USA TSTST 2023 Solutions

United States of America — TST Selection Test

Andrew Gu, Evan Chen, Gopal Goel

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§0 Problems

1. Let ABC be a triangle with centroid G. Points R and S are chosen on rays GB and GC, respectively, such that
$$\angle ABS = \angle ACR = 180^\circ - \angle BGC.$$
Prove that $\angle RAS + \angle BAC = \angle BGC$.

2. Let $n \ge m \ge 1$ be integers. Prove that
$$\sum_{k=m}^{n} \left(\frac{1}{k^2} + \frac{1}{k^3} \right) \ge m \cdot \left(\sum_{k=m}^{n} \frac{1}{k^2} \right)^2.$$

3. Find all positive integers n for which it is possible to color some cells of an infinite grid of unit squares red, such that each rectangle consisting of exactly n cells (and whose edges lie along the lines of the grid) contains an odd number of red cells.

4. Let $n \ge 3$ be an integer and let K_n be the complete graph on n vertices. Each edge of K_n is colored either red, green, or blue. Let A denote the number of triangles in K_n with all edges of the same color, and let B denote the number of triangles in K_n with all edges of different colors. Prove that
$$B \le 2A + \frac{n(n-1)}{3}.$$

5. Suppose a, b, and c are three complex numbers with product 1. Assume that none of a, b, and c are real or have absolute value 1. Define
$$p = (a+b+c) + \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) \qquad \text{and} \qquad q = \frac{a}{b} + \frac{b}{c} + \frac{c}{a}.$$
Given that both p and q are real numbers, find all possible values of the ordered pair (p, q).

6. Let ABC be a scalene triangle and let P and Q be two distinct points in its interior. Suppose that the angle bisectors of $\angle PAQ$, $\angle PBQ$, and $\angle PCQ$ are the altitudes of triangle ABC. Prove that the midpoint of $\overline{PQ}$ lies on the Euler line of ABC.

7. The Bank of Pittsburgh issues coins that have a heads side and a tails side. Vera has a row of 2023 such coins alternately tails-up and heads-up, with the leftmost coin tails-up.

   In a *move*, Vera may flip over one of the coins in the row, subject to the following rules:

   - On the first move, Vera may flip over any of the 2023 coins.
   - On all subsequent moves, Vera may only flip over a coin adjacent to the coin she flipped on the previous move. (We do not consider a coin to be adjacent to itself.)

   Determine the smallest possible number of moves Vera can make to reach a state in which every coin is heads-up.

8. Let ABC be an equilateral triangle with side length 1. Points A_1 and A_2 are chosen on side BC, points B_1 and B_2 are chosen on side CA, and points C_1 and C_2 are chosen on side AB such that $BA_1 < BA_2$, $CB_1 < CB_2$, and $AC_1 < AC_2$.

Suppose that the three line segments B_1C_2, C_1A_2, and A_1B_2 are concurrent, and the perimeters of triangles AB_2C_1, BC_2A_1, and CA_2B_1 are all equal. Find all possible values of this common perimeter.

9. Let p be a fixed prime and let $a \geq 2$ and $e \geq 1$ be fixed integers. Given a function $f \colon \mathbb{Z}/a\mathbb{Z} \to \mathbb{Z}/p^e\mathbb{Z}$ and an integer $k \geq 0$, the *kth finite difference*, denoted $\Delta^k f$, is the function from $\mathbb{Z}/a\mathbb{Z}$ to $\mathbb{Z}/p^e\mathbb{Z}$ defined recursively by

$$\begin{aligned} \Delta^0 f(n) &= f(n) \\ \Delta^k f(n) &= \Delta^{k-1} f(n+1) - \Delta^{k-1} f(n) \qquad \text{for } k = 1, 2, \dots. \end{aligned}$$

Determine the number of functions f such that there exists some $k \geq 1$ for which $\Delta^k f = f$.

§1 Solutions to Day 1

§1.1 TSTST 2023/1, proposed by Merlijn Staps

Available online at https://aops.com/community/p28015679.

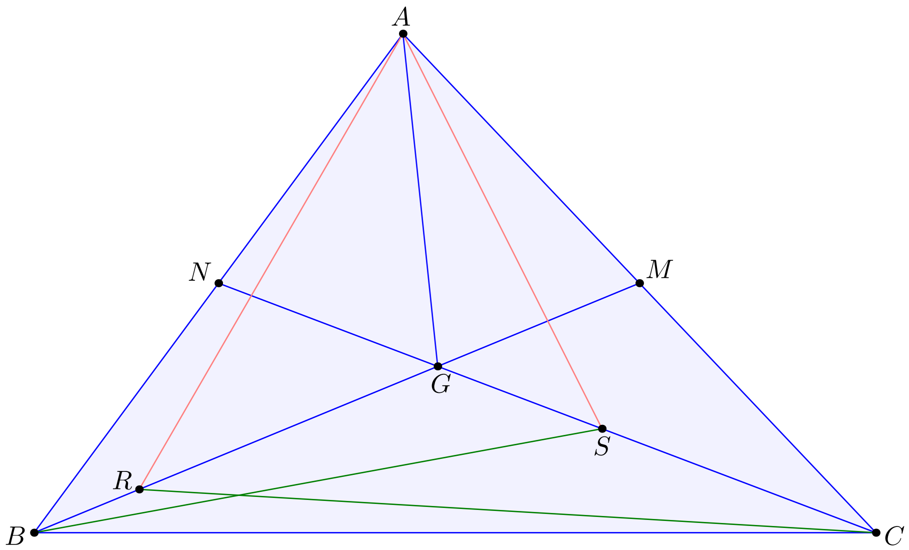
Problem statement

Let ABC be a triangle with centroid G. Points R and S are chosen on rays GB and GC, respectively, such that

$$\angle ABS = \angle ACR = 180^\circ - \angle BGC.$$

Prove that $\angle RAS + \angle BAC = \angle BGC$.

In all the following solutions, let M and N denote the midpoints of $\overline{AC}$ and $\overline{AB}$, respectively.



¶ Solution 1 using power of a point. From the given condition that $\measuredangle ACR = \measuredangle CGM$, we get that

$$MA^2 = MC^2 = MG \cdot MR \implies \measuredangle RAC = \measuredangle MGA.$$

Analogously,

$$\measuredangle BAS = \measuredangle AGN.$$

Hence,

$$\measuredangle RAS + \measuredangle BAC = \measuredangle RAC + \measuredangle BAS = \measuredangle MGA + \measuredangle AGN = \measuredangle MGN = \measuredangle BGC.$$

¶ Solution 2 using similar triangles. As before, $\triangle MGC \sim \triangle MCR$ and $\triangle NGB \sim \triangle NBS$. We obtain

$$\frac{|AC|}{|CR|} = \frac{2|MC|}{|CR|} = \frac{2|MG|}{|GC|} = \frac{|GB|}{2|NG|} = \frac{|BS|}{2|BN|} = \frac{|BS|}{|AB|}$$

which together with $\angle ACR = \angle ABS$ yields

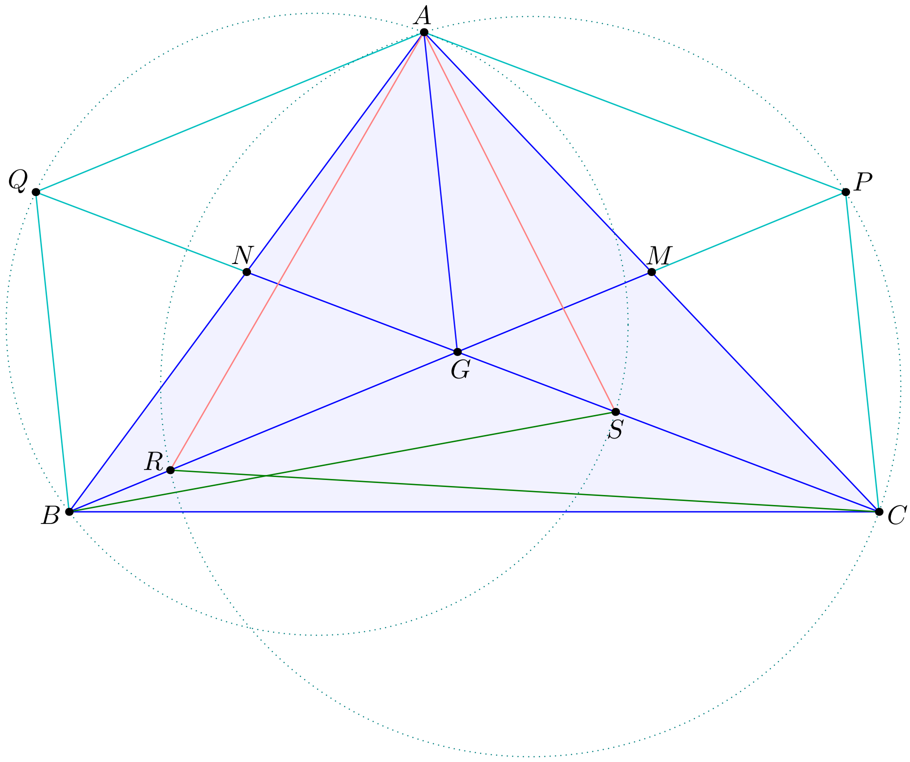
$$\triangle ACR \sim \triangle SBA \implies \measuredangle BAS = \measuredangle CRA.$$

Hence

$$\measuredangle RAS + \measuredangle BAC = \measuredangle RAC + \measuredangle BAS = \measuredangle RAC + \measuredangle CRA = -\measuredangle ACR = \measuredangle BGC,$$

which proves the statement.

¶ Solution 3 using parallelograms. Let M and N be defined as above. Let P be the reflection of G in M and let Q the reflection of G in N. Then $AGCP$ and $AGBQ$ are parallelograms.



Claim — Quadrilaterals $APCR$ and $AQBS$ are concyclic.

Proof. Because $\measuredangle APR = \measuredangle APG = \measuredangle CGP = -\measuredangle BGC = \measuredangle ACR$. □

Thus from $\overline{PC} \parallel \overline{GA}$ we get

$$\measuredangle RAC = \measuredangle RPC = \measuredangle GPC = \measuredangle PGA$$

and similarly

$$\measuredangle BAS = \measuredangle BQS = \measuredangle BQG = \measuredangle AGQ.$$

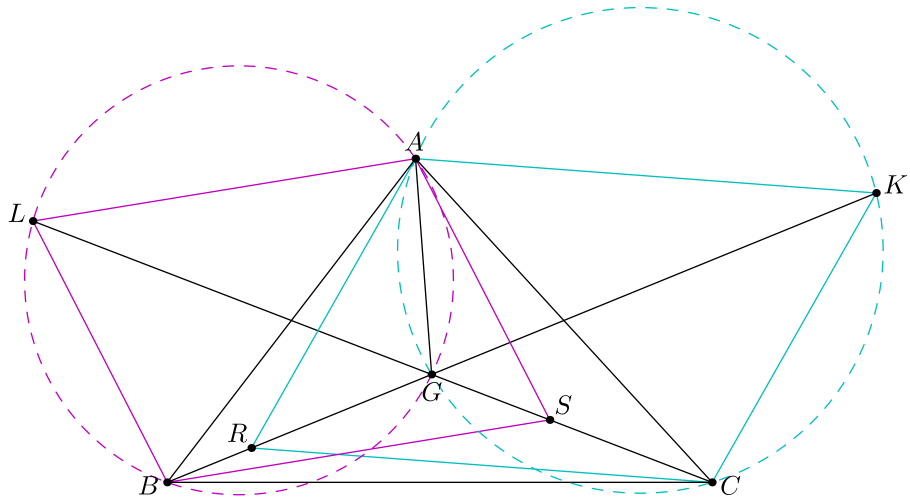
We conclude that

$$\measuredangle RAS + \measuredangle BAC = \measuredangle RAC + \measuredangle BAS = \measuredangle PGA + \measuredangle AGQ = \measuredangle PGQ = \measuredangle BGC.$$

¶ Solution 4 also using parallelograms, by Ankan Bhattacharya. Construct parallelograms $ARCK$ and $ASBL$. Since

$$\measuredangle CAK = \measuredangle ACR = \measuredangle CGB = \measuredangle CGK,$$

it follows that $AGCK$ is cyclic. Similarly, $AGBL$ is also cyclic.



Finally, observe that

$$\begin{aligned}\angle RAS + \angle BAC &= \measuredangle BAS + \measuredangle RAC \\ &= \measuredangle ABL + \measuredangle KCA \\ &= \measuredangle AGL + \measuredangle KGA \\ &= \measuredangle KGL \\ &= \angle BGC\end{aligned}$$

as requested.

¶ Solution 5 using complex numbers, by Milan Haiman. Note that $\angle RAS + \angle BAC = \angle BAS + \angle RAC$. We compute $\angle BAS$ in complex numbers; then $\angle RAC$ will then be known by symmetry.

Let a, b, c be points on the unit circle representing A, B, C respectively. Let $g = \frac{1}{3}(a+b+c)$ represent the centroid G, and let s represent S.

Claim — We have
$$\frac{s-a}{b-a} = \frac{ab-2bc+ca}{2ab-bc-ca}.$$

Proof. Since S is on line CG, which passes through the midpoint of segment AB, we have that
$$s = \frac{a+b}{2} + t(c-g)$$
for some $t \in \mathbb{R}$.

By the given angle condition, we have that
$$\frac{(s-b)/(b-a)}{(c-g)/(g-b)} \in \mathbb{R}.$$

Note that
$$\frac{s-b}{b-a} = t\frac{c-g}{b-a} - \frac{1}{2}.$$

So,
$$t\frac{g-b}{b-a} - \frac{g-b}{2(c-g)} \in \mathbb{R}.$$

Thus

$$t = \frac{\operatorname{Im}\left(\frac{g-b}{2(c-g)}\right)}{\operatorname{Im}\left(\frac{g-b}{b-a}\right)} = \frac{1}{2} \cdot \frac{\left(\frac{g-b}{c-g}\right) - \overline{\left(\frac{g-b}{c-g}\right)}}{\left(\frac{g-b}{b-a}\right) - \overline{\left(\frac{g-b}{b-a}\right)}}.$$

Let N and D be the numerator and denominator of the second factor above.

We want to compute

$$\frac{s-a}{b-a} = \frac{1}{2} + t\frac{c-g}{b-a} = \frac{(b-a)+2t(c-g)}{2(b-a)} = \frac{(b-a)D+(c-g)N}{2(b-a)D}.$$

We have

$$\begin{aligned}
(c-g)N &= g-b-(c-g)\overline{\left(\frac{g-b}{c-g}\right)} \\
&= \frac{a+b+c}{3} - b - \left(c - \frac{a+b+c}{3}\right)\frac{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}-\frac{3}{b}}{\frac{3}{c}-\frac{1}{a}-\frac{1}{b}-\frac{1}{c}} \\
&= \frac{(a+c-2b)(2ab-bc-ca)-(2c-a-b)(ab+bc-2ca)}{3(2ab-bc-ca)} \\
&= \frac{3(a^2b+b^2c+c^2a-ab^2-bc^2-ca^2)}{3(2ab-bc-ca)} \\
&= \frac{(a-b)(b-c)(a-c)}{2ab-bc-ca}
\end{aligned}$$

We also compute

$$\begin{aligned}
(b-a)D &= g-b-(b-a)\overline{\left(\frac{g-b}{b-a}\right)} \\
&= \frac{a+b+c}{3} - b - (b-a)\frac{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}-\frac{3}{b}}{\frac{3}{b}-\frac{3}{a}} \\
&= \frac{(a+c-2b)c+(ab+bc-2ca)}{3c} \\
&= \frac{ab-bc-ca+c^2}{3c} \\
&= \frac{(a-c)(b-c)}{3c}
\end{aligned}$$

So, we obtain

$$\frac{s-a}{b-a} = \frac{\frac{1}{3c}+\frac{a-b}{2ab-bc-ca}}{\frac{2}{3c}} = \frac{2ab-bc-ca+3c(a-b)}{2(2ab-bc-ca)} = \frac{ab-2bc+ca}{2ab-bc-ca}.$$

□

By symmetry,

$$\frac{r-a}{c-a} = \frac{ab-2bc+ca}{2ca-ab-bc}.$$

Hence their ratio

$$\frac{s-a}{b-a} \div \frac{r-a}{c-a} = \frac{2ab-bc-ca}{2ca-ab-bc}$$

has argument $\angle RAC + \angle BAS$.

We also have that $\angle BGC$ is the argument of

$$\frac{b-g}{c-g} = \frac{2b-a-c}{2c-a-b}.$$

Note that these two complex numbers are inverse-conjugates, and thus have the same argument. So we're done.

§1.2 TSTST 2023/2, proposed by Raymond Feng, Luke Robitaille

Available online at https://aops.com/community/p28015692.

Problem statement

Let $n \geq m \geq 1$ be integers. Prove that

$$\sum_{k=m}^{n} \left(\frac{1}{k^2} + \frac{1}{k^3} \right) \geq m \cdot \left(\sum_{k=m}^{n} \frac{1}{k^2} \right)^2 .$$

We show several approaches.

¶ First solution (authors). By Cauchy-Schwarz, we have

$$\begin{aligned} \sum_{k=m}^{n} \frac{k+1}{k^3} &= \sum_{k=m}^{n} \frac{\left(\frac{1}{k^2}\right)^2}{\frac{1}{k(k+1)}} \\ &\geq \frac{\left(\frac{1}{m^2} + \frac{1}{(m+1)^2} + \dots + \frac{1}{n^2}\right)^2}{\frac{1}{m(m+1)} + \frac{1}{(m+1)(m+2)} + \dots + \frac{1}{n(n+1)}} \\ &= \frac{\left(\frac{1}{m^2} + \frac{1}{(m+1)^2} + \dots + \frac{1}{n^2}\right)^2}{\frac{1}{m} - \frac{1}{n+1}} \\ &> \frac{\left(\sum_{k=m}^{n} \frac{1}{k^2}\right)^2}{\frac{1}{m}} \end{aligned}$$

as desired.

Remark (Bound on error)**.** Let $A = \sum_{k=m}^{n} k^{-2}$ and $B = \sum_{k=m}^{n} k^{-3}$. The inequality above becomes tighter for large m and $n \gg m$. If we use Lagrange's identity in place of Cauchy-Schwarz, we get

$$A + B - mA^2 = m \cdot \sum_{m \leq a < b} \frac{(a-b)^2}{a^3 b^3 (a+1)(b+1)}.$$

We can upper bound this error by

$$\leq m \cdot \sum_{m \leq a < b} \frac{1}{a^3 (a+1) b (b+1)} = m \cdot \sum_{m \leq a} \frac{1}{a^3 (a+1)^2} \approx m \cdot \frac{1}{m^4} = \frac{1}{m^3},$$

which is still generous as $(a-b)^2 \ll b^2$ for b not much larger than a, so the real error is probably around $\frac{1}{10m^3}$. This exhibits the tightness of the inequality since it implies

$$mA^2 + O(B/m) > A + B.$$

Remark (Construction commentary, from author)**.** My motivation was to write an inequality where Titu could be applied creatively to yield a telescoping sum. This can be difficult because most of the time, such a reverse-engineered inequality will be so loose it's trivial anyways. My first attempt was the not-so-amazing inequality

$$\frac{n^2+3n}{2} = \sum_1^n i+1 = \sum_1^n \frac{\frac{1}{i}}{\frac{1}{i(i+1)}} > \left(\sum_1^n \frac{1}{\sqrt{i}}\right)^2,$$

which is really not surprising given that $\sum \frac{1}{\sqrt{i}} \ll \frac{n}{\sqrt{2}}$. The key here is that we need "near-equality" as dictated by the Cauchy-Schwarz equality case, i.e. the square root of the numerators should be approximately proportional to the denominators.

This motivates using $\frac{1}{i^4}$ as the numerator, which works like a charm. After working out the resulting statement, the LHS and RHS even share a sum, which adds to the simplicity of the problem.

The final touch was to unrestrict the starting value of the sum, since this allows the strength of the estimate $\frac{1}{i^2} \approx \frac{1}{i(i+1)}$ to be fully exploited.

¶ Second approach by inducting down, Luke Robitaille and Carl Schildkraut. Fix n; we'll induct downwards on m. For the base case of $n = m$ the result is easy, since the left side is $\frac{m+1}{m^3}$ and the right side is $\frac{m}{m^4} = \frac{1}{m^3}$.

For the inductive step, suppose we have shown the result for $m+1$. Let

$$A = \sum_{k=m+1}^{n} \frac{1}{k^2} \qquad \text{and} \qquad B = \sum_{k=m+1}^{n} \frac{1}{k^3}.$$

We know $A + B \ge (m+1)A^2$, and we want to show

$$\left(A + \frac{1}{m^2}\right) + \left(B + \frac{1}{m^3}\right) \ge m\left(A + \frac{1}{m^2}\right)^2.$$

Indeed,

$$\begin{aligned}\left(A + \frac{1}{m^2}\right) + \left(B + \frac{1}{m^3}\right) - m\left(A + \frac{1}{m^2}\right)^2 &= A + B + \frac{m+1}{m^3} - mA^2 - \frac{2A}{m} - \frac{1}{m^3} \\ &= \left(A + B - (m+1)A^2\right) + \left(A - \frac{1}{m}\right)^2 \ge 0,\end{aligned}$$

and we are done.

¶ Third approach by reducing $n \to \infty$, Michael Ren and Carl Schildkraut. First, we give:

Claim (Reduction to $n \to \infty$) — If the problem is true when $n \to \infty$, it is true for all n.

Proof. Let $A = \sum_{k=m}^n k^{-2}$ and $B = \sum_{k=m}^n k^{-3}$. Consider the region of the xy-plane defined by $y > mx^2 - x$. We are interested in whether (A, B) lies in this region.

However, the region is bounded by a convex curve, and the sequence of points $(0,0)$, $\left(\frac{1}{m^2}, \frac{1}{m^3}\right)$, $\left(\frac{1}{m^2} + \frac{1}{(m+1)^2}, \frac{1}{m^3} + \frac{1}{(m+1)^3}\right)$, ... has successively decreasing slopes between consecutive points. Thus it suffices to check that the inequality is true when $n \to \infty$. □

Set $n = \infty$ henceforth. Let

$$A = \sum_{k=m}^{\infty} \frac{1}{k^2} \text{ and } B = \sum_{k=m}^{\infty} \frac{1}{k^3};$$

we want to show $B \geq mA^2 - A$, which rearranges to

$$1 + 4mB \geq (2mA - 1)^2.$$

Write

$$C = \sum_{k=m}^{\infty} \frac{1}{k^2(2k-1)(2k+1)} \text{ and } D = \sum_{k=m}^{\infty} \frac{8k^2-1}{k^3(2k-1)^2(2k+1)^2}.$$

Then

$$\frac{2}{2k-1} - \frac{2}{2k+1} = \frac{1}{k^2} + \frac{1}{k^2(2k-1)(2k+1)},$$

and

$$\frac{2}{(2k-1)^2} - \frac{2}{(2k+1)^2} = \frac{1}{k^3} + \frac{8k^2-1}{k^3(2k-1)^2(2k+1)^2},$$

so that

$$A = \frac{2}{2m-1} - C \text{ and } B = \frac{2}{(2m-1)^2} - D.$$

Our inequality we wish to show becomes

$$\frac{2m+1}{2m-1} C \geq D + mC^2.$$

We in fact show two claims:

Claim — We have

$$\frac{2m+1/2}{2m-1} C \geq D.$$

Proof. We compare termwise; we need

$$\frac{2m+1/2}{2m-1} \cdot \frac{1}{k^2(2k-1)(2k+1)} \geq \frac{8k^2-1}{k^3(2k-1)^2(2k+1)^2}$$

for $k \geq m$. It suffices to show

$$\frac{2k+1/2}{2k-1} \cdot \frac{1}{k^2(2k-1)(2k+1)} \geq \frac{8k^2-1}{k^3(2k-1)^2(2k+1)^2},$$

which is equivalent to $k(2k+1/2)(2k+1) \geq 8k^2 - 1$. This holds for all $k \geq 1$. □

Claim — We have

$$\frac{1/2}{2m-1} C \geq mC^2.$$

Proof. We need $C \leq 1/(2m(2m-1))$; indeed,

$$\frac{1}{2m(2m-1)} = \sum_{k=m}^{\infty} \left(\frac{1}{2k(2k-1)} - \frac{1}{2(k+1)(2k+1)} \right) = \sum_{k=m}^{\infty} \frac{4k+1}{2k(2k-1)(k+1)(2k+1)};$$

comparing term-wise with the definition of C and using the inequality $k(4k+1) \geq 2(k+1)$ for $k \geq 1$ gives the desired result. □

Combining the two claims finishes the solution.

¶ Fourth approach by bashing, Carl Schildkraut. With a bit more work, the third approach can be adapted to avoid the $n \to \infty$ reduction. Similarly to before, define

$$A = \sum_{k=m}^{n} \frac{1}{k^2} \text{ and } B = \sum_{k=m}^{n} \frac{1}{k^3};$$

we want to show $1 + 4mB \geq (2mA - 1)^2$. Writing

$$C = \sum_{k=m}^{n} \frac{1}{k^2(2k-1)(2k+1)} \text{ and } D = \sum_{k=m}^{n} \frac{8k^2-1}{k^3(2k-1)^2(2k+1)^2}.$$

We compute

$$A = \frac{2}{2m-1} - \frac{2}{2n+1} - C \text{ and } B = \frac{2}{(2m-1)^2} - \frac{2}{(2n+1)^2} - D.$$

Then, the inequality we wish to show reduces (as in the previous solution) to

$$\frac{2m+1}{2m-1}C + \frac{2(2m+1)}{(2m-1)(2n+1)} \geq D + mC^2 + \frac{2(2m+1)}{(2n+1)^2} + \frac{4m}{2n+1}C.$$

We deal first with the terms not containing the variable n, i.e. we show that

$$\frac{2m+1}{2m-1}C \geq D + mC^2.$$

For this part, the two claims from the previous solution go through exactly as written above, and we have $C \leq 1/(2m(2m-1))$. We now need to show

$$\frac{2(2m+1)}{(2m-1)(2n+1)} \geq \frac{2(2m+1)}{(2n+1)^2} + \frac{4m}{2n+1}C$$

(this is just the inequality between the remaining terms); our bound on C reduces this to proving

$$\frac{4(2m+1)(n-m+1)}{(2m-1)(2n+1)^2} \geq \frac{2}{(2m-1)(2n+1)}.$$

Expanding and writing in terms of n, this is equivalent to

$$n \geq \frac{1+2(m-1)(2m+1)}{4m} = m - \frac{2m+1}{4m},$$

which holds for all $n \geq m$.

§1.3 TSTST 2023/3, proposed by Merlijn Staps

Available online at https://aops.com/community/p28015682.

Problem statement

Find all positive integers n for which it is possible to color some cells of an infinite grid of unit squares red, such that each rectangle consisting of exactly n cells (and whose edges lie along the lines of the grid) contains an odd number of red cells.

We claim that this is possible for all positive integers n. Call a positive integer for which such a coloring is possible *good*. To show that all positive integers n are good we prove the following:

(i) If n is good and p is an odd prime, then pn is good;

(ii) For every $k \geq 0$, the number $n = 2^k$ is good.

Together, (i) and (ii) imply that all positive integers are good.

¶ Proof of (i). We simply observe that if every rectangle consisting of n cells contains an odd number of red cells, then so must every rectangle consisting of pn cells. Indeed, because p is prime, a rectangle consisting of pn cells must have a dimension (length or width) divisible by p and can thus be subdivided into p rectangles consisting of n cells.

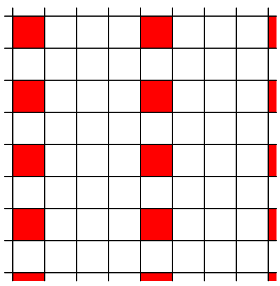
Thus every coloring that works for n automatically also works for pn.

¶ Proof of (ii). Observe that rectangles with $n = 2^k$ cells have $k+1$ possible shapes: $2^m \times 2^{k-m}$ for $0 \leq m \leq k$.

Claim — For each of these $k+1$ shapes, there exists a coloring with two properties:

- Every rectangle with n cells and shape $2^m \times 2^{k-m}$ contains an odd number of red cells.
- Every rectangle with n cells and a different shape contains an even number of red cells.

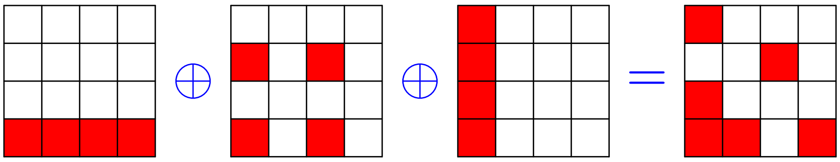
Proof. This can be achieved as follows: assuming the cells are labeled with $(x, y) \in \mathbb{Z}^2$, color a cell red if $x \equiv 0 \pmod{2^m}$ and $y \equiv 0 \pmod{2^{k-m}}$. For example, a 4×2 rectangle gets the following coloring:



A $2^m \times 2^{k-m}$ rectangle contains every possible pair $(x \mod 2^m, y \mod 2^{k-m})$ exactly once, so such a rectangle will contain one red cell (an odd number).

On the other hand, consider a $2^\ell \times 2^{k-\ell}$ rectangle with $\ell > m$. The set of cells this covers is (x, y) where x covers a range of size 2^ℓ and y covers a range of size $2^{k-\ell}$. The number of red cells is the count of x with $x \equiv 0 \mod 2^m$ multiplied by the count of y with $y \equiv 0 \mod 2^{k-m}$. The former number is exactly $2^{\ell-k}$ because 2^k divides 2^ℓ (while the latter is 0 or 1) so the number of red cells is even. The $\ell < m$ case is similar. $\square$

Finally, given these $k+1$ colorings, we can add them up modulo 2, i.e. a cell will be colored red if it is red in an odd number of these $k+1$ colorings. We illustrate $n = 4$ as an example; the coloring is 4-periodic in both axes so we only show one 4×4 cell.



This solves the problem.

Remark. The final coloring can be described as follows: color (x, y) red if

$$\max(0, \min(\nu_2(x), k) + \min(\nu_2(y), k) - k + 1)$$

is odd.

Remark (Luke Robitaille)**.** Alternatively for (i), if $n = 2^e k$ for odd k then one may dissect an $a \times b$ rectangle with area n into k rectangles of area 2^e, each $2^{\nu_2(a)} \times 2^{\nu_2(b)}$. This gives a way to deduce the problem from (ii) without having to consider odd prime numbers.

¶ Alternate proof of (ii) using generating functions. We will commit to constructing a coloring which is n-periodic in both directions. (This is actually forced, so it's natural to do so.) With that in mind, let

$$f(x, y) = \sum_{i=0}^{2^k-1} \sum_{j=0}^{2^k-1} \lambda_{i,j} x^i y^j$$

denote its generating function, where $f \in \mathbb{F}_2[x, y]$.

For this to be valid, we need that for any $2^p \times 2^q$ rectangle with area n, the sum of the coefficients of f over it should be one, modulo $x^{2^k} = y^{2^k} = 1$. In other words, whenever $p + q = k$, we must have

$$f(x, y)(1 + \dots + x^{2^p-1})(1 + \dots + y^{2^q-1}) = (1 + \dots + x^{2^k-1})(1 + \dots + y^{2^k-1}),$$

taken modulo $x^{2^k} = y^{2^k} = 1$. The idea is to rewrite these expressions: because we're in characteristic 2, the given assertion is $(x+1)^{2^k} = (y+1)^{2^k} = 0$, and the requested property is

$$f(x, y)(x+1)^{2^p-1}(y+1)^{2^q-1} = (x+1)^{2^k-1}(y+1)^{2^k-1}.$$

This suggests the substitution $g(x, y) = f(x+1, y+1)$: then we can replace $(x+1, y+1) \mapsto (x, y)$ to simplify the requested property significantly:

Whenever $p + q = k$, we must have

$$g(x, y) x^{2^p-1} y^{2^q-1} = x^{2^k-1} y^{2^k-1},$$

modulo x^{2^k} and y^{2^k}.

However, now the construction of g is very simple: for example, the choice

$$g(x,y) = \sum_{p+q=k} x^{2^k-2^p} y^{2^k-2^q}$$

works. The end.

> **Remark.** Unraveling the substitutions seen here, it's possible to show that this is actually the same construction provided in the first solution.

§2 Solutions to Day 2

§2.1 TSTST 2023/4, proposed by Ankan Bhattacharya

Available online at https://aops.com/community/p28015691.

Problem statement

Let $n \geq 3$ be an integer and let K_n be the complete graph on n vertices. Each edge of K_n is colored either red, green, or blue. Let A denote the number of triangles in K_n with all edges of the same color, and let B denote the number of triangles in K_n with all edges of different colors. Prove that

$$B \leq 2A + \frac{n(n-1)}{3}.$$

Consider all unordered pairs of different edges which share exactly one vertex (call these *vees* for convenience). Assign each vee a *charge* of $+2$ if its edge colors are the same, and a charge of -1 otherwise.

We compute the total charge in two ways.

¶ Total charge by summing over triangles. Note that

- each monochromatic triangle has a charge of $+6$,
- each bichromatic triangle has a charge of 0, and
- each trichromatic triangle has a charge of -3.

Since each vee contributes to exactly one triangle, we obtain that the total charge is $6A - 3B$.

¶ Total charge by summing over vertices. We can also calculate the total charge by examining the centers of the vees. If a vertex has a red edges, b green edges, and c blue edges, the vees centered at that vertex contribute a total charge of

$$\begin{aligned}
&2\left[\binom{a}{2} + \binom{b}{2} + \binom{c}{2}\right] - (ab + ac + bc) \\
&= (a^2 - a + b^2 - b + c^2 - c) - (ab + ac + bc) \\
&= (a^2 + b^2 + c^2 - ab - ac - bc) - (a + b + c) \\
&= (a^2 + b^2 + c^2 - ab - ac - bc) - (n-1) \\
&\geq -(n-1).
\end{aligned}$$

In particular, the total charge is at least $-n(n-1)$.

¶ Conclusion. Thus, we obtain

$$6A - 3B \geq -n(n-1) \iff B \leq 2A + \frac{n(n-1)}{3}$$

as desired.

§2.2 TSTST 2023/5, proposed by David Altizio

Available online at https://aops.com/community/p28015713.

Problem statement

Suppose a, b, and c are three complex numbers with product 1. Assume that none of a, b, and c are real or have absolute value 1. Define

$$p = (a+b+c) + \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \qquad \text{and} \qquad q = \frac{a}{b} + \frac{b}{c} + \frac{c}{a}.$$

Given that both p and q are real numbers, find all possible values of the ordered pair (p, q).

We show $(p, q) = (-3, 3)$ is the only possible ordered pair.

¶ First solution.

Setup for proof Let us denote $a = y/x$, $b = z/y$, $c = x/z$, where x, y, z are nonzero complex numbers. Then

$$\begin{aligned}
p + 3 &= 3 + \sum_{\text{cyc}} \left(\frac{x}{y} + \frac{y}{x}\right) = 3 + \frac{x^2(y+z) + y^2(z+x) + z^2(x+y)}{xyz} \\
&= \frac{(x+y+z)(xy+yz+zx)}{xyz}. \\
q - 3 &= -3 + \sum_{\text{cyc}} \frac{y^2}{zx} = \frac{x^3+y^3+z^3-3xyz}{xyz} \\
&= \frac{(x+y+z)(x^2+y^2+z^2-xy-yz-zx)}{xyz}.
\end{aligned}$$

It follows that

$$\begin{aligned}
\mathbb{R} &\ni 3(p+3) + (q-3) \\
&= \frac{(x+y+z)(x^2+y^2+z^2+2(xy+yz+zx))}{xyz} \\
&= \frac{(x+y+z)^3}{xyz}.
\end{aligned}$$

Now, note that if $x + y + z = 0$, then $p = -3$, $q = 3$ so we are done.

Main proof We will prove that if $x+y+z \neq 0$ then we contradict either the hypothesis that $a, b, c \notin \mathbb{R}$ or that a, b, c do not have absolute value 1.

Scale x, y, z in such a way that $x + y + z$ is nonzero and real; hence so is xyz. Thus, as $p + 3 \in \mathbb{R}$, we conclude $xy + yz + zx \in \mathbb{R}$ as well. Hence, x, y, z are the roots of a cubic with real coefficients. Thus,

- either all three of $\{x, y, z\}$ are real (which implies $a, b, c \in \mathbb{R}$),
- or two of $\{x, y, z\}$ are a complex conjugate pair (which implies one of a, b, c has absolute value 1).

Both of these were forbidden by hypothesis.

Construction As we saw in the setup, $(p,q)=(-3,3)$ will occur as long as $x+y+z=0$, and no two of x, y, z to share the same magnitude or are collinear with the origin. This is easy to do; for example, we could choose $(x,y,z)=(3,4i,-(3+4i))$. Hence $a=\frac{3}{4i}$, $b=-\frac{4i}{3+4i}$, $c=-\frac{3+4i}{3}$ satisfies the hypotheses of the problem statement.

¶ Second solution, found by contestants. The main idea is to make the substitution

$$x=a+\frac{1}{c}, \qquad y=b+\frac{1}{a}, \qquad z=c+\frac{1}{b}.$$

Then we can check that

$$\begin{aligned} x+y+z&=p \\ xy+yz+zx&=p+q+3 \\ xyz&=p+2. \end{aligned}$$

Therefore x, y, z are the roots of a cubic with real coefficients. As in the previous solution, we note that this cubic must either have all real roots, or a complex conjugate pair of roots. We also have the relation $a(y+1)=ab+a+1=x+1$, and likewise $b(z+1)=y+1$, $c(x+1)=z+1$. This means that if any of x, y, z are equal to -1, then all are equal to -1.

Assume for the sake of contradiction that none are equal to -1. In the case where the cubic has three real roots, $a=\frac{x+1}{y+1}$ would be real. On the other hand, if there is a complex conjugate pair (without loss of generality, x and y) then a has magnitude 1. Therefore this cannot occur.

We conclude that $x=y=z=-1$, so $p=-3$ and $q=3$. The solutions (a,b,c) can then be parameterized as $(a,-1-\frac{1}{a},-\frac{1}{1+a})$. To construct a solution, we need to choose a specific value of a such that none of the wrong conditions hold; when $a=2i$, say, we obtain the solution $(2i,-1+\frac{i}{2},\frac{-1+2i}{5})$.

¶ Third solution by Luke Robitaille and Daniel Zhu. The answer is $p=-3$ and $q=3$. Let's first prove that no other (p,q) work.

Let $e_1=a+b+c$ and $e_2=a^{-1}+b^{-1}+c^{-1}=ab+ac+bc$. Also, let $f=e_1e_2$. Note that $p=e_1+e_2$.

Our main insight is to consider the quantity $q'=\frac{b}{a}+\frac{c}{b}+\frac{a}{c}$. Note that $f=q+q'+3$. Also,

$$\begin{aligned} qq'&=3+\frac{a^2}{bc}+\frac{b^2}{ac}+\frac{c^2}{ab}+\frac{bc}{a^2}+\frac{ac}{b^2}+\frac{ab}{c^2} \\ &=3+a^3+b^3+c^3+a^{-3}+b^{-3}+c^{-3} \\ &=9+a^3+b^3+c^3-3abc+a^{-3}+b^{-3}+c^{-3}-3a^{-1}b^{-1}c^{-1} \\ &=9+e_1(e_1^2-3e_2)+e_2(e_2^2-3e_1) \\ &=9+e_1^3+e_2^3-6e_1e_2 \\ &=9+p(p^2-3f)-6f \\ &=p^3-(3p+6)f+9. \end{aligned}$$

As a result, the quadratic with roots q and q' is $x^2-(f-3)x+(p^3-(3p+6)f+9)$, which implies that

$$q^2-qf+3q+p^3-(3p+6)f+9=0 \iff (3p+q+6)f=p^3+q^2+3q+9.$$

At this point, two miracles occur. The first is the following claim:

Claim — f is not real.

Proof. Suppose f is real. Since $(x-e_1)(x-e_2) = x^2 - px + f$, there are two cases:

- e_1 and e_2 are real. Then, a, b, and c are the roots of $x^3 - e_1x^2 + e_2x - 1$, and since every cubic with real coefficients has at least one real root, at least one of a, b, and c is real, contradiction.

- e_1 and e_2 are conjugates. Then, the polynomial $x^3 - \bar{e}_2x^2 + \bar{e}_1x - 1$, which has roots $\bar{a}^{-1}$, $\bar{b}^{-1}$, and $\bar{c}^{-1}$, is the same as the polynomial with a, b, c as roots. We conclude that the multiset $\{a, b, c\}$ is invariant under inversion about the unit circle, so one of a, b, and c must lie on the unit circle. This is yet another contradiction. $\square$

As a result, we know that $3p + q + 6 = p^3 + q^2 + 3q + 9 = 0$. The second miracle is that substituting $q = -3p - 6$ into $q^2 + 3q + p^3 + 9 = 0$, we get

$$0 = p^3 + 9p^2 + 27p + 27 = (p+3)^3,$$

so $p = -3$. Thus $q = 3$.

It remains to construct valid a, b, and c. To do this, let's pick some e_1, let $e_2 = -3 - e_1$, and let a, b, and c be the roots of $x^3 - e_1x^2 + e_2x - 1$. It is clear that this guarantees $p = -3$. By our above calculations, q and q' are the roots of the quadratic $x^2 - (f-3)x + (3f - 18)$, so one of q and q' must be 3; by changing the order of a, b, and c if needed, we can guarantee this to be q. It suffices to show that for some choice of e_1, none of a, b, or c are real or lie on the unit circle.

To do this, note that we can rewrite $x^3 - e_1x^2 + (-3 - e_1)x - 1 = 0$ as

$$e_1 = \frac{x^3 - 3x - 1}{x^2 + x},$$

so all we need is a value of e_1 that is not $\frac{x^3-3x-1}{x^2+x}$ for any real x or x on the unit circle. One way to do this is to choose any nonreal e_1 with $|e_1| < 1/2$. This clearly rules out any real x. Also, if $|x| = 1$, by the triangle inequality $|x^3 - 3x - 1| \ge |3x| - |x^3| - |1| = 1$ and $|x^2 + x| \le 2$, so $\left|\frac{x^3-3x-1}{x^2+x}\right| \ge \frac{1}{2}$.

§2.3 TSTST 2023/6, proposed by Holden Mui

Available online at https://aops.com/community/p28015708.

Problem statement

Let ABC be a scalene triangle and let P and Q be two distinct points in its interior. Suppose that the angle bisectors of $\angle PAQ$, $\angle PBQ$, and $\angle PCQ$ are the altitudes of triangle ABC. Prove that the midpoint of $\overline{PQ}$ lies on the Euler line of ABC.

We present three approaches.

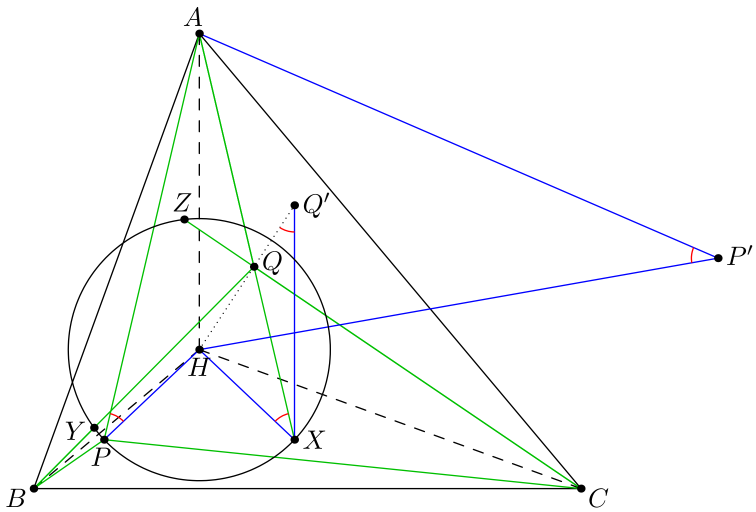
¶ Solution 1 (Ankit Bisain). Let H be the orthocenter of ABC, and construct P' using the following claim.

Claim — There is a point P' for which

$$\measuredangle APH + \measuredangle AP'H = \measuredangle BPH + \measuredangle BP'H = \measuredangle CPH + \measuredangle CP'H = 0.$$

Proof. After inversion at H, this is equivalent to the fact that P's image has an isogonal conjugate in ABC's image. □

Now, let X, Y, and Z be the reflections of P over $\overline{AH}$, $\overline{BH}$, and $\overline{CH}$ respectively. Additionally, let Q' be the image of Q under inversion about $(PXYZ)$.



Claim — $ABCP' \overset{-}{\sim} XYZQ'$.

Proof. Since

$$\measuredangle YXZ = \measuredangle YPZ = \measuredangle(\overline{BH}, \overline{CH}) = -\measuredangle BAC$$

and cyclic variants, triangles ABC and XYZ are similar. Additionally,

$$\measuredangle HQ'X = -\measuredangle HXQ = -\measuredangle HXA = \measuredangle HPA = -\measuredangle HP'A$$

and cyclic variants, so summing in pairs gives $\measuredangle YQ'Z = -\measuredangle BP'C$ and cyclic variants; this implies the similarity. □

Claim — Q' lies on the Euler line of triangle XYZ.

Proof. Let O be the circumcenter of ABC so that $ABCOP' \overset{-}{\sim} XYZHQ'$. Then $\measuredangle HP'A = -\measuredangle HQ'X = \measuredangle OP'A$, so P' lies on $\overline{OH}$. By the similarity, Q' must lie on the Euler line of XYZ. $\square$

To finish the problem, let G_1 be the centroid of ABC and G_2 be the centroid of XYZ. Then with signed areas,

$$\begin{aligned}
[G_1HP] + [G_1HQ] &= \frac{[AHP]+[BHP]+[CHP]}{3} + \frac{[AHQ]+[BHQ]+[CHQ]}{3} \\
&= \frac{[AHQ]-[AHX]+[BHQ]-[BHY]+[CHQ]-[CHZ]}{3} \\
&= \frac{[HQX]+[HQY]+[HQZ]}{3} \\
&= [QG_2H] \\
&= 0
\end{aligned}$$

where the last line follows from the last claim. Therefore $\overline{G_1H}$ bisects $\overline{PQ}$, as desired.

Remark. This solution characterizes the set of all points P for which such a point Q exists. It is the image of the Euler line under the mapping described in the first claim.

¶ Solution 2 using complex numbers (Carl Schildkraut and Milan Haiman). Let (ABC) be the unit circle in the complex plane, and let $A = a$, $B = b$, $C = c$ such that $|a| = |b| = |c| = 1$. Let $P = p$ and $Q = q$, and $O = 0$ and $H = h = a + b + c$ be the circumcenter and orthocenter of ABC respectively.

The first step is to translate the given geometric conditions into a single usable equation:

Claim — We have the equation

$$(p+q)\sum_{\text{cyc}} a^3(b^2-c^2) = (\overline{p}+\overline{q})abc\sum_{\text{cyc}}(bc(b^2-c^2)). \tag{1}$$

Proof. The condition that the altitude $\overline{AH}$ bisects $\angle PAQ$ is equivalent to

$$\begin{aligned}
&\frac{(p-a)(q-a)}{(h-a)^2} = \frac{(p-a)(q-a)}{(b+c)^2} \in \mathbb{R} \\
\implies &\frac{(p-a)(q-a)}{(b+c)^2} = \overline{\left(\frac{(p-a)(q-a)}{(b+c)^2}\right)} = \frac{(a\overline{p}-1)(a\overline{q}-1)b^2c^2}{(b+c)^2a^2} \\
\implies &a^2(p-a)(q-a) = b^2c^2(a\overline{p}-1)(a\overline{q}-1) \\
\implies &a^2pq - a^2b^2c^2\overline{pq} + (a^4 - b^2c^2) = a^3(p+q) - ab^2c^2(\overline{p}+\overline{q}).
\end{aligned}$$

Writing the symmetric conditions that $\overline{BH}$ and $\overline{CH}$ bisect $\angle PBQ$ and $\angle PCQ$ gives three equations:

$$\begin{aligned}
a^2pq - a^2b^2c^2\overline{pq} + (a^4 - b^2c^2) &= a^3(p+q) - ab^2c^2(\overline{p}+\overline{q}) \\
b^2pq - a^2b^2c^2\overline{pq} + (b^4 - c^2a^2) &= b^3(p+q) - bc^2a^2(\overline{p}+\overline{q})
\end{aligned}$$

$$c^2pq - a^2b^2c^2\overline{pq} + (c^4 - a^2b^2) = c^3(p+q) - ca^2b^2(\overline{p} + \overline{q}).$$

Now, sum (b^2-c^2) times the first equation, (c^2-a^2) times the second equation, and (a^2-b^2) times the third equation. On the left side, the coefficients of pq and $\overline{pq}$ are 0. Additionally, the coefficient of 1 (the parenthesized terms on the left sides of each equation) sum to 0, since

$$\sum_{\text{cyc}}(a^4 - b^2c^2)(b^2-c^2) = \sum_{\text{cyc}}(a^4b^2 - b^4c^2 - a^4c^2 + c^4b^2).$$

This gives (1) as desired. $\square$

We can then factor (1):

Claim — The left-hand side of (1) factors as

$$-(p+q)(a-b)(b-c)(c-a)(ab+bc+ca)$$

while the right-hand side factors as

$$-(\overline{p}+\overline{q})(a-b)(b-c)(c-a)(a+b+c).$$

Proof. This can of course be verified by direct expansion, but here is a slightly more economic indirect proof.

Consider the cyclic sum on the left as a polynomial in a, b, and c. If $a=b$, then it simplifies as $a^3(a^2-c^2)+a^3(c^2-a^2)+c^3(a^2-a^2)=0$, so $a-b$ divides this polynomial. Similarly, $a-c$ and $b-c$ divide it, so it can be written as $f(a,b,c)(a-b)(b-c)(c-a)$ for some symmetric quadratic polynomial f, and thus it is some linear combination of $a^2+b^2+c^2$ and $ab+bc+ca$. When $a=0$, the whole expression is $b^2c^2(b-c)$, so $f(0,b,c)=-bc$, which implies that $f(a,b,c)=-(ab+bc+ca)$.

Similarly, consider the cyclic sum on the right as a polynomial in a, b, and c. If $a=b$, then it simplifies as $ac(a^2-c^2)+ca(c^2-a^2)+a^2(a^2-a^2)=0$, so $a-b$ divides this polynomial. Similarly, $a-c$ and $b-c$ divide it, so it can be written as $g(a,b,c)(a-b)(b-c)(c-a)$ where g is a symmetric linear polynomial; hence, it is a scalar multiple of $a+b+c$. When $a=0$, the whole expression is $bc(b^2-c^2)$, so $g(0,b,c)=-b-c$, which implies that $g(a,b,c)=-(a+b+c)$. $\square$

Since A, B, and C are distinct, we may divide by $(a-b)(b-c)(c-a)$ to obtain

$$(p+q)(ab+bc+ca) = (\overline{p}+\overline{q})abc(a+b+c) \implies (p+q)\overline{h} = (\overline{p}+\overline{q})h.$$

This implies that $\frac{\frac{p+q}{2}-0}{h-0}$ is real, so the midpoint of $\overline{PQ}$ lies on line $\overline{OH}$.

¶ Solution 3 also using complex numbers (Michael Ren). We use complex numbers as in the previous solution. The angle conditions imply that $\frac{(a-p)(a-q)}{(b-c)^2}$, $\frac{(b-p)(b-q)}{(c-a)^2}$, and $\frac{(c-p)(c-q)}{(a-b)^2}$ are real numbers. Take a linear combination of these with real coefficients X, Y, and Z to be determined; after expansion, we obtain

$$\begin{aligned}&\left[\frac{X}{(b-c)^2}+\frac{Y}{(c-a)^2}+\frac{Z}{(a-b)^2}\right]pq\\ -&\left[\frac{aX}{(b-c)^2}+\frac{bY}{(c-a)^2}+\frac{cZ}{(a-b)^2}\right](p+q)\end{aligned}$$

$$+\left[\frac{a^2X}{(b-c)^2}+\frac{b^2Y}{(c-a)^2}+\frac{c^2Z}{(a-b)^2}\right]$$

which is a real number. To get something about the midpoint of PQ, the pq coefficient should be zero, which motivates the following lemma.

Lemma

There exist real X, Y, Z for which

$$\begin{aligned}\frac{X}{(b-c)^2}+\frac{Y}{(c-a)^2}+\frac{Z}{(a-b)^2}&=0 \text{ and}\\ \frac{aX}{(b-c)^2}+\frac{bY}{(c-a)^2}+\frac{cZ}{(a-b)^2}&\neq 0.\end{aligned}$$

Proof. Since $\mathbb{C}$ is a 2-dimensional vector space over $\mathbb{R}$, there exist real X, Y, Z such that $(X, Y, Z) \neq (0,0,0)$ and the first condition holds. Suppose for the sake of contradiction that $\frac{aX}{(b-c)^2}+\frac{bY}{(c-a)^2}+\frac{cZ}{(a-b)^2}=0$. Then

$$\begin{aligned}&\frac{(b-a)Y}{(c-a)^2}+\frac{(c-a)Z}{(a-b)^2}\\ =&\frac{aX}{(b-c)^2}+\frac{bY}{(c-a)^2}+\frac{cZ}{(a-b)^2}-a\left(\frac{X}{(b-c)^2}+\frac{Y}{(c-a)^2}+\frac{Z}{(a-b)^2}\right)\\ =&0.\end{aligned}$$

We can easily check that $(Y, Z) = (0,0)$ is impossible, therefore $\frac{(b-a)^3}{(c-a)^3} = -\frac{Z}{Y}$ is real. This means $\angle BAC = 60^\circ$ or 120°. By symmetry, the same is true of $\angle CBA$ and $\angle ACB$. This is impossible because ABC is scalene. $\square$

With the choice of X, Y, Z as in the lemma, there exist complex numbers α and β, depending only on a, b, and c, such that $\alpha \neq 0$ and $\alpha(p+q)+\beta$ is real. Therefore the midpoint of PQ, which corresponds to $\frac{p+q}{2}$, lies on a fixed line. It remains to show that this line is the Euler line. First, choose $P = Q$ to be the orthocenter to show that the orthocenter lies on the line. Secondly, choose P and Q to be the foci of the Steiner circumellipse to show that the centroid lies on the line. (By some ellipse properties, the external angle bisector of $\angle PAQ$ is the tangent to the circumellipse at A, which is the line through A parallel to BC. Therefore these points are valid.) Therefore the fixed line of the midpoint is the Euler line.

Remark. This solution does not require fixing the origin of the complex plane or setting (ABC) to be the unit circle.

§3 Solutions to Day 3

§3.1 TSTST 2023/7, proposed by Luke Robitaille

Available online at https://aops.com/community/p28015706.

Problem statement

The Bank of Pittsburgh issues coins that have a heads side and a tails side. Vera has a row of 2023 such coins alternately tails-up and heads-up, with the leftmost coin tails-up.

In a *move*, Vera may flip over one of the coins in the row, subject to the following rules:

- On the first move, Vera may flip over any of the 2023 coins.
- On all subsequent moves, Vera may only flip over a coin adjacent to the coin she flipped on the previous move. (We do not consider a coin to be adjacent to itself.)

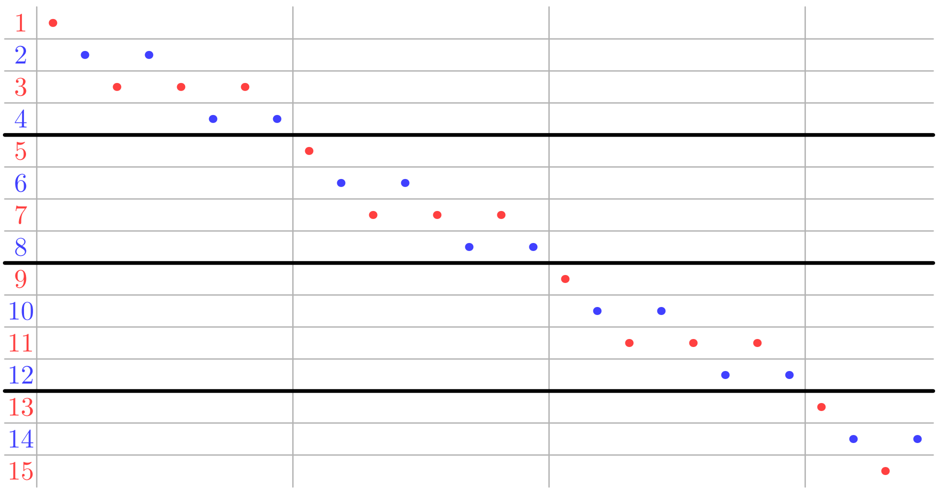
Determine the smallest possible number of moves Vera can make to reach a state in which every coin is heads-up.

The answer is $\boxed{4044}$. In general, replacing 2023 with $4n + 3$, the answer is $8n + 4$.

¶ Bound. Observe that the first and last coins must be flipped, and so every coin is flipped at least once. Then, the $2n + 1$ even-indexed coins must be flipped at least twice, so they are flipped at least $4n + 2$ times.

The $2n + 2$ odd-indexed coins must then be flipped at least $4n + 1$ times. Since there are an even number of these coins, the total flip count must be even, so they are actually flipped a total of at least $4n + 2$ times, for a total of at least $8n + 4$ flips in all.

¶ Construction. For $k = 0, 1, \dots, n-1$, flip $(4k+1, 4k+2, 4k+3, 4k+2, 4k+3, 4k+4, 4k+3, 4k+4)$ in that order; then at the end, flip $4n + 1$, $4n + 2$, $4n + 3$, $4n + 2$. This is illustrated below for $4n + 3 = 15$.



It is easy to check this works, and there are 4044 flips, as desired.

§3.2 TSTST 2023/8, proposed by Ankan Bhattacharya

Available online at https://aops.com/community/p28015680.

Problem statement

Let ABC be an equilateral triangle with side length 1. Points A_1 and A_2 are chosen on side BC, points B_1 and B_2 are chosen on side CA, and points C_1 and C_2 are chosen on side AB such that $BA_1 < BA_2$, $CB_1 < CB_2$, and $AC_1 < AC_2$.

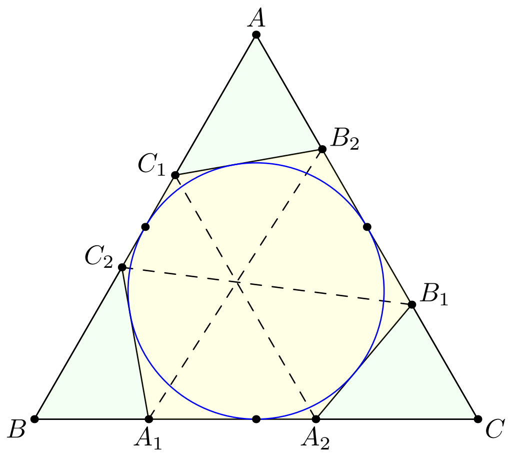
Suppose that the three line segments B_1C_2, C_1A_2, and A_1B_2 are concurrent, and the perimeters of triangles AB_2C_1, BC_2A_1, and CA_2B_1 are all equal. Find all possible values of this common perimeter.

The only possible value of the common perimeter, denoted p, is 1.

¶ Synthetic approach (from author). We prove the converse of the problem first:

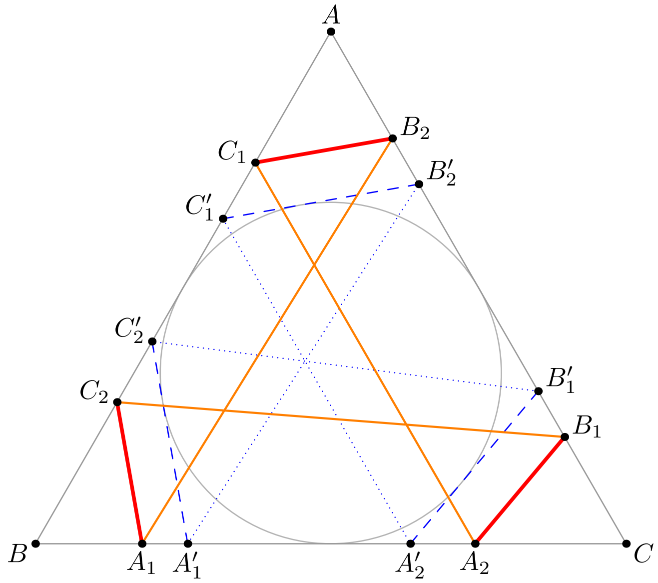
Claim ($p = 1$ implies concurrence) — Suppose the six points are chosen so that triangles AB_2C_1, BC_2A_1, CA_2B_1 all have perimeter 1. Then lines $\overline{B_1C_2}$, $\overline{C_1A_2}$, and $\overline{A_1B_2}$ are concurrent.

Proof. The perimeter conditions mean that $\overline{B_2C_1}$, $\overline{C_2A_1}$, and $\overline{A_2B_1}$ are tangent to the incircle of $\triangle ABC$.



Hence the result follows by *Brianchon's theorem*. □

Now suppose $p \neq 1$. Let $\overline{B_2'C_1'}$ be the dilation of $\overline{B_2C_1}$ with ratio $\frac{1}{p}$ at center A, and define C_2', A_1', A_2', B_1' similarly. The following diagram showcases the situation $p < 1$.



By the reasoning in the $p = 1$ case, note that $\overline{B_1'C_2'}$, $\overline{C_1'A_2'}$, and $\overline{A_1'B_2'}$ are concurrent. However, $\overline{B_1C_2}$, $\overline{C_1A_2}$, $\overline{A_1B_2}$ lie in the interior of quadrilaterals $BCB_1'C_2'$, $CAC_1'A_2'$, and $ABA_1'B_2'$, and these quadrilaterals do not share an interior point, a contradiction.

Thus $p \geq 1$. Similarly, we can show $p \leq 1$, and so $p = 1$ is forced (and achieved if, for example, the three triangles are equilateral with side length $1/3$).

¶ Barycentric solution (by Carl, Krit, Milan). We show that, if the common perimeter is 1, then the lines concur. To do this, we use barycentric coordinates. Let $A = (1 : 0 : 0)$, $B = (0 : 1 : 0)$, and $C = (0 : 0 : 1)$. Let $A_1 = (0 : 1 - a_1 : a_1)$, $A_2 = (0 : a_2 : 1 - a_2)$, $B_1 = (b_1 : 0 : 1 - b_1)$, $B_2 = (1 - b_2 : 0 : b_2)$, $C_1 = (1 - c_1 : c_1 : 0)$, and $C_2 = (c_2 : 1 - c_2 : 0)$. The line B_1C_2 is defined by the equation

$$\det \begin{bmatrix} x & y & z \\ b_1 & 0 & 1 - b_1 \\ c_2 & 1 - c_2 & 0 \end{bmatrix} = 0;$$

i.e.

$$x\big(-(1 - b_1)(1 - c_2)\big) + y\big((1 - b_1)c_2\big) + z\big(b_1(1 - c_2)\big) = 0.$$

Computing the equations for the other lines cyclically, we get that the lines B_1C_2, C_1A_2, and A_1B_2 concur if and only if

$$\det \begin{bmatrix} -(1 - b_1)(1 - c_2) & (1 - b_1)c_2 & b_1(1 - c_2) \\ c_1(1 - a_2) & -(1 - c_1)(1 - a_2) & (1 - c_1)a_2 \\ (1 - a_1)b_2 & a_1(1 - b_2) & -(1 - a_1)(1 - b_2) \end{bmatrix} = 0.$$

Let this matrix be M. We also define the similar matrix

$$N = \begin{bmatrix} -(1 - b_2)(1 - c_1) & (1 - b_2)c_1 & b_2(1 - c_1) \\ c_2(1 - a_1) & -(1 - c_2)(1 - a_1) & (1 - c_2)a_1 \\ (1 - a_2)b_1 & a_2(1 - b_1) & -(1 - a_2)(1 - b_1) \end{bmatrix}.$$

Geometrically, $\det N = 0$ if and only if $B_2'C_1'$, $C_2'A_1'$, and $A_2'B_1'$ concur, where for a point P on a side of triangle ABC, P' denotes its reflection over that side's midpoint.

Claim — We have $\det M = \det N$.

Proof. To show $\det M = \det N$, it suffices to demonstrate that the determinant above is invariant under swapping subscripts of "1" and "2," an operation we call Ψ.

We use the definition of the determinant as a sum over permutations. The even permutations give us the following three terms:

$$\begin{aligned} -(1-b_1)(1-c_2)(1-c_1)(1-a_2)(1-a_1)(1-b_2) &= -\prod_{i=1}^{2}\left((1-a_i)(1-b_i)(1-c_i)\right) \\ (1-a_1)b_2(1-b_1)c_2(1-c_1)a_2 &= \big((1-a_1)(1-b_1)(1-c_1)\big)\big(a_2b_2c_2\big) \\ c_1(1-a_2)a_1(1-b_2)b_1(1-c_2) &= \big((1-a_2)(1-b_2)(1-c_2)\big)\big(a_1b_1c_1\big). \end{aligned}$$

The first term is invariant under Ψ, while the second and third terms are swapped under Ψ. For the odd permutations, we have a contribution to the determinant of

$$\sum_{\text{cyc}}(1-b_1)(1-c_2)(1-c_1)a_2a_1(1-b_2);$$

each summand is invariant under Ψ. This finishes the proof of our claim. □

Now, it suffices to show that, if AB_2C_1, BC_2A_1, and CA_2B_1 each have perimeter 1, then

$$\det\begin{bmatrix} -(1-b_2)(1-c_1) & (1-b_2)c_1 & b_2(1-c_1) \\ c_2(1-a_1) & -(1-c_2)(1-a_1) & (1-c_2)a_1 \\ (1-a_2)b_1 & a_2(1-b_1) & -(1-a_2)(1-b_1). \end{bmatrix} = 0.$$

Indeed, we have $AB_2 = b_2$ and $AC_1 = c_1$, so by the law of cosines,

$$1-b_2-c_1 = 1-AB_2-AC_1 = B_2C_1 = \sqrt{b_2^2+c_1^2-b_2c_1}.$$

This gives

$$(1-b_2-c_1)^2 = b_2^2+c_1^2-b_2c_1 \implies 1-2b_2-2c_1+3b_2c_1 = 0.$$

Similarly, $1-2c_2-2a_1+3c_2a_1 = 0$ and $1-2a_2-2b_1+3a_2b_1 = 0$.

Now,

$$\begin{aligned} N\begin{bmatrix}1\\1\\1\end{bmatrix} &= \begin{bmatrix} -(1-b_2)(1-c_1)+(1-b_2)c_1+b_2(1-c_1) \\ -(1-c_2)(1-a_1)+(1-c_2)a_1+c_2(1-a_1) \\ -(1-a_2)(1-b_1)+(1-a_2)b_1+a_2(1-b_1) \end{bmatrix} \\ &= \begin{bmatrix} -1+2b_2+2c_1-3b_2c_1 \\ -1+2c_2+2a_1-3c_2a_1 \\ -1+2a_2+2b_1-2a_2b_1 \end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}. \end{aligned}$$

So it follows $\det N = 0$, as desired.

§3.3 TSTST 2023/9, proposed by Holden Mui

Available online at https://aops.com/community/p28015688.

Problem statement

Let p be a fixed prime and let $a \geq 2$ and $e \geq 1$ be fixed integers. Given a function $f\colon \mathbb{Z}/a\mathbb{Z} \to \mathbb{Z}/p^e\mathbb{Z}$ and an integer $k \geq 0$, the *kth finite difference*, denoted $\Delta^k f$, is the function from $\mathbb{Z}/a\mathbb{Z}$ to $\mathbb{Z}/p^e\mathbb{Z}$ defined recursively by

$$\begin{aligned} \Delta^0 f(n) &= f(n) \\ \Delta^k f(n) &= \Delta^{k-1} f(n+1) - \Delta^{k-1} f(n) \qquad \text{for } k = 1, 2, \dots. \end{aligned}$$

Determine the number of functions f such that there exists some $k \geq 1$ for which $\Delta^k f = f$.

The answer is

$$(p^e)^a \cdot p^{-ep^{\nu_p(a)}} = p^{e(a - p^{\nu_p(a)})}.$$

¶ **First solution by author.** For convenience in what follows, set $d = \nu_p(a)$, let $a = p^d \cdot b$, and let a function $f\colon \mathbb{Z}/a\mathbb{Z} \to \mathbb{Z}/p^e\mathbb{Z}$ be *essential* if it equals one of its iterated finite differences.

The key claim is the following.

Claim (Characterization of essential functions) — A function f is essential if and only if

$$f(x) + f(x+p^d) + \cdots + f(x+(b-1)p^d) = 0 \tag{2}$$

for all x.

As usual, we split the proof into two halves.

Proof that essential implies the equation First, suppose that f is essential, with $\Delta^N f = f$. Observe that f is in the image of Δ^k for any k, because $\Delta^{mN} f = f$ for any m. The following lemma will be useful.

Lemma

Let $g\colon \mathbb{Z}/a\mathbb{Z} \to \mathbb{Z}/p^e\mathbb{Z}$ be any function, and let $h = \Delta^{p^d} g$. Then

$$h(x) + h(x+p^d) + \cdots + h(x+(b-1)p^d) \equiv 0 \pmod p$$

for all x.

Proof. By definition,

$$h(x) = \Delta^{p^d} g(x) = \sum_{k=0}^{p^d} (-1)^k \binom{p^d}{k} g(x + p^d - k).$$

However, it is known that $\binom{p^d}{k}$ is a multiple of p if $1 \leq k \leq p^d - 1$, so

$$h(x) \equiv g(x+p^d) + (-1)^{p^d} g(x) \pmod p.$$

Using this, we easily obtain

$$\begin{aligned} & h(x)+h(x+p^d)+\cdots+h(x+(b-1)p^d) \\ \equiv & \begin{cases} 0 & p>2 \\ 2(g(x)+g(x+p^d)+\cdots+g(x+(b-1)p^d)) & p=2 \end{cases} \\ \equiv & \ 0 \pmod p, \end{aligned}$$

as desired. □

Corollary

Let $g\colon \mathbb{Z}/a\mathbb{Z} \to \mathbb{Z}/p^e\mathbb{Z}$ be any function, and let $h = \Delta^{ep^d} g$. Then

$$h(x)+h(x+p^d)+\cdots+h(x+(b-1)p^d) = 0$$

for all x.

Proof. Starting with the lemma, define

$$h_1(x) = \frac{h(x)+h(x+p^d)+\cdots+h(x+(b-1)p^d)}{p}.$$

Applying the lemma to h_1 shows the corollary for $e=2$, since $h_1(x)$ is divisible by p, hence the numerator is divisible by p^2. Continue in this manner to get the result for general $e>2$. □

This immediately settles this direction, since f is in the image of Δ^{ep^d}.

Proof the equation implies essential Let $\mathcal{S}$ be the set of all functions satisfying 2; then it's easy to see that Δ is a function on $\mathcal{S}$. To show that all functions in $\mathcal{S}$ are essential, it's equivalent to show that Δ is a permutation on $\mathcal{S}$.

We will show that Δ is injective on $\mathcal{S}$. Suppose otherwise, and consider two functions f, g in $\mathcal{S}$ with $\Delta f = \Delta g$. Then, we obtain that f and g differ by a constant; say $g = f + \lambda$. However, then

$$\begin{aligned} & g(0)+g(p^e)+\cdots+g((b-1)p^e) \\ = & (f(0)+\lambda)+(f(p^e)+\lambda)+\cdots+(f((b-1)p^e)+\lambda) \\ = & b\lambda. \end{aligned}$$

This should also be zero. Since $p \nmid b$, we obtain $\lambda = 0$, as desired.

Counting Finally, we can count the essential functions: all but the last p^d entries can be chosen arbitrarily, and then each remaining entry has exactly one possible choice. This leads to a count of

$$(p^e)^{a-p^d} = p^{e(a-p^{\nu_p(a)})},$$

as promised.

¶ Second solution by Daniel Zhu. There are two parts to the proof: solving the $e=1$ case, and using the $e=1$ result to solve the general problem by induction on e. These parts are independent of each other.

The case $e = 1$ Represent functions f as elements

$$\alpha_f := \sum_{k \in \mathbb{Z}/a\mathbb{Z}} f(-k)x^k \in \mathbb{F}_p[x]/(x^a - 1)$$

Then, since $\alpha_{\Delta f} = (x-1)\alpha_f$, we wish to find the number of $\alpha \in \mathbb{F}_p[x]/(x^a - 1)$ such that $(x-1)^m \alpha = \alpha$ for some m.

Now, make the substitution $y = x - 1$ and let $P(y) = (y+1)^a - 1$; we want to find $\alpha \in \mathbb{F}_p[y]/(P(y))$ such that $y^m \alpha = \alpha$ for some m.

If we write $P(y) = y^d Q(y)$ with $Q(0) \neq 0$, then by the Chinese Remainder Theorem we have the ring isomorphism

$$\mathbb{F}_p[y]/(P(y)) \cong \mathbb{F}_p[y]/(y^d) \times \mathbb{F}_p[y]/(Q(y)).$$

Note that y is nilpotent in the first factor, while it is a unit in the second factor. So the α that work are exactly those that are zero in the first factor; thus there are p^{a-d} such α. We can calculate $d = p^{v_p(a)}$ (via, say, Lucas's Theorem), so we are done.

The general problem The general idea is as follows: call a $f \colon \mathbb{Z}/a\mathbb{Z} \to \mathbb{Z}/p^e\mathbb{Z}$ *e-good* if $\Delta^m f = f$ for some m. Our result above allows us to count the 1-good functions. Then, if $e \geq 1$, every $(e+1)$-good function, when reduced mod p^e, yields an e-good function, so we count $(e+1)$-good functions by counting how many reduce to any given e-good function.

Formally, we use induction on e, with the $e = 1$ case being treated above. Suppose now we have solved the problem for a given $e \geq 1$, and we now wish to solve it for $e+1$. For any function $g \colon \mathbb{Z}/a\mathbb{Z} \to \mathbb{Z}/p^{e+1}\mathbb{Z}$, let $\bar{g} \colon \mathbb{Z}/a\mathbb{Z} \to \mathbb{Z}/p^e\mathbb{Z}$ be its reduction mod p^e. For a given e-good f, let $n(f)$ be the number of $(e+1)$-good g with $\bar{g} = f$. The following two claims now finish the problem:

Claim — If f is e-good, then $n(f) > 0$.

Proof. Suppose m is such that $\Delta^m f = f$. Pick any g with $\bar{g} = f$, and consider the sequence of functions

$$g, \Delta^m g, \Delta^{2m} g, \dots.$$

Since there are finitely many functions $\mathbb{Z}/a\mathbb{Z} \to \mathbb{Z}/p^{e+1}\mathbb{Z}$, there must exist $a < b$ such that $\Delta^{am} g = \Delta^{bm} g$. We claim $\Delta^{am} g$ is the desired $(e+1)$-good function. To see this, first note that since $\overline{\Delta^k g} = \Delta^k \bar{g}$, we must have $\overline{\Delta^{am} g} = \Delta^{am} f = f$. Moreover,

$$\Delta^{(b-a)m}(\Delta^{am} g) = \Delta^{bm} g = \Delta^{am} g,$$

so $\Delta^{am} g$ is $(e+1)$-good. □

Claim — If f is e-good, and $n(f) > 0$, then $n(f)$ is exactly the number of 1-good functions, i.e. $p^{a - p^{v_p(a)}}$.

Proof. Let g be any $(e+1)$-good function with $\bar{g} = f$. We claim that the $(e+1)$-good g_1 with $\bar{g}_1 = f$ are exactly the functions of the form $g + p^e h$ for any 1-good h. Since these functions are clearly distinct, this characterization will prove the claim.

To show that this condition is sufficient, note that $\overline{g + p^e h} = \bar{g} = f$. Moreover, if $\Delta^m g = g$ and $\Delta^{m'} h = h$, then

$$\Delta^{mm'}(g + p^e h) = \Delta^{mm'} g + p^e \Delta^{mm'} h = g + p^e h.$$

To show that this condition is necessary, let g_1 be any $(e+1)$-good function such that $\bar{g}_1 = f$. Then $g_1 - g$ is also $(e+1)$-good, since if $\Delta^m g = g$, $\Delta^{m'} g_1 = g_1$, we have

$$\Delta^{mm'}(g_1 - g) = \Delta^{mm'} g_1 - \Delta^{mm'} g = g_1 - g.$$

On the other hand, we also know that $g_1 - g$ is divisible by p^e. This means that it must be $p^e h$ for some function $f \colon \mathbb{Z}/a\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$, and it is not hard to show that $g_1 - g$ being $(e+1)$-good means that h is 1-good. $\square$