The Incenter/Excenter Lemma

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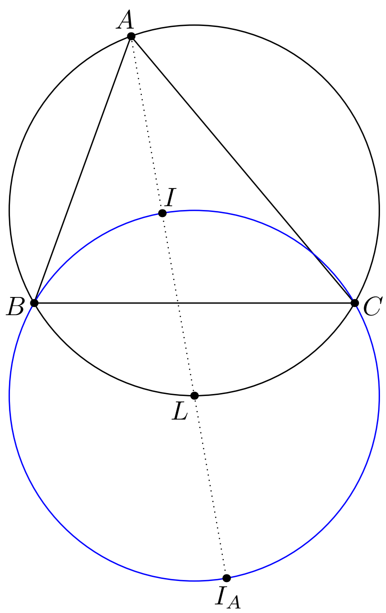
In this short note, we'll be considering the following useful lemma.

Lemma

Let ABC be a triangle with incenter I, A-excenter I_A, and denote by L the midpoint of arc BC. Show that L is the center of a circle through I, I_A, B, C.



Proof. This is just angle chasing. Let $A = \angle BAC$, $B = \angle CBA$, $C = \angle ACB$, and note that A, I, L are collinear (as L is on the angle bisector). We are going to show that $LB = LI$, the other cases being similar.

First, notice that

$$\angle LBI = \angle LBC + \angle CBI = \angle LAC + \angle CBI = \angle IAC + \angle CBI = \frac{1}{2}A + \frac{1}{2}B.$$

However,

$$\angle BIL = \angle BAI + \angle ABI = \frac{1}{2}A + \frac{1}{2}B.$$

Hence, $\triangle BIL$ is isosceles. So $LB = LI$. The rest of the proof proceeds along these lines. $\square$

Now, let's see where this lemma has come up before…

§1 Mild Embarrassments

Problem 1.1 (USAMO 1988)**.** Triangle ABC has incenter I. Consider the triangle whose vertices are the circumcenters of $\triangle IAB$, $\triangle IBC$, $\triangle ICA$. Show that its circumcenter coincides with the circumcenter of $\triangle ABC$.

Problem 1.2 (CGMO 2012)**.** The incircle of a triangle ABC is tangent to sides AB and AC at D and E respectively, and O is the circumcenter of triangle BCI. Prove that $\angle ODB = \angle OEC$.

Problem 1.3 (CHMMC Spring 2012)**.** In triangle ABC, the angle bisector of $\angle A$ meets the perpendicular bisector of $\overline{BC}$ at point D. The angle bisector of $\angle B$ meets the perpendicular bisector of $\overline{AC}$ at point E. Let F be the intersection of the perpendicular bisectors of $\overline{BC}$ and $\overline{AC}$. Find DF, given that $\angle ADF = 5^\circ$, $\angle BEF = 10^\circ$ and $AC = 3$.

Problem 1.4 (Nine-Point Circle)**.** Let ABC be an acute triangle with orthocenter H. Let D, E, F be the feet of the altitudes from A, B, C to the opposite sides. Show that the midpoint of $\overline{AH}$ lies on the circumcircle of $\triangle DEF$.

§2 Some Short-Answer Problems

Problem 2.1 (HMMT 2011)**.** Let $ABCD$ be a cyclic quadrilateral, and suppose that $BC = CD = 2$. Let I be the incenter of triangle ABD. If $AI = 2$ as well, find the minimum value of the length of diagonal BD.

Problem 2.2 (HMMT 2013)**.** Let triangle ABC satisfy $2BC = AB + AC$ and have incenter I and circumcircle ω. Let D be the intersection of AI and ω (with A, D distinct). Prove that I is the midpoint of AD.

Problem 2.3 (Online Math Open 2014/F19)**.** In triangle ABC, $AB = 3$, $AC = 5$, and $BC = 7$. Let E be the reflection of A over $\overline{BC}$, and let line BE meet the circumcircle of ABC again at D. Let I be the incenter of $\triangle ABD$. Compute $\cos\angle AEI$.

Problem 2.4 (NIMO 2012)**.** Let $ABXC$ be a cyclic quadrilateral such that $\angle XAB = \angle XAC$. Let I be the incenter of triangle ABC and by D the foot of I on $\overline{BC}$. Given $AI = 25$, $ID = 7$, and $BC = 14$, find XI.

§3 Intermediate Examples

Problem 3.1. Let ABC be an acute triangle such that $\angle A = 60^\circ$. Prove that $IH = IO$, where I, H, O are the incenter, orthocenter, and circumcenter.

Problem 3.2 (IMO 2006)**.** Let ABC be a triangle with incenter I. A point P in the interior of the triangle satisfies

$$\angle PBA + \angle PCA = \angle PBC + \angle PCB.$$

Show that $AP \geq AI$, and that equality holds if and only if $P = I$.

Problem 3.3 (APMO 2007)**.** In triangle ABC, we have $AB > AC$ and $\angle A = 60^\circ$. Let I and H denote the incenter and orthocenter of the triangle. Show that $2\angle AHI = 3\angle B$.

Problem 3.4 (ELMO 2013, Evan Chen)**.** Triangle ABC is inscribed in circle ω. A circle with chord BC intersects segments AB and AC again at S and R, respectively. Segments

BR and CS meet at L, and rays LR and LS intersect ω at D and E, respectively. The internal angle bisector of $\angle BDE$ meets line ER at K. Prove that if $BE = BR$, then $\angle ELK = \frac{1}{2}\angle BCD$.

Problem 3.5 (Online Math Open 2012/F27)**.** Let ABC be a triangle with circumcircle ω. Let the bisector of $\angle ABC$ meet segment AC at D and circle ω at $M \neq B$. The circumcircle of $\triangle BDC$ meets line AB at $E \neq B$, and CE meets ω at $P \neq C$. The bisector of $\angle PMC$ meets segment AC at $Q \neq C$. Given that $PQ = MC$, determine the degree measure of $\angle ABC$.

§4 Harder Tasks

Problem 4.1 (Iran 2001)**.** Let ABC be a triangle with incenter I and A-excenter I_A. Let M be the midpoint of arc BC not containing A, and let N denote the midpoint of arc MBA. Lines NI and NI_A intersect the circumcircle of ABC at S and T. Prove that the lines ST, BC and AI are concurrent.

Problem 4.2 (Online Math Open 2014/F26)**.** Let ABC be a triangle with $AB = 26$, $AC = 28$, $BC = 30$. Let X, Y, Z be the midpoints of arcs BC, CA, AB (not containing the opposite vertices) respectively on the circumcircle of ABC. Let P be the midpoint of arc BC containing point A. Suppose lines BP and XZ meet at M , while lines CP and XY meet at N. Find the square of the distance from X to MN.

Problem 4.3 (Euler)**.** Let ABC be a triangle with incenter I and circumcenter O. Show that $IO^2 = R(R - 2r)$, where R and r are the circumradius and inradius of $\triangle ABC$, respectively.

Problem 4.4 (IMO 2010)**.** Let I be the incenter of a triangle ABC and let Γ be its circumcircle. Let the line AI intersect Γ again at D. Let E be a point on the arc BDC and F a point on the side BC such that

$$\angle BAF = \angle CAE < \tfrac{1}{2}\angle BAC.$$

Finally, let G be the midpoint of $\overline{IF}$. Prove that $\overline{DG}$ and $\overline{EI}$ intersect on Γ.

§5 Bonus Problems

Problem 5.1 (Russia 2014)**.** Let ABC be a triangle with $AB > BC$ and circumcircle Ω. Points M, N lie on the sides AB, BC respectively, such that $AM = CN$. Lines MN and AC meet at K. Let P be the incenter of the triangle AMK, and let Q be the K-excenter of the triangle CNK. If R is midpoint of arc ABC of Ω then prove that $RP = RQ$.

Problem 5.2. Let ABC be a triangle with circumcircle Ω, and let D be any point on $\overline{BC}$. We draw a *curvilinear incircle* tangent to $\overline{AD}$ at L, to $\overline{BC}$ at K and internally tangent to Ω. Show that the incenter of triangle ABC lies on $\overline{KL}$.

§6 Hints to the Problems

1.1. Tautological.

1.2. Who is O?

1.3. Point F is the circumcenter of $\triangle ABC$. Who are D and E?

1.4. What is the incenter of $\triangle DEF$? What is the D-excenter?

2.1. Show that $AC = 4$.

2.2. Apply Ptolemy's Theorem.

2.3. Who is C? Erase E.

2.4. Apply Ptolemy's Theorem.

3.1. Since $\angle BHC = \angle BIC = \angle BOC = 120°$, points H and O now lie on the magic circle too. So $IH = IO$ is just an equality of certain arcs.

3.2. Use the angle condition to show that P also lies on the magic circle.

3.3. The point H lies on the magic circle. So $\angle IHC = 180° - \angle IBC$.

3.4. You need to do quite a bit of angle chasing. Show that R is the incenter of $\triangle CDE$. Who is B?

3.5. Both M and P are arc midpoints. (Why?)

4.1. First show that S, T, I, I_A are concyclic, say by $NI \cdot NS = NM^2 = NI_A \cdot NT$.

4.2. Add the incenter I. Line MN is a tangent.

4.3. Add in point L, the midpoint of arc BC. By Power of a Point, it's equivalent to prove $AI \cdot IL = 2Rr$, which can be done with similar triangles.

4.4. Take a homothety with ratio 2 at I. This sends G to F and D to the A-excenter.

5.1. Construct arc midpoints on the circumcircles of both $\triangle AMJ$ and $\triangle CNK$. Use spiral similarity at R.

5.2. Let the tangency point to Ω be T, let M be the midpoint of arc BC, and let lines KL and AM meet at I. Show that M, K, T are collinear. Show that $ALIT$ is cyclic. Prove that $MI^2 = MK \cdot MT = MC^2 = MI^2$.