A Brief Introduction to Olympiad Inequalities

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The goal of this document is to provide a easier introduction to olympiad inequalities than the standard exposition *Olympiad Inequalities*, by Thomas Mildorf. I was motivated to write it by feeling guilty for getting free 7's on problems by simply regurgitating a few tricks I happened to know, while other students were unable to solve the problem.

Warning: These are notes, not a full handout. Lots of the exposition is very minimal, and many things are left to the reader.

In a problem with n variables, these respectively mean to cycle through the n variables, and to go through all $n!$ permutations. To provide an example, in a three-variable problem we might write

$$\begin{aligned}\sum_{\text{cyc}} a^2 &= a^2+b^2+c^2\\ \sum_{\text{cyc}} a^2b &= a^2b+b^2c+c^2a\\ \sum_{\text{sym}} a^2 &= a^2+a^2+b^2+b^2+c^2+c^2\\ \sum_{\text{sym}} a^2b &= a^2b+a^2c+b^2c+b^2a+c^2a+c^2b.\end{aligned}$$

§1 Polynomial Inequalities

§1.1 AM-GM and Muirhead

Consider the following theorem.

Theorem 1.1 (AM-GM)

For nonnegative reals $a_1, a_2, \dots, a_n$ we have

$$\frac{a_1+a_2+\cdots+a_n}{n} \ge \sqrt[n]{a_1 \dots a_n}.$$

Equality holds if and only if $a_1 = a_2 = \cdots = a_n$.

For example, this implies

$$a^2+b^2 \ge 2ab, \quad a^3+b^3+c^3 \ge 3abc.$$

Adding such inequalities can give us some basic propositions.

Example 1.2

Prove that $a^2+b^2+c^2 \ge ab+bc+ca$ and $a^4+b^4+c^4 \ge a^2bc+b^2ca+c^2ab$.

Proof. By AM-GM,

$$\frac{a^2+b^2}{2} \ge ab \text{ and } \frac{2a^4+b^4+c^4}{4} \ge a^2bc.$$

Similarly,

$$\frac{b^2+c^2}{2} \ge bc \text{ and } \frac{2b^4+c^4+a^4}{4} \ge b^2ca.$$

$$\frac{c^2+a^2}{2} \ge ca \text{ and } \frac{2c^4+a^4+b^4}{4} \ge c^2ab.$$

Summing the above statements gives

$$a^2+b^2+c^2 \ge ab+bc+ca \text{ and } a^4+b^4+c^4 \ge a^2bc+b^2ca+c^2ab. \qquad \square$$

Exercise 1.3. Prove that $a^3+b^3+c^3 \ge a^2b+b^2c+c^2a$.

Exercise 1.4. Prove that $a^5+b^5+c^5 \ge a^3bc+b^3ca+c^3ab \ge abc(ab+bc+ca)$.

The fundamental intuition is being able to decide which symmetric polynomials of a given degree are bigger. For example, for degree 3, the polynomial $a^3+b^3+c^3$ is biggest and abc is the smallest. Roughly, the more "mixed" polynomials are the smaller. From this, for example, one can immediately see that the inequality

$$(a+b+c)^3 \ge a^3+b^3+c^3+24abc$$

must be true, since upon expanding the LHS and cancelling $a^3+b^3+c^3$, we find that the RHS contains only the piddling term $24abc$. That means a straight AM-GM will suffice.

A useful formalization of this is Muirhead's Inequality. Suppose we have two sequences $x_1 \ge x_2 \ge \dots \ge x_n$ and $y_1 \ge y_2 \ge \dots \ge y_n$ such that

$$x_1+x_2+\dots+x_n = y_1+y_2+\dots+y_n,$$

and for $k = 1, 2, \dots, n-1$

$$x_1+x_2+\dots+x_k \ge y_1+y_2+\dots+y_k,$$

Then we say that (x_n) *majorizes* (y_n), written $(x_n) \succ (y_n)$.

Using the above, we have the following theorem.

Theorem 1.5 (Muirhead's Inequality)

If$a_1, a_2, \dots, a_n$ are positive reals, and (x_n) majorizes (y_n) then we have the inequality.

$$\sum_{\text{sym}} a_1^{x_1}a_2^{x_2}\dots a_n^{x_n} \ge \sum_{\text{sym}} a_1^{y_1}a_2^{y_2}\dots a_n^{y_n}.$$

Example 1.6

Since $(5,0,0) \succ (3,1,1) \succ (2,2,1)$,

$$\begin{aligned} a^5+a^5+b^5+b^5+c^5+c^5 &\ge a^3bc+a^3bc+b^3ca+b^3ca+c^3ab+c^3ab \\ &\ge a^2b^2c+a^2b^2c+b^2c^2a+b^2c^2a+c^2a^2b+c^2a^2b. \end{aligned}$$

From this we derive $a^5+b^5+c^5 \ge a^3bc+b^3ca+c^3ab \ge abc(ab+bc+ca)$.

Notice that Muirhead is *symmetric*, not *cyclic*. For example, even though $(3,0,0) \succ (2,1,0)$, Muirhead's inequality only gives that

$$2(a^3+b^3+c^3) \ge a^2b+a^2c+b^2c+b^2a+c^2a+c^2b$$

and in particular this does *not* imply that $a^3+b^3+c^3 \ge a^2b+b^2c+c^2a$. These situations must still be resolved by AM-GM.

§1.2 Non-homogeneous inequalities

Consider the following example.

Example 1.7

Prove that if $abc = 1$ then $a^2+b^2+c^2 \ge a+b+c$.

Proof. AM-GM alone is hopeless here, because whenever we apply AM-GM, the left and right hand sides of the inequality all have the same degree. So we want to use the condition $abc = 1$ to force the problem to have the same degree. The trick is to notice that the given inequality can be rewritten as

$$a^2+b^2+c^2 \ge a^{1/3}b^{1/3}c^{1/3}\left(a+b+c\right).$$

Now the inequality is homogeneous. Observe that if we multiply a, b, c by any real number $k > 0$, all that happens is that both sides of the inequality are multiplied by k^2, which doesn't change anything. That means the condition $abc = 1$ can be ignored now. Since $(2,0,0) \succ (\frac{4}{3}, \frac{1}{3}, \frac{1}{3})$, applying Muirhead's Inequality solves the problem. $\square$

The importance of this problem is that it shows us how to eliminate a given condition by homogenizing the inequality; this is very important. (In fact, we will soon see that we can use this in reverse — we can impose an arbitrary condition on a homogeneous inequality.)

§1.3 Practice Problems

1. $a^7+b^7+c^7 \ge a^4b^3+b^4c^3+c^4a^3$.
2. If $a+b+c=1$, then $\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \le 3+2\cdot\frac{(a^3+b^3+c^3)}{abc}$.
3. $\frac{a^3}{bc}+\frac{b^3}{ca}+\frac{c^3}{ab} \ge a+b+c$.
4. If $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=1$, then $(a+1)(b+1)(c+1) \ge 64$.
5. (USA 2011) If $a^2+b^2+c^2+(a+b+c)^2 \le 4$, then
$$\frac{ab+1}{(a+b)^2}+\frac{bc+1}{(b+c)^2}+\frac{ca+1}{(c+a)^2} \ge 3.$$

6. If $abcd = 1$, then $a^4b + b^4c + c^4d + d^4a \geq a + b + c + d$.

§2 Inequalities in Arbitrary Functions

Let $f : (u, v) \to \mathbb{R}$ be a function and let $a_1, a_2, \dots, a_n \in (u, v)$. Suppose that we fix $\frac{a_1+a_2+\cdots+a_n}{n} = a$ (if the inequality is homogeneous, we will often insert such a condition) and we want to prove that

$$f(a_1) + f(a_2) + \cdots + f(a_n)$$

is at least (or at most) $nf(a)$. In this section we will provide three methods for doing so.

We say that function f is *convex* if $f''(x) \geq 0$ for all x; we say it is *concave* if $f''(x) \leq 0$ for all x. Note that f is convex if and only if $-f$ is concave.

§2.1 Jensen / Karamata

Theorem 2.1 (Jensen's Inequality)
If f is convex, then

$$\frac{f(a_1) + \cdots + f(a_n)}{n} \geq f\left(\frac{a_1 + \cdots + a_n}{n}\right).$$

The reverse inequality holds when f is concave.

Theorem 2.2 (Karamata's Inequality)
If f is convex, and (x_n) majorizes (y_n) then

$$f(x_1) + \cdots + f(x_n) \geq f(y_1) + \cdots + f(y_n).$$

The reverse inequality holds when f is concave.

Example 2.3 (Shortlist 2009)
Given $a + b + c = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$, prove that

$$\frac{1}{(2a+b+c)^2} + \frac{1}{(a+2b+c)^2} + \frac{1}{(a+b+2c)^2} \leq \frac{3}{16}.$$

Proof. First, we want to eliminate the condition. The original problem is equivalent to

$$\frac{1}{(2a+b+c)^2} + \frac{1}{(a+2b+c)^2} + \frac{1}{(a+b+2c)^2} \leq \frac{3}{16} \cdot \frac{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}{a+b+c}.$$

Now the inequality is homogeneous, so we can assume that $a + b + c = 3$. Now our original problem can be rewritten as

$$\sum_{\text{cyc}} \frac{1}{16a} - \frac{1}{(a+3)^2} \geq 0.$$

Set $f(x) = \frac{1}{16x} - \frac{1}{(x+3)^2}$. We can check that f over $(0, 3)$ is convex so Jensen completes the problem. $\square$

Example 2.4
Prove that

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \ge 2\left(\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\right) \ge \frac{9}{a+b+c}.$$

Proof. The problem is equivalent to

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \ge \frac{1}{\frac{a+b}{2}}+\frac{1}{\frac{b+c}{2}}+\frac{1}{\frac{c+a}{2}} \ge \frac{1}{\frac{a+b+c}{3}}+\frac{1}{\frac{a+b+c}{3}}+\frac{1}{\frac{a+b+c}{3}}.$$

Assume WLOG that $a \ge b \ge c$. Let $f(x) = 1/x$. Since

$$(a,b,c) \succ \left(\frac{a+b}{2}, \frac{a+c}{2}, \frac{b+c}{2}\right) \succ \left(\frac{a+b+c}{3}, \frac{a+b+c}{3}, \frac{a+b+c}{3}\right)$$

the conclusion follows by Karamata. □

Example 2.5 (APMO 1996)
If a, b, c are the three sides of a triangle, prove that

$$\sqrt{a+b-c}+\sqrt{b+c-a}+\sqrt{c+a-b} \le \sqrt{a}+\sqrt{b}+\sqrt{c}.$$

Proof. Again assume WLOG that $a \ge b \ge c$ and notice that $(a+b-c, c+a-b, b+c-a) \succ (a,b,c)$. Apply Karamata on $f(x) = \sqrt{x}$. □

§2.2 Tangent Line Trick

Again fix $a = \frac{a_1+\cdots+a_n}{n}$. If f is not convex, we can sometimes still prove the inequality

$$f(x) \ge f(a) + f'(a)\,(x-a)\,.$$

If this inequality manages to hold for all x, then simply summing the inequality will give us the desired conclusion. This method is called the *tangent line trick*.

Example 2.6 (David Stoner)
If $a+b+c = 3$, prove that

$$18\sum_{\text{cyc}} \frac{1}{(3-c)(4-c)} + 2(ab+bc+ca) \ge 15.$$

Proof. We can rewrite the given inequality as

$$\sum_{\text{cyc}} \left(\frac{18}{(3-c)(4-c)} - c^2\right) \ge 6.$$

Using the tangent line trick lets us obtain the magical inequality

$$\frac{18}{(3-c)(4-c)} - c^2 \ge \frac{c+3}{2} \iff c(c-1)^2(2c-9) \le 0$$

and the conclusion follows by summing. □

Example 2.7 (Japan)

Prove $\sum_{\text{cyc}} \frac{(b+c-a)^2}{a^2+(b+c)^2} \ge \frac{3}{5}$.

Proof. Since the inequality is homogeneous, we may assume WLOG that $a+b+c=3$. So the inequality we wish to prove is

$$\sum_{\text{cyc}} \frac{(3-2a)^2}{a^2+(3-a)^2} \ge \frac{3}{5}.$$

With some computation, the tangent line trick gives away the magical inequality:

$$\frac{(3-2a)^2}{(3-a)^2+a^2} \ge \frac{1}{5} - \frac{18}{25}(a-1) \iff \frac{18}{25}(a-1)^2 \frac{2a+1}{2a^2-6a+9} \ge 0. \qquad \square$$

§2.3 $n-1$ EV

The last such technique is $n-1$ EV. This is a brute force method involving much calculus, but it is nonetheless a useful weapon.

Theorem 2.8 ($n-1$ EV)

Let $a_1, a_2, \dots, a_n$ be real numbers, and suppose $a_1 + a_2 + \cdots + a_n$ is fixed. Let $f : \mathbb{R} \to \mathbb{R}$ be a function with exactly one inflection point. If

$$f(a_1) + f(a_2) + \cdots + f(a_n)$$

achieves a maximal or minimal value, then $n-1$ of the a_i are equal to each other.

Proof. See page 15 of *Olympiad Inequalities*, by Thomas Mildorf. The main idea is to use Karamata to "push" the a_i together. $\square$

Example 2.9 (IMO 2001 / APMOC 2014)

Let a, b, c be positive reals. Prove $1 \le \sum_{\text{cyc}} \frac{a}{\sqrt{a^2+8bc}} < 2$.

Proof. Set $e^x = \frac{bc}{a^2}$, $e^y = \frac{ca}{b^2}$, $e^z = \frac{ab}{c^2}$. We have the condition $x+y+z=0$ and want to prove

$$1 \le f(x) + f(y) + f(z) < 2$$

where $f(x) = \frac{1}{\sqrt{1+8e^x}}$. You can compute

$$f''(x) = \frac{4e^x\left(4e^x-1\right)}{\left(8e^x+1\right)^{\frac{5}{2}}}$$

so by $n-1$ EV, we only need to consider the case $x=y$. Let $t=e^x$; that means we want to show that

$$1 \le \frac{2}{\sqrt{1+8t}} + \frac{1}{\sqrt{1+8/t^2}} < 2.$$

Since this a function of one variable, we can just use standard Calculus BC methods. $\square$

Example 2.10 (Vietnam 1998)

Let x_1, x_2, ..., x_n be positive reals satisfying $\sum_{i=1}^{n} \frac{1}{1998+x_i} = \frac{1}{1998}$. Prove

$$\frac{\sqrt[n]{x_1 x_2 \dots x_n}}{n-1} \geq 1998.$$

Proof. Let $y_i = \frac{1998}{1998+x_i}$. Since $y_1 + y_2 + \dots + y_n = 1$, the problem becomes

$$\prod_{i=1}^{n} \left(\frac{1}{y_i} - 1 \right) \geq (n-1)^n .$$

Set $f(x) = \ln \left(\frac{1}{x} - 1 \right)$, so the inequality becomes $f(y_1) + \dots + f(y_n) \geq nf\left(\frac{1}{n}\right)$. We can prove that

$$f''(y) = \frac{1-2y}{(y^2-y)^2}.$$

So f has one inflection point, we can assume WLOG that $y_1 = y_2 = \dots y_{n-1}$. Let this common value be t; we only need to prove

$$(n-1) \ln \left(\frac{1}{t} - 1 \right) + \ln \left(\frac{1}{1-(n-1)t} - 1 \right) \geq n \ln(n-1).$$

Again, since this is a one-variable inequality, calculus methods suffice. □

§2.4 Practice Problems

1. Use Jensen to prove AM-GM.
2. If $a^2 + b^2 + c^2 = 1$ then $\frac{1}{a^2+2} + \frac{1}{b^2+2} + \frac{1}{c^2+2} \leq \frac{1}{6ab+c^2} + \frac{1}{6bc+a^2} + \frac{1}{6ca+b^2}$.
3. If $a + b + c = 3$ then
$$\sum_{\text{cyc}} \frac{a}{2a^2+a+1} \leq \frac{3}{4}.$$
4. (MOP 2012) If $a + b + c + d = 4$, then $\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + \frac{1}{d^2} \geq a^2 + b^2 + c^2 + d^2$.

§3 Eliminating Radicals and Fractions

§3.1 Weighted Power Mean

AM-GM has the following natural generalization.

Theorem 3.1 (Weighted Power Mean)

Let $a_1, a_2, \dots, a_n$ and w_1, w_2, ..., w_n be positive reals with $w_1 + w_2 + \dots + w_n = 1$. For any real number r, we define

$$\mathcal{P}(r) = \begin{cases} (w_1 a_1^r + w_2 a_2^r + \dots + w_n a_n^r)^{1/r} & r \neq 0 \\ a_1^{w_1} a_2^{w_2} \dots a_n^{w_n} & r = 0. \end{cases}$$

If $r > s$, then $\mathcal{P}(r) \geq \mathcal{P}(s)$ equality occurs if and only if $a_1 = a_2 = \dots = a_n$.

In particular, if $w_1 = w_2 = \cdots = w_n = \frac{1}{n}$, the above $\mathcal{P}(r)$ is just

$$\mathcal{P}(r) = \begin{cases} \left(\frac{a_1^r + a_2^r + \cdots + a_n^r}{n}\right)^{1/r} & r \neq 0 \\ \sqrt[n]{a_1 a_2 \dots a_n} & r = 0. \end{cases}$$

By setting $r = 2, 1, 0, -1$ we derive

$$\sqrt{\frac{a_1^2 + \cdots + a_n^2}{n}} \geq \frac{a_1 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \dots a_n} \geq \frac{n}{\frac{1}{a_1} + \cdots + \frac{1}{a_n}}$$

which is QM-AM-GM-HM. Moreover, AM-GM lets us "add" roots, like

$$\sqrt{a} + \sqrt{b} + \sqrt{c} \leq 3\sqrt{\frac{a+b+c}{3}}.$$

Example 3.2 (Taiwan TST Quiz)

Prove $3(a+b+c) \geq 8\sqrt[3]{abc} + \sqrt[3]{\frac{a^3+b^3+c^3}{3}}$.

Proof. By Power Mean with $r = 1$, $s = \frac{1}{3}$, $w_1 = \frac{1}{9}$, $w_2 = \frac{8}{9}$, we find that

$$\left(\frac{1}{9}\sqrt[3]{\frac{a^3+b^3+c^3}{3}} + \frac{8}{9}\sqrt[3]{abc}\right)^3 \leq \frac{1}{9}\left(\frac{a^3+b^3+c^3}{3}\right) + \frac{8}{9}(abc).$$

so we want to prove $a^3 + b^3 + c^3 + 24abc \leq (a+b+c)^3$, which is clear. □

§3.2 Cauchy and Hölder

Theorem 3.3 (Hölder's Inequality)

Let λ_a, λ_b, ..., λ_z be positive reals with $\lambda_a + \lambda_b + \cdots + \lambda_z = 1$. Let a_1, a_2, ..., a_n, b_1, b_2, ..., b_n, ..., z_1, z_2, ..., z_n be positive reals. Then

$$(a_1 + \cdots + a_n)^{\lambda_a} (b_1 + \cdots + b_n)^{\lambda_b} \dots (z_1 + \cdots + z_n)^{\lambda_z} \geq \sum_{i=1}^n a_i^{\lambda_a} b_i^{\lambda_b} \dots z_i^{\lambda_z}.$$

Equality holds if $a_1 : a_2 : \cdots : a_n \equiv b_1 : b_2 : \cdots : b_n \equiv \cdots \equiv z_1 : z_2 : \cdots : z_n$.

Proof. WLOG $a_1 + \cdots + a_n = b_1 + \cdots + b_n = \cdots = 1$ (note that the degree of the a_i on either side is λ_a). In that case, the LHS of the inequality is 1, and we just note

$$\sum_{i=1}^n a_i^{\lambda_a} b_i^{\lambda_b} \dots z_i^{\lambda_z} \leq \sum_{i=1}^n (\lambda_a a_i + \lambda_b b_i + \dots) = 1.$$ □

If we set $\lambda_a = \lambda_b = \frac{1}{2}$, we derive what is called the Cauchy-Schwarz inequality.

$$(a_1 + a_2 + \cdots + a_n)(b_1 + b_2 + \cdots + b_n) \geq \left(\sqrt{a_1 b_1} + \sqrt{a_2 b_2} + \cdots + \sqrt{a_n b_n}\right)^2.$$

Cauchy can be rewritten as

$$\frac{x_1^2}{y_1} + \frac{x_2^2}{y_2} + \dots + \frac{x_n^2}{y_n} \ge \frac{(x_1 + x_2 + \dots + x_n)^2}{y_1 + \dots + y_n}.$$

This form it is often called Titu's Lemma in the United States.

Cauchy and Hölder have at least two uses:

1. eliminating radicals,
2. eliminating fractions.

Let us look at some examples.

Example 3.4 (IMO 2001)

Prove

$$\sum_{\text{cyc}} \frac{a}{\sqrt{a^2+8bc}} \ge 1.$$

Proof. By Holder

$$\left(\sum_{\text{cyc}} a(a^2+8bc)\right)^{\frac13} \left(\sum_{\text{cyc}} \frac{a}{\sqrt{a^2+8bc}}\right)^{\frac23} \ge (a+b+c)$$

So it suffices to prove $(a+b+c)^3 \ge \sum_{\text{cyc}} a(a^2+8bc) = a^3+b^3+c^3+24abc$. Does this look familiar? $\square$

Example 3.5 (Balkan)

Prove $\frac{1}{a(b+c)} + \frac{1}{b(c+a)} + \frac{1}{c(a+b)} \ge \frac{27}{2(a+b+c)^2}$.

Proof. Again by Holder,

$$\left(\sum_{\text{cyc}} a\right)^{\frac13} \left(\sum_{\text{cyc}} b+c\right)^{\frac13} \left(\sum_{\text{cyc}} \frac{1}{a(b+c)}\right)^{\frac13} \ge 1+1+1 = 3.$$ $\square$

Example 3.6 (JMO 2012)

Prove $\sum_{\text{cyc}} \frac{a^3+5b^3}{3a+b} \ge \frac32 \left(a^2+b^2+c^2\right)$.

Proof. We use Cauchy (Titu) to obtain

$$\sum_{\text{cyc}} \frac{a^3}{3a+b} = \sum_{\text{cyc}} \frac{(a^2)^2}{3a^2+ab} \ge \frac{(a^2+b^2+c^2)^2}{\sum_{\text{cyc}} 3a^2+ab}.$$

We can easily prove this is at least $\frac14(a^2+b^2+c^2)$ (recall $a^2+b^2+c^2$ is the "biggest" sum, so we knew in advance this method would work). Similarly $\sum_{\text{cyc}} \frac{5b^3}{3a+b} \ge \frac54(a^2+b^2+c^2)$. $\square$

Example 3.7 (USA TST 2010)

If $abc = 1$, prove $\frac{1}{a^5(b+2c)^2} + \frac{1}{b^5(c+2a)^2} + \frac{1}{c^5(a+2b)^2} \ge \frac{1}{3}$.

Proof. We can use Hölder to eliminate the square roots in the denominator:

$$\left(\sum_{\text{cyc}} ab + 2ac\right)^2 \left(\sum_{\text{cyc}} \frac{1}{a^5(b+2c)^2}\right) \ge \left(\sum_{\text{cyc}} \frac{1}{a}\right)^3 \ge 3(ab+bc+ca)^2. \qquad \square$$

§3.3 Practice Problems

1. If $a+b+c=1$, then $\sqrt{ab+c} + \sqrt{bc+a} + \sqrt{ca+b} \ge 1 + \sqrt{ab} + \sqrt{bc} + \sqrt{ca}$.
2. If $a^2+b^2+c^2 = 12$, then $a \cdot \sqrt[3]{b^2+c^2} + b \cdot \sqrt[3]{c^2+a^2} + c \cdot \sqrt[3]{a^2+b^2} \le 12$.
3. (ISL 2004) If $ab+bc+ca = 1$, prove $\sqrt[3]{\frac{1}{a}+6b} + \sqrt[3]{\frac{1}{b}+6c} + \sqrt[3]{\frac{1}{c}+6a} \le \frac{1}{abc}$.
4. (MOP 2011) $\sqrt{a^2-ab+b^2} + \sqrt{b^2-bc+c^2} + \sqrt{c^2-ca+a^2} + 9\sqrt[3]{abc} \le 4(a+b+c)$.
5. (Evan Chen) If $a^3+b^3+c^3+abc = 4$, prove

$$\frac{(5a^2+bc)^2}{(a+b)(a+c)} + \frac{(5b^2+ca)^2}{(b+c)(b+a)} + \frac{(5c^2+ab)^2}{(c+a)(c+b)} \ge \frac{(10-abc)^2}{a+b+c}.$$

   When does equality hold?

§4 Problems

1. (MOP 2013) If $a+b+c=3$, then

$$\sqrt{a^2+ab+b^2} + \sqrt{b^2+bc+c^2} + \sqrt{c^2+ca+a^2} \ge \sqrt{3}.$$

2. (IMO 1995) If $abc = 1$, then $\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \ge \frac{3}{2}$.
3. (USA 2003) Prove $\sum_{\text{cyc}} \frac{(2a+b+c)^2}{2a^2+(b+c)^2} \le 8$.
4. (Romania) Let $x_1, x_2, \dots, x_n$ be positive reals with $x_1 x_2 \dots x_n = 1$. Prove that $\sum_{i=1}^n \frac{1}{n-1+x_i} \le 1$.
5. (USA 2004) Let a, b, c be positive reals. Prove that

$$\left(a^5-a^2+3\right)\left(b^5-b^2+3\right)\left(c^5-c^2+3\right) \ge (a+b+c)^3.$$

6. (Evan Chen) Let a, b, c be positive reals satisfying $a+b+c = \sqrt[7]{a} + \sqrt[7]{b} + \sqrt[7]{c}$. Prove $a^a b^b c^c \ge 1$.