finish this problem off.

Consider two points on this chord, $A = (1, 0, 0)$ and $P = (0 : b(b-\ell) : -c(c-\ell))$.

Then, the (scaled) displacement vector $\overrightarrow{PA}$ is $(b(b-\ell) - c(c-\ell) : -b(b-\ell) : c(c-\ell))$.

Let the circumcenter be the null vector. Then, the displacement vector $\overrightarrow{MH}$ is $\vec{A}+\vec{B}+\vec{C}-\frac{\vec{B}+\vec{C}}{2} = \left(1, \frac{1}{2}, \frac{1}{2}\right)$ which can be scaled as $(2 : 1 : 1)$

Since PA has coefficient-sum 0, we can apply Strong EFFT:

$$MH \perp PA \Leftrightarrow a^2[c(c-\ell) - b(b-\ell)] + b^2[c(c-\ell) + b(b-\ell)] + c^2[-b(b-\ell) - c(c-\ell)] = 0$$

which becomes

$$b(a^2 - b^2 + c^2)(b-\ell) = c(a^2 + b^2 - c^2)(c-\ell)$$

But we have

$$b - \ell = \frac{b(a+b+c)(b+c-a) - (b^2+c^2-a^2)(b+c)}{(a+b+c)(b+c-a)} = \frac{c(a^2+b^2-c^2)}{(a+b+c)(b+c-a)}$$

Similarly,

$$c - \ell = \frac{b(a^2-b^2+c^2)}{(a+b+c)(b+c-a)}$$

Thus,

$$\begin{aligned} b(a^2-b^2+c^2)(b-\ell) &= b(a^2-b^2+c^2)\frac{c(a^2+b^2-c^2)}{(a+b+c)(b+c-a)} \\ &= c(a^2+b^2-c^2)(c-\ell) \\ &\Rightarrow MH \perp PA \end{aligned}$$

□

2.5.2 Commentary

While we eventually did have to use the somewhat messy form of H, Strong EFFT permits us to avoid doing so when finding the condition for perpendicular chords. The other useful insight was computing the radical axis by simply subtracting the two equations: keep this in mind!

By the way, it is true in general (as the diagram implies) that M, H and F are collinear. To my best knowledge, the easiest way to prove this is to first prove the original problem.

2.6 USA TSTST 2011/4

The solution to this problem involves a synthetic observation, followed by an application of Vieta. Normally, solving for P involves an incredibly painful quadratic. However, by first computing the other intersection, we can actually use Vieta to deduce the coordinate of the point we care about.

2.6.1 Problem and Solution

Problem (TSTST Problem 4)**.** Acute triangle ABC is inscribed in circle ω. Let H and O denote its orthocenter and circumcenter, respectively. Let M and N be the midpoints of sides AB and AC, respectively. Rays MH and NH meet ω at P and Q, respectively. Lines MN and PQ meet at R. Prove that $OA \perp RA$.

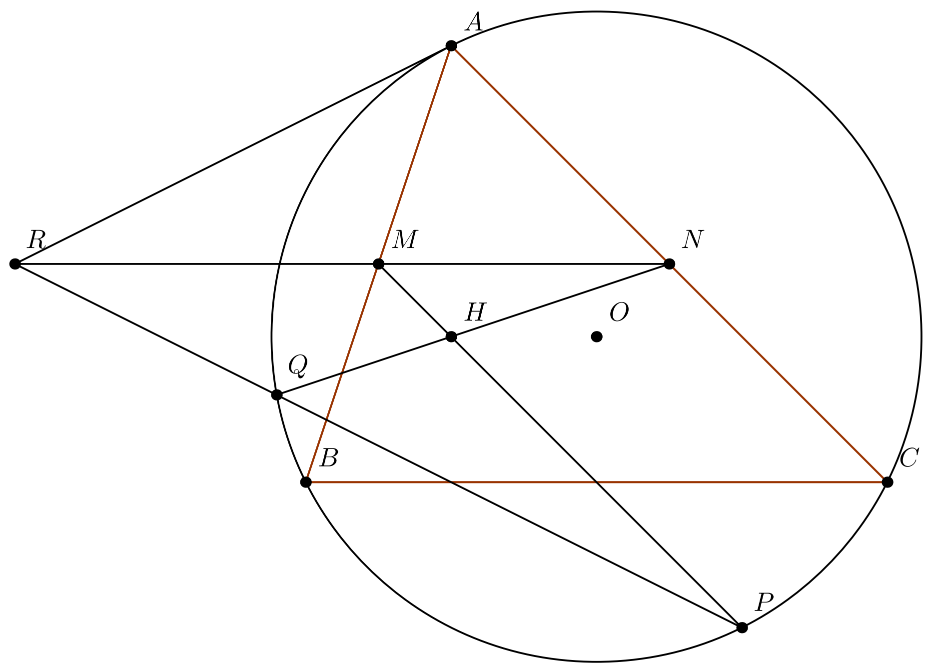

Figure 6: 2011 TSTST, Problem 4

Solution. (Evan Chen) We use barycentric coordinates. Set $A = (1, 0, 0)$, $B = (0, 1, 0)$ and $C = (0, 0, 1)$. Let $T_A = \frac{1}{S_A}$ in Conway's Notation, so that $H = (T_A : T_B : T_C)$.

First we will compute the point $R' = (x : y : z)$ on MN with $R'A \perp OA$. Setting $\vec{O} = 0$, and applying Strong EFFT, we find that the point satisfies equations $c^2y + b^2z = 0$ (per EFFT) and $x = y + z$ (per midline). Thus, we may write $\boxed{R' = (b^2 - c^2 : b^2 : -c^2)}$.

Next, we will compute the coordinates of P. Check that we have the line

$$HM : x - y + \frac{T_B - T_A}{T_C} z = 0$$

Furthermore, the circumcircle has equation $z[a^2y + b^2x] + c^2xy = 0$. WLOG, homogenize so that $z = -T_C$; then the first equation becomes $y = x + T_A - T_B$. Substituting in the second equation, this becomes simply

$$c^2x(x + (T_A - T_B)) + a^2(-T_C)(x + T_A - T_B) + b^2(-T_C)(x) = 0$$

Expanding, and collecting coefficients of x, this reduces to

$$c^2x^2 + [c^2(T_A - T_B) - a^2T_C - b^2T_C]x + a^2(-T_C)(T_A - T_B) = 0$$

Now here's the trick: it's well known that the reflection of H across M lies on the circle. So the other solution to this equation is $\vec{H}' = \vec{A} + \vec{B} - \vec{H}$ (since $H'AHB$ is a parallelogram). We can thus quickly deduce that, in barycentric coordinates, $H' = (T_B + T_C : T_A + T_C : -T_C)$. So by Vieta's Formulas, the x-coordinate we seek is just

$$x = -\frac{c^2(T_A - T_B) - a^2T_C - b^2T_C}{c^2} - (T_B + T_C) = \frac{a^2 + b^2 - c^2}{c^2}T_C - T_A$$

Hence,

$$y = \frac{a^2 + b^2 - c^2}{c^2}T_C - T_B$$

So finally, we obtain

$$\boxed{P = \left(\frac{a^2 + b^2 - c^2}{c^2}T_C - T_A : \frac{a^2 + b^2 - c^2}{c^2}T_C - T_B : -T_C\right)}$$

Similarly,

$$\boxed{Q = \left(\frac{a^2 + c^2 - b^2}{b^2}T_B - T_A : -T_B : \frac{a^2 + c^2 - b^2}{b^2}T_B - T_C\right)}$$

It remains to show that P, Q and R' are collinear. This occurs if and only if

$$0 = \det \begin{pmatrix} \frac{a^2+b^2-c^2}{c^2}T_C - T_A & \frac{a^2+b^2-c^2}{c^2}T_C - T_B & -T_C \\ \frac{a^2+c^2-b^2}{c^2}T_B - T_A & -T_B & \frac{a^2+c^2-b^2}{b^2}T_B - T_C \\ b^2 - c^2 & b^2 & -c^2 \end{pmatrix}$$

Subtracting the second and third column from the first, this is equivalent to

$$0 = \det \begin{pmatrix} T_C + T_B - T_A & \frac{a^2+b^2-c^2}{c^2}T_C - T_B & -T_C \\ T_C + T_B - T_A & -T_B & \frac{a^2+c^2-b^2}{b^2}T_B - T_C \\ 0 & b^2 & -c^2 \end{pmatrix}$$

Expanding the determinant, and factoring out the term $T_C + T_B - T_A$, we see that it suffices to show that

$$0 = c^2T_B - b^2T_C - \left[-(c^2)\left(\frac{a^2 + b^2 - c^2}{c^2}T_C - T_B\right) + (b^2)\left(\frac{a^2 + c^2 - b^2}{b^2}T_B - T_C\right)\right]$$

Expanding, this is just

$$-(a^2 - b^2 + c^2)T_B + (a^2 + b^2 - c^2)T_C$$

But $T_B = \frac{1}{S_B} = \frac{2}{a^2-b^2+c^2}$ and $T_C = \frac{2}{a^2+b^2-c^2}$. Since $-2 + 2 = 0$, we conclude that the three points are collinear.

2.6.2 Commentary

The application of Vieta played a critical role in this problem. BTW, during the actual TSTST, only one person successfully bashed this problem (via complex numbers). Had I known about this technique at the time, I think I could have gotten a positive score on Day 2.

2.7 USA TSTST 2011/2

Finally, to complete the section, we present an example of a bash which is possibly not feasible by hand. Incidentally, this is the only problem in the section that was not done by hand. Max Schindler claims this is doable if you have the computational fortitude.

The problem has been inverted around the point A, and the constant c has been replaced by $1/c$.

2.7.1 Problem and Solution

Problem (TSTST Problem 2, Inverted)**.** Let APQ be an obtuse triangle with $\angle A > 90°$. The tangents at P and Q to the circumcircle of APQ meet at the point B. Points X and Y are selected on segments PB and QS, respectively, such that $PX = c \cdot QA$, and $QY = c \cdot PA$. Then, PA and QA are extended through A to R and S, so that $AR = AS = c \cdot \frac{PA \cdot QA}{PQ}$. O is the point such that $AROS$ is a parallelogram. Given that $AXOY$ is cyclic, prove that $\angle PXA = \angle QYA$.

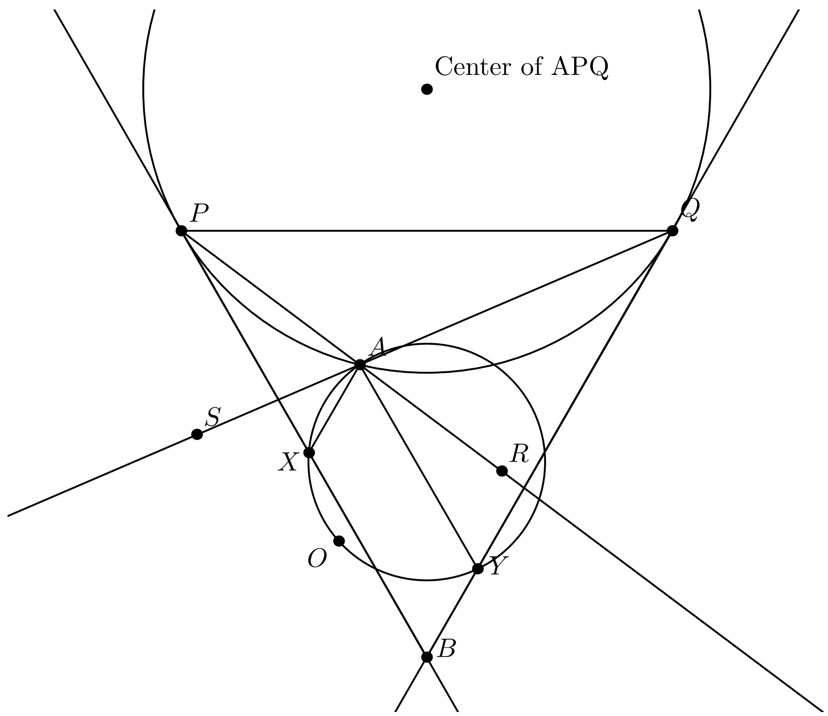

Figure 7: 2011 TSTST, Problem 2

Solution. The reference triangle is APQ, with $A = (1,0,0)$, $P = (0,1,0)$ and $Q = (0,0,1)$. To avoid confusion, let $a = PQ$, $q = AP$ and $p = AQ$.

It is not hard at all to compute R and S. Since $AR = AS = \frac{cpq}{a}$, and $A = (1,0,0)$, $P = (0,1,0)$ and $Q = (0,0,1)$, a quick application of ratios of lengths yields

$$R = \left(\frac{cq}{a} + 1, \frac{-cq}{a}, 0\right)$$

and

$$S = \left(\frac{cp}{a} + 1, 0, \frac{-cp}{a}\right)$$

Next we want to compute $X = (x, y, z)$. Since X lies on the tangent PB, it satisfies $0 = b^2y + c^2z$ (see the earlier problem, 2.6). Letting $x + y + z = 1$, and invoking the distance formula, we find that $(cq)^2 = a^2(y-1)z + b^2zx + c^2x(y-1)$. Solving this system by writing everything in z, and taking the negative root, gives $X = \left(\frac{ac}{p}, \frac{-a^2c+ap+cq^2}{ap}, -\frac{cq^2}{ap}\right)$.

Similarly, $Y = \left(\frac{ac}{q}, -\frac{cp^2}{aq}, \frac{a^2(-c)+aq+cp^2}{aq}\right)$. We then obtain that $O = \left(1 + \frac{cp}{a} + \frac{cq}{a}, -\frac{cp}{a}, -\frac{cq}{a}\right)$

We'd like to find criteria for when $\angle PXA = \angle QYA$. Noting the equal angles $\angle BPQ = \angle BQP$, we extend XA and YA through A to points X' and Y' respectively. Then $X' = (0 : pa + q^2c - a^2c : -q^2c)$ and $Y' = (0 : -p^2c : qa + p^2c - a^2c)$ (aren't cevians wonderful?). In that case, we obtain the lengths

$$PX' = \frac{-q^2c}{pa - a^2c}a = \frac{-q^2c}{p - ac}$$

and similarly,

$$QY' = \frac{-p^2c}{q - ac}$$

Now $\angle PXA = \angle QYA$ if and only if the triangles XPX' and YQY' are similar! Recall that $PX = cq$ and $QY = cp$; by SAS, this occurs if and only if

$$\frac{\frac{-q^2c}{p-ac}}{cq} = \frac{\frac{-p^2c}{q-ac}}{cp} \Leftrightarrow q(q-ac) = p(p-ac) \Leftrightarrow c = \frac{p+q}{a}$$

So we just need to show that if $AROS$ is cyclic, then $c = \frac{p+q}{a}$.

Unfortunately, this is where the calculations get heavy.[2]

Putting A, X, Y into the circle formula and solving for u, v, w yields

$$\begin{aligned} u &= 0 \\ v &= (c^2pq^2(-a^2c + aq + cp(p+q)))/(-a^3c^2 + a^2c(p+q) - c(p^3+q^3) + a(-pq + c^2(p^2+q^2))) \\ w &= (c^2p^2q(-a^2c + ap + cq(p+q)))/(-a^3c^2 + a^2c(p+q) - c(p^3+q^3) + a(-pq + c^2(p^2+q^2))) \end{aligned}$$

Substituting the point O into the equation and simplifying, we get that

$$\frac{-(cpq(a^5c^3-2a^4c^2(p+q)-c^2(p+q)^3(p^2-pq+q^2)+a^3c((1-2c^2)p^2+3pq+(1-2c^2)q^2)+ac(p+q)^2((-1+c^2)p^2-c^2pq+(-1+c^2)q^2)+a^2(p+q)(-pq+3c^2(p^2+q^2))))}{(a^2(a^3c^2-a^2c(p+q)+c(p^3+q^3)-a(-pq+c^2(p^2+q^2))))}=0$$

Clearing the denominator, this is just

$$\begin{aligned} 0 = &- cpq(a^5c^3 - 2a^4c^2(p+q) \\ &- c^2(p+q)^3(p^2 - pq + q^2) \\ &+ a^3c((1-2c^2)p^2 + 3pq + (1-2c^2)q^2) \\ &+ ac(p+q)^2((-1+c^2)p^2 - c^2pq + (-1+c^2)q^2) \\ &+ a^2(p+q)(-pq + 3c^2(p^2+q^2))) \end{aligned}$$

Noting that $c = 0$ and $c = \frac{p+q}{a}$ are solutions (the latter "by inspection") we can divide this out and collect terms in c to get

$$c^2\left(-a^4 - p^4 - p^3q - pq^3 - q^4 + 2a^2\left(p^2+q^2\right)\right) + c\left(a^3(p+q) - a\left(p^3 + p^2q + pq^2 + q^3\right)\right) - a^2pq = 0$$

[2] I wonder if there's any synthetic interpretation that would lead directly to this expression for c. That would be nice too. Tell me if you find one!

We need to show that there are no positive reals c satisfying this. A quick glance at the constant term tells us that we should probably use the discriminant; our only hope is that there are no real roots at all. Computing, we get

$$\begin{aligned}\Delta =& a^6p^2 - 2a^4p^4 + a^2p^6 - 2a^6pq + 4a^4p^3q - 2a^2p^5q \\ &+ a^6q^2 - 4a^4p^2q^2 - a^2p^4q^2 + 4a^4pq^3 + 4a^2p^3q^3 \\ &- 2a^4q^4 - a^2p^2q^4 - 2a^2pq^5 + a^2q^6\end{aligned}$$

which factors as

$$\Delta = -a^2(-a+p+q)(p-q)^2(a+p-q)(a-p+q)(a+p+q)$$

Since a, p and q are the sides of a triangle, this is always negative.

□

2.7.2 Commentary

The calculations are relatively easy up until incorporating the condition that $AROS$ is cyclic. Despite the fact that the "cyclic" condition was fairly brutal, it illustrates a few points worth making: (i) similar triangles are your friends in barycentrics too, because they can yield equal angles, (ii) inversion can provide a way to eliminate awkward circle conditions, (iii) unusual side length conditions related to the sides of the triangles is not a problem, and (iv) cyclic quadrilaterals suck.

Please let us know if you find a cyclic criteria which completes a reasonable solution! We'd like to have a human to attribute the solution to...

□

3 Problems

There is light at the end of the tunnel, but it is moving away at speed c.

Here, we provide some problems for the reader to attempt. Some are easier than others. Infinity [1] has a longer list than is included here; see `http://www.bmoc.maths.org/home/areals.pdf` for the relevant excerpt.

For practice "in real life", i.e. determining when barycentric coordinates are feasible for a problem (here, they are), you can often simply take an arbitrary geometry problem that begins with the words "triangle ABC", and see if it's possible to get a barycentric solution. Some will, of course, take much longer than others, but a good portion will have a reasonable solution.

3.1 Problems

1. Prove Stewart's Theorem.

2. Prove Routh's Theorem.

3. (USA TST 2003/2) Let ABC be a triangle and let P be a point in its interior. Lines PA, PB, PC intersect sides BC, CA, AB at D, E, F, respectively. Prove that

$$[PAF] + [PBD] + [PCE] = \frac{1}{2}[ABC]$$

   if and only if P lies on at least one of the medians of triangle ABC. (Here $[XYZ]$ denotes the area of triangle XYZ.)

4. (Mongolia TST 2000/6) In a triangle ABC, the angle bisector at A,B,C meet the opposite sides at A_1,B_1,C_1, respectively. Prove that if the quadrilateral $BA_1B_1C_1$ is cyclic, then

$$\frac{AC}{AB+BC} = \frac{AB}{AC+BC} + \frac{BC}{BA+AC}$$

5. (ISL 2005/G1) Given a triangle ABC satisfying $AC + BC = 3 \cdot AB$. The incircle of triangle ABC has center I and touches the sides BC and CA at the points D and E, respectively. Let K and L be the reflections of the points D and E with respect to I. Prove that the points A, B, K, L lie on one circle.

6. (APMO 2005/5) In a triangle ABC, points M and N are on sides AB and AC, respectively, such that $MB = BC = CN$. Let R and r denote the circumradius and the inradius of the triangle ABC, respectively. Express the ratio MN/BC in terms of R and r.

7. (ISL 2008/G4) In an acute triangle ABC segments BE and CF are altitudes. Two circles passing through the point A and F and tangent to the line BC at the points P and Q so that B lies between C and Q. Prove that lines PE and QF intersect on the circumcircle of triangle AEF.

8. (ISL 2001/G6) Let ABC be a triangle and P an exterior point in the plane of the triangle. Suppose the lines AP, BP, CP meet the sides BC, CA, AB (or extensions thereof) in D, E, F, respectively. Suppose further that the areas of triangles PBD, PCE, PAF are all equal. Prove that each of these areas is equal to the area of triangle ABC itself.

9. (IMO 2007/2) Consider five points A, B, C, D and E such that $ABCD$ is a parallelogram and $BCED$ is a cyclic quadrilateral. Let ℓ be a line passing through A. Suppose that ℓ intersects the interior of the segment DC at F and intersects line BC at G. Suppose also that $EF = EG = EC$. Prove that ℓ is the bisector of angle DAB.

3.2 Hints

Hints have been obfuscated by ROT13.

1. ABG UNEQ
2. FGENVTUGSBEJNEQ
3. HFR NERN SBEZ
4. PVEPYR SBEZ FBYIR HIJ
5. PVEPF GURA J VF ZVAHF N O
6. QVFG SNPGBE GURA TRB VQ
7. CBJRE CBVAG FUBJF NRCD PLPYVP GURA NATYR PUNFR
8. PNERSHY FVTARQ NERNF
9. ERS OPQ... SVAQ S T FLF RDA RYVZ P M SNPGBE

4 Additional Problems from MOP 2012

If our solution has at table of contents, how should we number the pages?
– Evan Chen, MOP 2012

Here's a more recent collection of geometry problems solvable using barycentric coordinates. These are all problems which appeared at MOP 2012,[3] along with an IMO question from that year.

These are in very roughly order, based both on the amount of computation and the difficulty of setup

1. (IMO 2012) Given triangle ABC the point J is the center of the excircle opposite the vertex A. This excircle is tangent to the side BC at M, and to the lines AB and AC at K and L, respectively. The lines LM and BJ meet at F, and the lines KM and CJ meet at G. Let S be the point of intersection of the lines AF and BC, and let T be the point of intersection of the lines AG and BC. Prove that M is the midpoint of ST.

2. (USA TST 2012) In acute triangle ABC, $\angle A < \angle B$ and $\angle A < \angle C$. Let P be a variable point on side BC. Points D and E lie on sides AB and AC, respectively, such that $BP = PD$ and $CP = PE$. Prove that if P moves along side BC, the circumcircle of triangle ADE passes through a fixed point other than A.

3. (ELMO[4] 2012/5) Let ABC be an acute triangle with $AB < AC$, and let D and E be points on side BC such that $BD = CE$ and D lies between B and E. Suppose there exists a point P inside ABC such that $PD \parallel AE$ and $\angle PAB = \angle EAC$. Prove that $\angle PBA = \angle PCA$.

4. (Saudi Arabia TST 2012) Let $ABCD$ be a convex quadrilateral such that $AB = AC = BD$. The lines AC and BD meet at point O, the circles ABC and ADO meet again at point P, and the lines AP and BC meet at point Q. Show that $\angle COQ = \angle DOQ$.

5. (ISL 2011/G2) Let $A_1A_2A_3A_4$ be a non-cyclic quadrilateral. For $1 \le i \le 4$, let O_i and r_i be the circumcenter and the circumradius of triangle $A_{i+1}A_{i+2}A_{i+3}$ (where $A_{i+4} = A_i$). Prove that

$$\frac{1}{O_1A_1^2 - r_1^2} + \frac{1}{O_2A_2^2 - r_2^2} + \frac{1}{O_3A_3^2 - r_3^2} + \frac{1}{O_4A_4^2 - r_4^2} = 0$$

6. (USA TSTST 2012) Triangle ABC is inscribed in circle Ω. The interior angle bisector of angle A intersects side BC and Ω at D and L (other than A), respectively. Let M be the midpoint of side BC. The circumcircle of triangle ADM intersects sides AB and AC again at Q and P (other than A), respectively. Let N be the midpoint of segment PQ, and let H be the foot of the perpendicular from L to line ND. Prove that line ML is tangent to the circumcircle of triangle HMN.

7. (ISL 2011/G4) Acute triangle ABC is inscribed in circle Ω. Let B_0, C_0, D lie on sides AC, AB, BC, respectively, such that $AB_0 = CB_0$, $AC_0 = BC_0$, and $AD \perp BC$. Let G denote the centroid of triangle ABC, and let ω be a circle through B_0 and C_0 that is tangent to Ω at a point X other than A. Prove that D, G, X are collinear.

8. Let ABC be a triangle with circumcenter O and let the angle bisector of $\angle BAC$ intersect BC at D. The point M is such that $MC \perp BC$ and $MA \perp AD$. Lines BM and OA intersect at the point P. Show that the circle centered at P and passing through A is tangent to segment BC.

9. (USA TSTST 2012) Let $ABCD$ be a quadrilateral with $AC = BD$. Diagonals AC and BD meet at P. Let ω_1 and O_1 denote the circumcircle and circumcenter of triangle ABP. Let ω_2 and O_2 denote the circumcircle and circumcenter of triangle CDP. Segment BC meets ω_1 and ω_2 again at S and T (other than B and C), respectively. Let M and N be the midpoints of minor arcs SP (not including B) and TP (not including C). Prove that $MN \parallel O_1O_2$.

[3] MOP, occasionally called MOSP, stands for Mathematical Olympiad Program; it is the USA training program for the IMO.
[4] This year's test was called "Every Little Mistake $\Rightarrow$ 0".

5 References

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