USEMO 2021 Solutions

United States Ersatz Math Olympiad

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Contents

§1 Solutions to Day 1

§1.1 Solution to USEMO 1, by Holden Mui

Let n be a positive integer and consider an $n \times n$ grid of real numbers. Determine the greatest possible number of cells c in the grid such that the entry in c is both strictly *greater* than the average of c's column and strictly *less* than the average of c's row.

The answer is $(n-1)^2$. An example is given by the following construction, shown for $n = 5$, which generalizes readily. Here, the lower-left $(n-1) \times (n-1)$ square gives a bound.

$$\begin{bmatrix} -1 & -1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

We give two proofs of the bound. Call a cell *good* if it satisfies the condition of the problem.

Coloring proof, from the author We now prove that no more than $(n-1)^2$ squares can be good. The cells are weakly ordered by $\geq$ (there may be some ties due to equal elements); we arbitrarily extend it to a total ordering $\gtrdot$, breaking all ties. (Alternatively, one can phrase this as perturbing the grid entries in such a way that they become distinct.)

- For every column, we color red the $\gtrdot$-largest element in that column.
- For every row, we color blue the $\gtrdot$-smallest element in that row.

This means that there are exactly n red and n blue cells. Note that these cells are never good.

Claim — There is at most one cell that is both red and blue.

Proof. Assume for contradiction that P_1 and P_2 are two "purple" cells (both red and blue). Look at the resulting picture

$$\begin{bmatrix} P_1 & x \\ y & P_2 \end{bmatrix}.$$

By construction, we have $P_1 \gtrdot x \gtrdot P_2 \gtrdot y \gtrdot P_1$. This is a contradiction. □

Thus at least $2n - 1$ cells cannot be good. This proves the bound.

Proof using König's theorem, from Ankan Bhattacharya This proof is based on the following additional claim:

Claim — No column/row can be all-good, and no transversal can be all-good.

Proof. The first part is obvious. As for the second, let r_i and c_j denote the column sums. If cell (i, j) is good, then

$$r_i < a_{i,j} < c_j.$$

If we have a good transversal, summing the inequality $r_i < c_j$ over the cells in this transversal gives a contradiction (as $\sum r_\bullet = \sum c_\bullet$). □

This claim alone is enough to imply the desired bound.

Claim — There exists a choice of a columns and b rows, with $a + b = n + 1$, such that no good cells lie on the intersection of the columns and rows.

Proof. Follows by *König's theorem*, and the previous claim. Alternatively, quote the contrapositive of Hall's marriage theorem: because there was no all-good transversal, there must be a set of a columns with more than $n - a$ "compatible" rows. □

Suppose $a \le \frac{n}{2} \le b$; the other case is similar. Now we bound:

- The $a \times b$ cells of the claim are given to be all non-good.
- In the $n - a = b - 1$ remaining rows, there is at least one more non-good cell.

Thus the number of non-good cells is at least

$$ab + (b - 1) = (a + 1)b - 1 \ge 2 \cdot n - 1 = n^2 - (n - 1)^2,$$

and so there are at most $(n - 1)^2$ good cells.

§1.2 Solution to USEMO 2, by Ankan Bhattacharya

Find all integers $n \geq 1$ such that $2^n - 1$ has exactly n positive integer divisors.

The valid n are 1, 2, 4, 6, 8, 16, 32. They can be verified to work through inspection, using the well known fact that the Fermat prime $F_i = 2^{2^i} + 1$ is indeed prime for $i = 0, 1, \dots, 4$ (but not prime when $i = 5$).

We turn to the proof that these are the only valid values of n. In both solutions that follow, $d(n)$ is the divisor counting function.

First approach (from author) Let d be the divisor count function. Now suppose n works, and write $n = 2^k m$ with m odd. Observe that

$$2^n - 1 = (2^m - 1)(2^m + 1)(2^{2m} + 1) \cdots (2^{2^{k-1}m} + 1),$$

and all $k+1$ factors on the RHS are pairwise coprime. In particular,

$$d(2^m - 1)d(2^m + 1)d(2^{2m} + 1) \cdots d(2^{2^{k-1}m} + 1) = 2^k m.$$

Recall the following fact, which follows from Milaheescu's theorem.

Lemma

$2^r - 1$ is a square if and only if $r = 1$, and $2^r + 1$ is a square if and only if $r = 3$.

Now, if $m \geq 5$, then all $k+1$ factors on the LHS are even, a contradiction. Thus $m \leq 3$. We deal with both cases.

If $m = 1$, then the inequalities

$$\begin{aligned} d(2^{2^0} - 1) &= 1 \\ d(2^{2^0} + 1) &\geq 2 \\ d(2^{2^1} + 1) &\geq 2 \\ &\vdots \\ d(2^{2^{k-1}} + 1) &\geq 2 \end{aligned}$$

mean that it is necessary and sufficient for all of $2^{2^0} + 1,\ 2^{2^1} + 1,\ \dots,\ 2^{2^{k-1}} + 1$ to be prime. As mentioned at the start of the problem, this happens if and only if $k \leq 5$, giving the answers $n \in \{1, 2, 4, 8, 16, 32\}$.

If $m = 3$, then the inequalities

$$\begin{aligned} d(2^{3 \cdot 2^0} - 1) &= 2 \\ d(2^{3 \cdot 2^0} + 1) &= 3 \\ d(2^{3 \cdot 2^1} + 1) &\geq 4 \\ &\vdots \\ d(2^{3 \cdot 2^{k-1}} + 1) &\geq 4 \end{aligned}$$

mean that $k \geq 2$ does not lead to a solution. Thus $k \leq 1$, and the only valid possibility turns out to be $n = 6$.

Consolidating both cases, we obtain the claimed answer $n \in \{1, 2, 4, 6, 8, 16, 32\}$.

Second approach using Zsigmondy (suggested by reviewers) There are several variations of this Zsigmondy solution; we present the approach found by Nikolai Beluhov. Assume $n \geq 7$, and let $n = \prod_1^m p_i^{e_i}$ be the prime factorization with $e_i > 0$ for each i. Define the numbers

$$\begin{aligned} T_1 &= 2^{p_1^{e_1}} - 1 \\ T_2 &= 2^{p_2^{e_2}} - 1 \\ &\vdots \\ T_m &= 2^{p_m^{e_m}} - 1. \end{aligned}$$

We are going to use two facts about T_i.

Claim — The T_i are pairwise relatively prime and

$$\prod_{i=1}^m T_i \mid 2^n - 1.$$

Proof. Each T_i divides $2^n - 1$, and the relatively prime part follows from the identity $\gcd(2^x - 1, 2^y - 1) = 2^{\gcd(x,y)} - 1$. □

Claim — The number T_i has at least e_i distinct prime factors.

Proof. This follows from Zsigmondy's theorem: each successive quotient $(2^{p^{k+1}} - 1)/(2^{p^k} - 1)$ has a new prime factor. □

Claim (Main claim) — Assume n satisfies the problem conditions. Then both the previous claims are sharp in the following sense: each T_i has *exactly* e_i distinct prime divisors, and

$$\left\{ \text{primes dividing } \prod_{i=1}^m T_i \right\} = \left\{ \text{primes dividing } 2^n - 1 \right\}.$$

Proof. Rather than try to give a size contradiction directly from here, the idea is to define an ancillary function

$$s(x) = \sum_{p \text{ prime}} \nu_p(x)$$

which computes the sum of the exponents in the prime factorization. For example

$$s(n) = e_1 + e_2 + \dots + e_m.$$

On the other hand, using the earlier claim, we get

$$s(d(2^n - 1)) \geq s\left(d\left(\prod T_i\right)\right) \geq e_1 + e_2 + \dots + e_m = s(n).$$

But we were told that $d(2^n - 1) = n$; hence equality holds in all our estimates, as needed. □

At this point, we may conclude directly that $m = 1$ in any solution; indeed if $m \geq 2$ and $n \geq 7$, Zsigmondy's theorem promises a primitive prime divisor of $2^n - 1$ not dividing any of the T_i.

Now suppose $n = p^e$, and $d(2^{p^e} - 1) = n = p^e$. Since $2^{p^e} - 1$ has exactly e distinct prime divisors, this can only happen if in fact

$$2^{p^e} - 1 = q_1^{p-1} q_2^{p-1} \dots q_e^{p-1}$$

for some distinct primes $q_1, q_2, \dots, q_e$. This is impossible modulo 4 unless $p = 2$.

So we are left with just the case $n = 2^e$, and need to prove $e \leq 5$. The proof consists of simply remarking that $2^{2^5} + 1$ is known to not be prime, and hence for $e \geq 6$ the number $2^{2^e} - 1$ always has at least $e + 1$ distinct prime factors.

§1.3 Solution to USEMO 3, by Ankan Bhattacharya

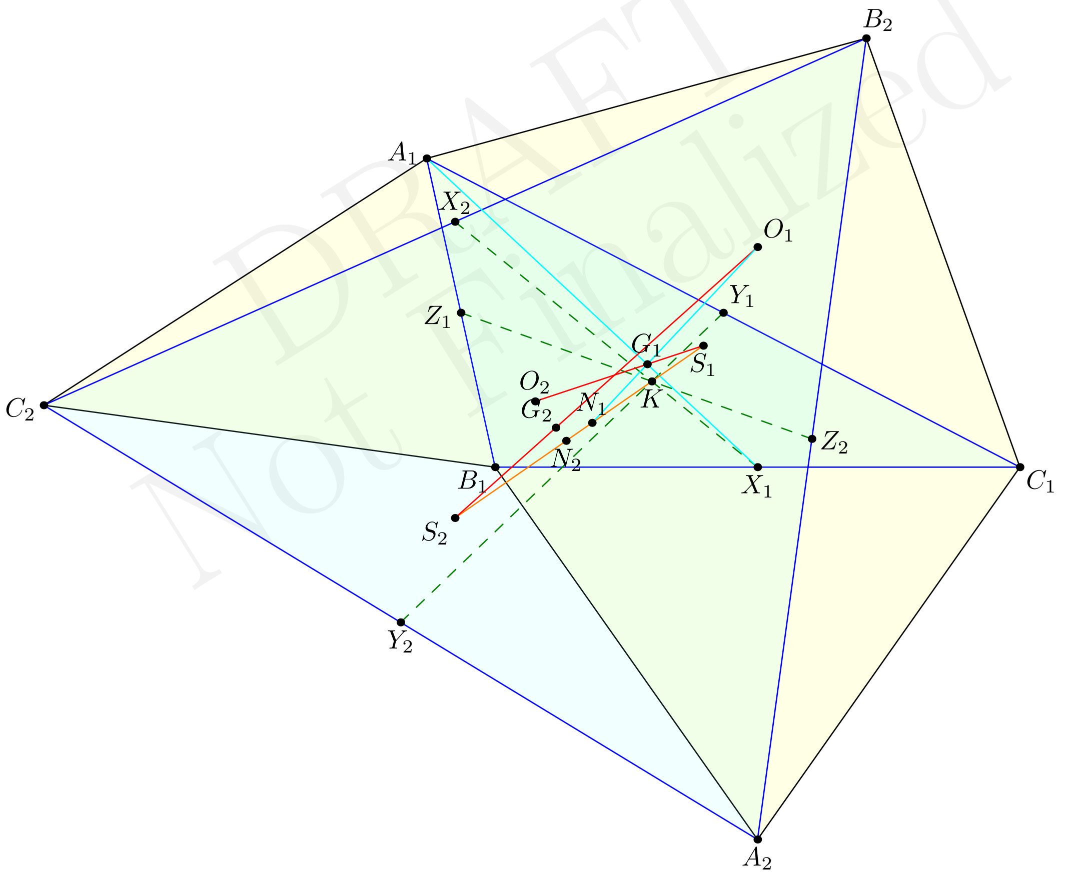
Let $A_1C_2B_1A_2C_1B_2$ be an equilateral hexagon. Let O_1 and H_1 denote the circumcenter and orthocenter of $\triangle A_1B_1C_1$, and let O_2 and H_2 denote the circumcenter and orthocenter of $\triangle A_2B_2C_2$. Suppose that $O_1 \neq O_2$ and $H_1 \neq H_2$. Prove that the lines O_1O_2 and H_1H_2 are either parallel or coincide.

Let $\triangle X_1Y_1Z_1$ and $\triangle X_2Y_2Z_2$ be the medial triangles of $\triangle A_1B_1C_1$ and $\triangle A_2B_2C_2$. The first simple observation is as follows.

Claim — Y_1, Z_1, Y_2, Z_2 are concyclic.

Proof. The distance from each of Y_1, Z_1, Y_2, Z_2 to the midpoint of $\overline{A_1A_2}$ is half the side length of the hexagon. □

Hence by radical axis argument, we obtain that $\overline{X_1X_2}$, $\overline{Y_1Y_2}$, $\overline{Z_1Z_2}$ are concurrent, except possibly when all six points lie on a circle. In this case, $\triangle A_1B_1C_1$ and $\triangle A_2B_2C_2$ share the same nine-point center, so clearly $\overline{O_1O_2} \parallel \overline{H_1H_2}$. So we will assume going forward that $(X_1Y_1Z_1)$ and $(X_2Y_2Z_2)$ are distinct circles.



The heart of the proof revolves around the following two claims.

Claim (Perspectivity) — The two triangles $\triangle X_1Y_1Z_1$ and $\triangle X_2Y_2Z_2$ are perspective at some point K.

Proof. As mentioned above, $\overline{X_1X_2}$, $\overline{Y_1Y_2}$, $\overline{Z_1Z_2}$ are concurrent. □

Let N_1 and G_1 be the circumcenter and centroid of $\triangle X_1Y_1Z_1$; define N_2 and G_2 similarly.

Claim (Orthology) — Triangles $\triangle X_1Y_1Z_1$ and $\triangle X_2Y_2Z_2$ are orthologic. In fact, the orthology center S_1 is the image of O_2 under a homothety centered at G_1 with ratio $-\frac{1}{2}$.

Proof. Since the mentioned homothety takes $\overline{A_1O_2} \to \overline{X_1S_1}$, so

$$\overline{Y_2Z_2} \parallel \overline{B_2C_2} \perp \overline{A_1O_2} \parallel \overline{X_1S_1}$$

as desired. □

We have obtained that $\triangle X_1Y_1Z_1$ and $\triangle X_2Y_2Z_2$ are both orthologic (with centers S_1 and S_2) and perspective (through K). Hence it follows by **Sondat's theorem** that S_1, S_2, and K lie on a line perpendicular to the perspectrix.

To finish, we follow up with two more claims:

Claim (Perspectrix is radical axis) — The perspectrix of the two triangles is exactly the radical axis of their circumcircles, hence perpendicular to $\overline{N_1N_2}$.

Proof. This follows from the earlier observation that Y_1, Y_2, Z_1, Z_2 was cyclic, etc. □

Claim (Degenerate parallelogram) — $N_1S_1N_2S_2$ is a (possibly degenerate) parallelogram.

Proof. Because $\overrightarrow{S_1N_2} \overset{O_2}{=} \frac{3}{2}\overrightarrow{G_1G_2} \overset{O_1}{=} \overrightarrow{N_1S_2}$. □

In this way we can conclude that $\overline{N_1N_2} \parallel \overline{S_1S_2}$ through the former claim, but they have the same midpoint by the latter claim, so ultimately all N_i and S_i are collinear.

Finally, note that

$$\overline{N_1N_2} \parallel \overline{N_1S_1} \overset{G_1}{\parallel} \overline{O_1O_2}.$$

It easily follows that $\overline{O_1O_2} \parallel \overline{H_1H_2}$, as wanted.

Remark. An amusing corollary of the above solution is the following:

> Assuming $A_1C_2B_1A_2C_1B_2$ is not self-intersecting, the midpoints of $\overline{A_1A_2}$, $\overline{B_1B_2}$, $\overline{C_1C_2}$ cannot be collinear (unless two of them coincide).

To see this, let M_A, M_B, M_C be said midpoints. If they are different and lie on line ℓ, then $M_BX_1M_CX_2$ is a rhombus with side length $\frac{1}{2}s$, so X_1 and X_2 are reflections in ℓ.

Similarly, $\triangle X_1Y_1Z_1$ and $\triangle X_2Y_2Z_2$ are reflections in ℓ, so $\triangle A_1B_1C_1$ and $\triangle A_2B_2C_2$ are as well. This is not possible if $A_1C_2B_1A_2C_1B_2$ is not self-intersecting, because some side will intersect ℓ: then its opposite side will intersect this side at the intersection point.

§2 Solutions to Day 2

§2.1 Solution to USEMO 4, by Sayandeep Shee

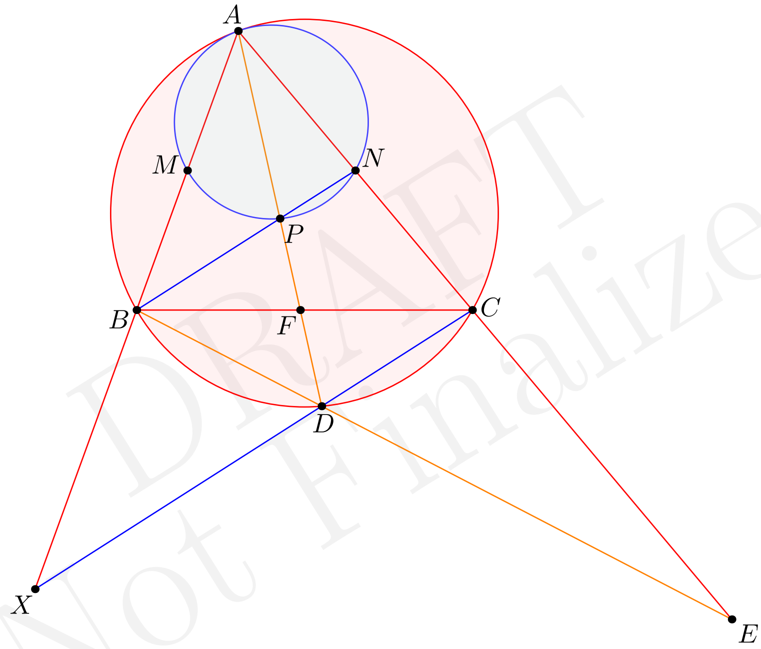
Let ABC be a triangle with circumcircle ω, and let X be the reflection of A in B. Line CX meets ω again at D. Lines BD and AC meet at E, and lines AD and BC meet at F. Let M and N denote the midpoints of AB and AC.

Can line EF share a point with the circumcircle of triangle AMN?

The answer is no, they never intersect.

Classical solution, by author Let P denote the midpoint of $\overline{AD}$, which

- lies on $\overline{BN}$, since $\overline{BN} \parallel \overline{CX}$; and
- lies on (AMN), since it's homothetic to (ABC) through A with factor $\frac{1}{2}$.



Now, note that

$$\measuredangle FBP = \measuredangle CBN = \measuredangle BCD = \measuredangle BAD = \measuredangle BAF \implies FB^2 = FP \cdot FA$$
$$\measuredangle EBN = \measuredangle EDC = \measuredangle BDC = \measuredangle BAC = \measuredangle BAE \implies EB^2 = EN \cdot EA.$$

This means that line EF is the radical axis of the circle centered at B with radius zero, and the circumcircle of triangle AMN. Since B obviously lies outside (AMN), the disjointness conclusion follows.

Projective solution, by Ankit Bisain In this approach we are still going to prove that $\overline{EF}$ is the radical axis of (AMN) and the circle of radius zero at B, but we are not going to use the point P, or even points E and F.

Instead, let $Y = \overline{EF} \cap \overline{AB}$, which by Brokard's theorem on $ABDC$ satisfies $(AB;XY) = -1$. Since $XB = XA$, it follows that $AY : YB = 2$. From here it is straightforward to verify that

$$YB^2 = \frac{1}{9}AB^2 = YM \cdot YA.$$

Thus Y lies on the radical axis.

Finally, by Brokard's theorem again, if O is the center of ω then $\overline{OX} \perp \overline{EF}$. Taking a homothety with scale factor 2 at A, it follows that the line through B and the center of (AMN) is perpendicular to $\overline{EF}$.

Since $\overline{EF}$ contains Y, it now follows that $\overline{EF}$ is the radical axis, as claimed.

Solution with inversion, projective, and Cartesian coordinates, by Ankan Bhattacharya

In what follows, let O be the center of ω. Note that Brokard's theorem gives that $\overline{EF}$ is the polar of X.

Note that since none of E, F, X are points at infinity, O is different from all three.

We consider inversion in ω to eliminate the polar:

- The circumcircle of $\triangle AMN$, i.e. the circle with diameter $\overline{AO}$, is sent to the line ℓ tangent to ω at A.
- The line EF, as the polar of X, is sent to the circle with diameter $\overline{OX}$. (It is indeed a circle, because O does not lie on line EF.)

Thus, if the posed question is true, then we see that ℓ intersects (OX). We claim this is impossible.

Establish Cartesian coordinates with $A = (0,0)$ and $O = (2,0)$, so ℓ is the y-axis. Let T be the center of (OX): the midpoint of $\overline{OX}$. Observe:

- B lies on the circle with center $(2,0)$ and radius 2.
- X lies on the circle with center $(4,0)$ and radius 4.
- T lies on the circle with center $(3,0)$ and radius 2.

Thus, let the coordinates of T be (x,y), with $(x-3)^2 + y^2 = 4$. The intersection of ℓ and (OX) being nonempty is equivalent to

$$\begin{aligned} & d(T,\ell)^2 \le OT^2 \\ \iff & x^2 \le (x-2)^2 + y^2 \\ \iff & x^2 \le (x-2)^2 + [4 - (x-3)^2] \\ \iff & (x-1)^2 \le 0, \end{aligned}$$

or $x = 1$ (which forces $y = 0$); i.e. $T = (1,0)$. However, this forces

$$B = (0,0) = A,$$

which is not permitted. Thus, line ℓ cannot share a point with (OX), and so line EF cannot share a point with (AMN).

§2.2 Solution to USEMO 5, by Bhavya Tiwari

Given a polynomial $p(x)$ with real coefficients, we denote by $S(p)$ the sum of the squares of its coefficients. For example, $S(20x+21) = 20^2 + 21^2 = 841$.

Prove that if $f(x)$, $g(x)$, and $h(x)$ are polynomials with real coefficients satisfying the identity $f(x) \cdot g(x) = h(x)^2$, then

$$S(f) \cdot S(g) \geq S(h)^2.$$

The following write-up is due to Ankan Bhattacharya, and is the same as the solution proposed by the author.

Claim — Let p be a polynomial with real coefficients, and $n > \deg p$ an integer. Then

$$S(p) = \frac{1}{n} \sum_{k=0}^{n-1} |p(e^{2\pi i k/n})|^2.$$

Proof. Note that

$$|p(e^{2\pi i k/n})|^2 = p(e^{2\pi i k/n}) \cdot p(e^{-2\pi i k/n})$$

so if we define $q(x) = p(x)p(1/x)$, the right-hand side is the sum of q across the nth roots of unity.

Applying a roots of unity filter, the right-hand side is the constant coefficient of $q(x)$. But that constant coefficient is exactly equal to $S(p)$. □

To solve the problem, choose $n > \max\{\deg f, \deg g, \deg h\}$, set $\omega = e^{2\pi i/n}$, and apply the key claim to all three to get that the desired inequality is equivalent to

$$\left[\frac{1}{n}\sum |f(\omega^k)|^2\right] \cdot \left[\frac{1}{n}\sum |g(\omega^k)|^2\right] \geq \left[\frac{1}{n}\sum |h(\omega^k)|^2\right]^2$$

$$\iff \left[\sum |f(\omega^k)|^2\right] \cdot \left[\sum |g(\omega^k)|^2\right] \geq \left[\sum |f(\omega^k)| \cdot |g(\omega^k)|\right]^2.$$

This is just Cauchy-Schwarz, so we are done.

Remark (Continuous version of above solution)**.** To avoid the arbitrary choice of parameter n, one can make the same argument to show that for any $p \in \mathbb{R}[x]$,

$$S(p) = \frac{1}{2\pi} \int_0^{2\pi} |p(e^{ix})|^2 \, dx.$$

Using Cauchy's inequality for integrals, we obtain a continuous version of the above solution. However, this is technically out of scope for high-school olympiad, despite the fact it is really just the limit as $n \to \infty$ of the above solution.

§2.3 Solution to USEMO 6, by Nikolai Beluhov

A *bagel* is a loop of $2a + 2b + 4$ unit squares which can be obtained by cutting a concentric $a \times b$ hole out of an $(a + 2) \times (b + 2)$ rectangle, for some positive integers a and b. (The side of length a of the hole is parallel to the side of length $a + 2$ of the rectangle.)

Consider an infinite grid of unit square cells. For each even integer $n \geq 8$, a *bakery of order n* is a finite set of cells S such that, for every n-cell bagel B in the grid, there exists a congruent copy of B all of whose cells are in S. (The copy can be translated and rotated.) We denote by $f(n)$ the smallest possible number of cells in a bakery of order n.

Find a real number α such that, for all sufficiently large even integers $n \geq 8$, we have

$$\frac{1}{100} < \frac{f(n)}{n^\alpha} < 100.$$

The answer is $\alpha = 3/2$.

In what follows, "Y is about X" means that $Y = [1+o(1)]X$. Equivalently, $\lim_{n\to\infty} Y/X = 1$. Intuitively, both of these say that X and Y become closer and closer together as n grows. This is fine for the problem since only sufficiently large n are involved.

Bound First we prove that every bakery S of order n contains at least about $n^{3/2}/8$ cells.

We say that a bagel is *horizontal* or *vertical* depending on the orientation of its pair of longer sides. (A square bagel is both.) For each $a < b$ with $2a + 2b + 4 = n$, take one bagel in S whose hole is of size either $a \times b$ or $b \times a$. Without loss of generality, at least about $n/8$ of our bagels are horizontal.

Say that there are a total of k rows which contain a longer side of at least one of our horizontal bagels. Note that the shorter side length of a horizontal bagel depends only on the distance between the rows of its longer sides. Since the shorter side lengths of all of our bagels are pairwise distinct, we obtain that $\binom{k}{2}$ is at least about $n/8$. Consequently, k is at least about $\sqrt{n}/2$.

On the other hand, each such row contains at least about $n/4$ cells in S. Therefore, $|S|$ is at least about $n^{3/2}/8$, as needed.

Construction To complete the solution, we construct a bakery S of order n with at most about $\sqrt{2} \cdot n^{3/2}$ cells. Define

$$\ell = \left\lceil \sqrt{n/2} \right\rceil \qquad \text{and} \qquad D = \{-\ell^2, -(\ell-1)\ell, \dots, -3\ell, -2\ell, -\ell, 0, 1, 2, \dots, \ell\}.$$

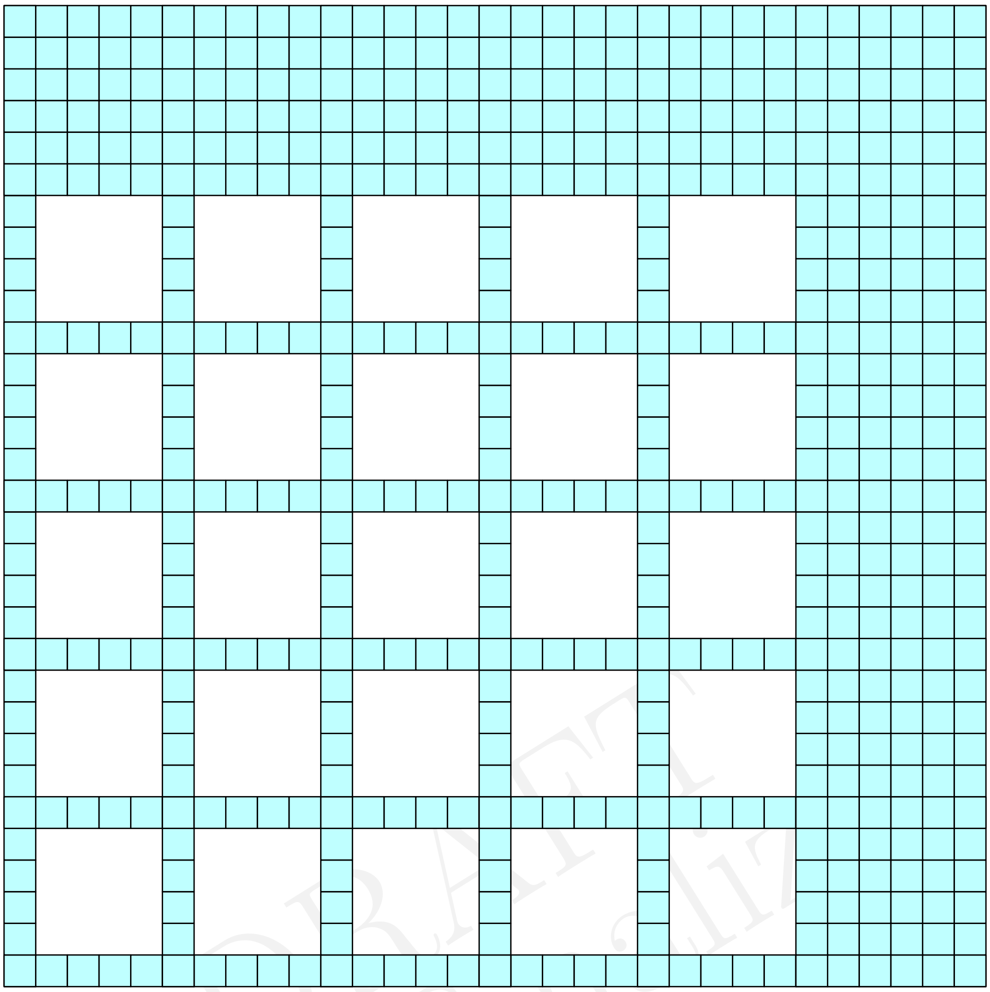
Then $|D|$ is about $\sqrt{2n}$.

We refer to the set D as a *ruler* in the sense that for any $1 \leq m < n/2$, there are x_1 and x_2 in D with $x_2 - x_1 = m$. Indeed, one lets x_2 be the remainder when m is divided by ℓ, so that $x_1 = x_2 - m \leq 0$ is a multiple of ℓ.

Now, if we let $T = \{-\ell^2, -\ell^2 + 1, \dots, \ell\}$ then we may define

$$S = (D \times T) \cup (T \times D).$$

An illustration below is given for $\ell = 5$.



Note that $|S|$ is at most about $n|D|$, that is, at most about $\sqrt{2} \cdot n^{3/2}$.

Claim — The set S is a bakery of order n.

Proof. Let a and b be any positive integers with $2a + 2b + 4 = n$. By the choice of D, there are x_1 and x_2 in D such that $x_2 - x_1 = a + 1$, as well as y_1 and y_2 in D such that $y_2 - y_1 = b + 1$. Then the bagel with opposite corner cells (x_1, y_1) and (x_2, y_2) has a hole with side lengths a and b and all of its cells are in S, as needed. □

Remark. Let us call a ruler *sparse* when a lot of its marks are missing but we can still measure out each one of the distances 1, 2, ..., N. Then for the set D in the solution essentially we need a sparse ruler with about $c\sqrt{N}$ marks, for some reasonably small positive real constant c. The construction above is simple but also far from optimal. Other constructions are known which are more complicated but yield smaller values of c. See, for example, Ed Pegg Jr, Hitting All The Marks.

4 Marking schemes

§4.1 Rubric for USEMO1

In all approaches, **1 point** is awarded for a correct construction, and **6 points** for the proof that $(n-1)^2$ is best possible; hence $1+6=7$. No points are awarded for answer alone.

The 6 points can be decomposed according to the following approaches. All the items, including the deductions, are additive within each approach. But the marks from different approaches are not additive.

Finally, **one point is deducted** from a 7^- solution if the proof is only correct when all the cells are distinct, and the student fails to deal with situations in which a subset of cells have the same number. (This is quite forgiving.)

First approach

- 1 point for the claim that there exists only 1 cell which is the largest and smallest in its row and column respectively
- 5 points for the proof and by showing that otherwise implies a contradictory chain of inequalities.

Second Approach

- 2 points for proving that not all cells in a row/column and no transversal can satisfy the problem condition
- 2 points for proving that there exists a choice of a columns and b rows such that $a+b=n+1$
- 2 points for correctly bounding the number of sapphire cells

§4.2 Rubric for USEMO2

In this rubric, none of the items are additive: neither the positive items nor the deductions. Hence an incomplete solution receives the largest positive item, while a complete solution receives 7 minus the largest deduction. Deductions do *not* apply to 0^+ solutions.

Common items for both solutions

- **0 points** for correct solution set.
- **0 points** for proving that special cases of n (odd n, or prime power n) don't work.
- **-1 points** for not mentioning anywhere that all n in the solution set work.
- **-1 points** for a solution which claims that all powers of 2 work (but resolves the other cases correctly). (Stating that it's well-known F_i is prime for $i=1,\dots,4$ but not 5, where F_i are the Fermat primes, counts as a correct proof.)
- **-1 points** for a solution which has an incorrect solution set but is otherwise correct (unless the only error is missing $n=1$, in which case there is no deduction).

First official solution

- **0 points** for just writing down $2^n - 1 = (2^m - 1)(2^m + 1)(2^{2m} + 1) \cdots$.
- **2 points** for proving that one of $2^m - 1$, $2^m + 1$, $2^{2m} + 1$, ... is a square.
- **3 points** for proving that $m = 1$ or 3.
- **7 points** for a complete solution.
- **-1 points** for an incorrect proof, or statement without proof, that $2^r - 1$ is only a square when $r = 1$, and/or that $2^r + 1$ is only a square when $r = 3$, if the solution is otherwise correct. (Citing Catalan/Mihailescu counts as a correct proof.)

Second official solution

- **0 points** for just writing down a result of Zsigmondy (that $2^n - 1$ has at least $e_1 + \cdots + e_m$, or $d(n) - 2$, distinct prime factors).
- **2 points** for proving that equality holds in the estimate $s(d(2^n - 1)) \geq \cdots$ or something similar.
- **3 points** for proving n is 6 or a prime power.
- **7 points** for a complete solution.

§4.3 Rubric for USEMO 3

Because the number of solutions with any substantial progress was low, there was no official rubric written for this problem. Instead, the graders first identified all papers that were plausible candidates for partial marks, if any. Then they discussed each individual paper case by case.

§4.4 Rubric for USEMO 4

The following things might happen:

(a) It is stated/conjectured that answer is NO.

(b) It is mentioned that $\overline{EF}$ is the polar of X wrt ω.

(c) Point $K = \overline{EF} \cap \overline{AB}$ is introduced.

(d) It is mentioned that $\frac{AK}{KB} = 2$.

(e) Center of O of ω is defined.

(f) As we only need to show distance between midpoint of segment AO and line EF is $R/2$ (where R is the radius of ω), so it is mentioned that it suffices to show

$$d(A, \overline{EF}) + d(O, \overline{EF}) > R \qquad (1)$$

(g) Now in the above step, we require that A, O lie on the same side of line EF. So some students might not mention that.

(h) Proving (1).

Now we can give the following marks:

(i) **0 marks** for only $(a), (b)$.

(ii) **1 marks** for $(c), (d)$.

(iii) **2 marks** for mentioning (1) without doing the mistake in (g) and also proving $d(A, \overline{EF}) = \frac{d(X, \overline{EF})}{2}$.

(iv) **1 marks** If mistake in (g) is also done in (f).

(v) **6 marks** If the solution is correct till end but mistake in (g) is done.

(vi) **7 marks** for a perfect solution.

§4.5 Rubric for USEMO 5

Most solutions probably are worth 0 or 7 points. As always there will be a 1 point deduction to essentially complete solutions with minor errors that could be easily be fixed.

Roots of unity solution

The following items apply but are not additive.

- 0 points for proving the inequality for special cases. For example when f is a quadratic polynomial, cubic polynomial etc.
- 0 points for expressing the polynomials as $f(x) = p(x)^2 r(x)$, $g(x) = q(x)^2 r(x)$ and $h(x) = p(x)q(x)r(x)$.
- 1 point for proving that $s(p)$ is equal to the independent coefficient of $p(x)p(1/x)$
- 5 points for using a roots of unity filter to characterize $S(p)$ as

$$S(p) = \frac{1}{n} \sum_{i=0}^{n-1} |p(e^{2\pi i k/n})|^2 \quad \text{or} \quad S(p) = \frac{1}{2\pi} \int_0^{2\pi} |p(e^{ix})|^2 \mathrm{d}x$$

- 7 points for a complete solution

As always there will be a 1 point deduction to essentially complete solutions with minor errors that could be easily be fixed such as:

- not picking a big enough n when proving or applying the discrete characterisation of $S(p)$.

Coefficient manipulation solution

The following items apply but are not additive.

- 0 points for proving the inequality for special cases. For example when f is a quadratic polynomial, cubic polynomial etc.
- 0 points for expressing the polynomials as $f(x) = p(x)^2 r(x)$, $g(x) = q(x)^2 r(x)$ and $h(x) = p(x)q(x)r(x)$.
- 1 point for proving that $s(p)$ is equal to the independent coefficient of $p(x)p(1/x)$
- 5 points for proving the identities

$$S(r(x)p(x)^2) = \sum_k \left(r(x)p(x)p\left(\frac{1}{x}\right)[x^k] \right)^2$$
$$S(r(x)q(x)^2) = \sum_k \left(r\left(\frac{1}{x}\right)q(x)q\left(\frac{1}{x}\right)[x^{-k}] \right)^2$$

- 7 points for a complete solution

§4.6 Rubric for USEMO 6

No points are awarded for the answer $\alpha = 3/2$ alone. However, the following items are possible:

- A complete costruction is worth **2 points**. A student can earn **1 point** of this item for stating the answer $\alpha = 3/2$ roughly describing the idea of the construction (that is, to find a "ruler" D, and hope that the ruler has about $\Theta(\sqrt{n})$ numbers). This can be given even if the student has no idea how to actually find such a ruler.
- Meanwhile, proving the lower bound $\alpha \geq 3/2$ is worth **2 points**.

These two items are additive, meaning $1 + 2 = 3$ while $2 + 2 = 7$.