1. Let $ABC$ be a triangle. Prove that
\[
\sin \frac{3A}{2} + \sin \frac{3B}{2} + \sin \frac{3C}{2} \leq \cos \frac{A - B}{2} + \cos \frac{B - C}{2} + \cos \frac{C - A}{2}.
\]

2. Let $p$ be a prime number greater than 5. For any integer $x$, define
\[
f_p(x) = \sum_{k=1}^{p-1} \frac{1}{(px + k)^2}.
\]
Prove that for all positive integers $x$ and $y$ the numerator of $f_p(x) - f_p(y)$, when written in lowest terms, is divisible by $p^3$.

3. Let $n$ be an integer greater than 2, and $P_1, P_2, \ldots, P_n$ distinct points in the plane. Let $S$ denote the union of all segments $P_1P_2, P_2P_3, \ldots, P_{n-1}P_n$. Determine if it is always possible to find points $A$ and $B$ in $S$ such that $P_1P_n \parallel AB$ (segment $AB$ can lie on line $P_1P_n$) and $P_1P_n = kAB$, where (1) $k = 2.5$; (2) $k = 3$. 

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4. Let $n$ be a positive integer and let $S$ be a set of $2^n + 1$ elements. Let $f$ be a function from the set of two-element subsets of $S$ to $\{0, \ldots, 2^{n-1} - 1\}$. Assume that for any elements $x, y, z$ of $S$, one of $f(\{x, y\}), f(\{y, z\}), f(\{z, x\})$ is equal to the sum of the other two. Show that there exist $a, b, c$ in $S$ such that $f(\{a, b\}), f(\{b, c\}), f(\{c, a\})$ are all equal to 0.

5. Consider the family of non-isosceles triangles $ABC$ satisfying the property $AC^2 + BC^2 = 2AB^2$. Points $M$ and $D$ lie on side $AB$ such that $AM = BM$ and $\angle ACD = \angle BCD$. Point $E$ is in the plane such that $D$ is the incenter of triangle $CEM$. Prove that exactly one of the ratios

$$\frac{CE}{EM}, \frac{EM}{MC}, \frac{MC}{CE}$$

is constant.

6. Find in explicit form all ordered pairs of positive integers $(m, n)$ such that $mn - 1$ divides $m^2 + n^2$. 

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