USA TSTST 2021 Solutions

United States of America — TST Selection Test

Andrew Gu and Evan Chen

63rd IMO 2022 Norway and 11th EGMO 2022 Hungary

Contents

§0 Problems

1. Let $ABCD$ be a quadrilateral inscribed in a circle with center O. Points X and Y lie on sides AB and CD, respectively. Suppose the circumcircles of ADX and BCY meet line XY again at P and Q, respectively. Show that $OP = OQ$.

2. Let $a_1 < a_2 < a_3 < a_4 < \cdots$ be an infinite sequence of real numbers in the interval $(0, 1)$. Show that there exists a number that occurs exactly once in the sequence
$$\frac{a_1}{1}, \frac{a_2}{2}, \frac{a_3}{3}, \frac{a_4}{4}, \dots.$$

3. Find all positive integers $k > 1$ for which there exists a positive integer n such that $\binom{n}{k}$ is divisible by n, and $\binom{n}{m}$ is not divisible by n for $2 \le m < k$.

4. Let a and b be positive integers. Suppose that there are infinitely many pairs of positive integers (m, n) for which $m^2 + an + b$ and $n^2 + am + b$ are both perfect squares. Prove that a divides $2b$.

5. Let T be a tree on n vertices with exactly k leaves. Suppose that there exists a subset of at least $\frac{n+k-1}{2}$ vertices of T, no two of which are adjacent. Show that the longest path in T contains an even number of edges.

6. Triangles ABC and DEF share circumcircle Ω and incircle ω so that points A, F, B, D, C, and E occur in this order along Ω. Let Δ_A be the triangle formed by lines AB, AC, and EF, and define triangles $\Delta_B, \Delta_C, \dots, \Delta_F$ similarly. Furthermore, let Ω_A and ω_A be the circumcircle and incircle of triangle Δ_A, respectively, and define circles $\Omega_B, \omega_B, \dots, \Omega_F, \omega_F$ similarly.
   (a) Prove that the two common external tangents to circles Ω_A and Ω_D and the two common external tangents to circles ω_A and ω_D are either concurrent or pairwise parallel.
   (b) Suppose that these four lines meet at point T_A, and define points T_B and T_C similarly. Prove that points T_A, T_B, and T_C are collinear.

7. Let M be a finite set of lattice points and n be a positive integer. A *mine-avoiding path* is a path of lattice points with length n, beginning at $(0, 0)$ and ending at a point on the line $x + y = n$, that does not contain any point in M. Prove that if there exists a mine-avoiding path, then there exist at least $2^{n-|M|}$ mine-avoiding paths.

8. Let ABC be a scalene triangle. Points A_1, B_1 and C_1 are chosen on segments BC, CA, and AB, respectively, such that $\triangle A_1B_1C_1$ and $\triangle ABC$ are similar. Let A_2 be the unique point on line B_1C_1 such that $AA_2 = A_1A_2$. Points B_2 and C_2 are defined similarly. Prove that $\triangle A_2B_2C_2$ and $\triangle ABC$ are similar.

9. Let $q = p^r$ for a prime number p and positive integer r. Let $\zeta = e^{\frac{2\pi i}{q}}$. Find the least positive integer n such that
$$\sum_{\substack{1 \le k \le q \\ \gcd(k,p)=1}} \frac{1}{(1-\zeta^k)^n}$$
is not an integer. (The sum is over all $1 \le k \le q$ with p not dividing k.)

§1 Solutions to Day 1

§1.1 TSTST 2021/1, proposed by Holden Mui

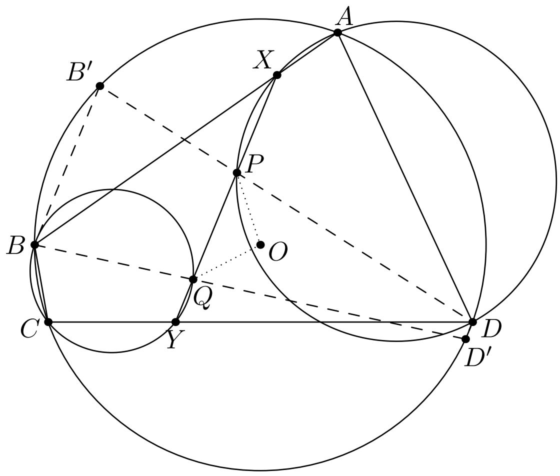
Available online at https://aops.com/community/p23586650.

Problem statement

Let $ABCD$ be a quadrilateral inscribed in a circle with center O. Points X and Y lie on sides AB and CD, respectively. Suppose the circumcircles of ADX and BCY meet line XY again at P and Q, respectively. Show that $OP = OQ$.

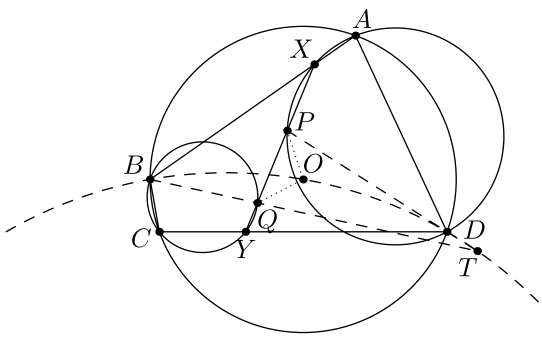
We present many solutions.

¶ First solution, angle chasing only (Ankit Bisain). Let lines BQ and DP meet $(ABCD)$ again at D' and B', respectively.



Then $BB' \parallel PX$ and $DD' \parallel QY$ by Reim's theorem. Segments BB', DD', and PQ share a perpendicular bisector which passes through O, so $OP = OQ$.

¶ Second solution via isosceles triangles (from contestants). Let $T = \overline{BQ} \cap \overline{DP}$.



Note that PQT is isosceles because

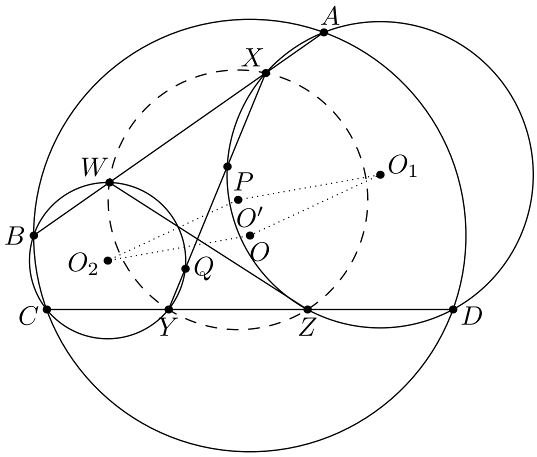
$$\measuredangle PQT = \measuredangle YQB = \measuredangle BCD = \measuredangle BAD = \measuredangle XPD = \measuredangle TPQ.$$

Then $(BODT)$ is cyclic because

$$\measuredangle BOD = 2\measuredangle BCD = \measuredangle PQT + \measuredangle TPQ = \measuredangle BTD.$$

Since $BO = OD$, $\overline{TO}$ is an angle bisector of $\measuredangle BTD$. Since $\triangle PQT$ is isosceles, $\overline{TO} \perp \overline{PQ}$, so $OP = OQ$.

¶ Third solution using a parallelogram (from contestants). Let (BCY) meet $\overline{AB}$ again at W and let (ADX) meet $\overline{CD}$ again at Z. Additionally, let O_1 be the center of (ADX) and O_2 be the center of (BCY).



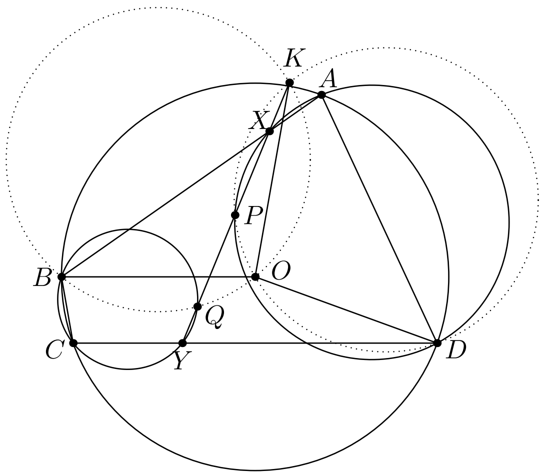
Note that $(WXYZ)$ is cyclic since

$$\measuredangle XWY + \measuredangle YZX = \measuredangle YWB + \measuredangle XZD = \measuredangle YCB + \measuredangle XAD = 0^\circ,$$

so let O' be the center of $(WXYZ)$. Since $\overline{AD} \parallel \overline{WY}$ and $\overline{BC} \parallel \overline{XZ}$ by Reim's theorem, $OO_1O'O_2$ is a parallelogram.

To finish the problem, note that projecting O_1, O_2, and O' onto $\overline{XY}$ gives the midpoints of $\overline{PX}$, $\overline{QY}$, and $\overline{XY}$. Since $OO_1O'O_2$ is a parallelogram, projecting O onto $\overline{XY}$ must give the midpoint of $\overline{PQ}$, so $OP = OQ$.

¶ Fourth solution using congruent circles (from contestants). Let the angle bisector of $\measuredangle BOD$ meet $\overline{XY}$ at K.

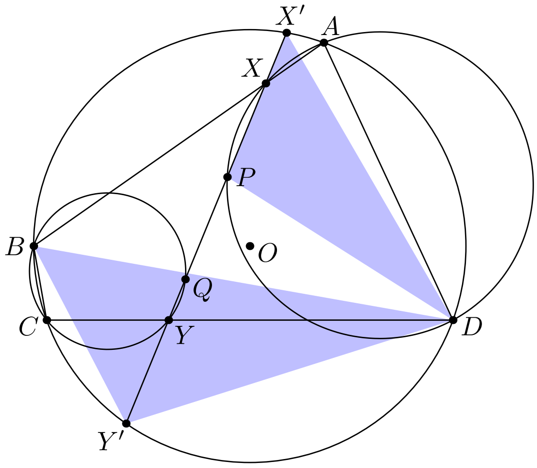


Then $(BQOK)$ is cyclic because $\measuredangle KOD = \measuredangle BAD = \measuredangle KPD$, and $(DOPK)$ is cyclic similarly. By symmetry over KO, these circles have the same radius r, so

$$OP = 2r \sin \angle OKP = 2r \sin \angle OKQ = OQ$$

by the Law of Sines.

¶ Fifth solution by ratio calculation (from contestants). Let $\overline{XY}$ meet $(ABCD)$ at X' and Y'.



Since $\measuredangle Y'BD = \measuredangle PX'D$ and $\measuredangle BY'D = \measuredangle BAD = \measuredangle X'PD$,

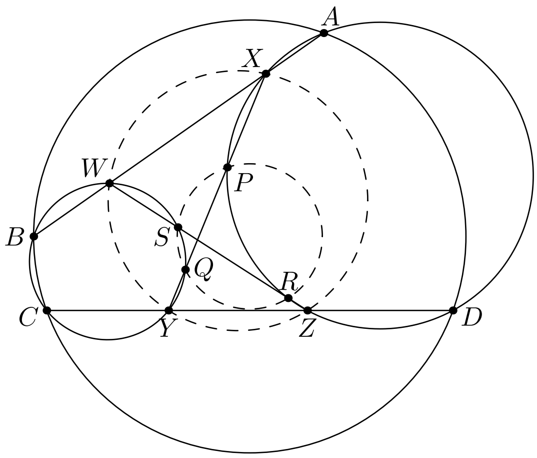
$$\triangle BY'D \sim \triangle XP'D \implies PX' = BY' \cdot \frac{DX'}{BD}.$$

Similarly,

$$\triangle BX'D \sim \triangle BQY' \implies QY' = DX' \cdot \frac{BY'}{BD}.$$

Thus $PX' = QY'$, which gives $OP = OQ$.

¶ Sixth solution using radical axis (from author). Without loss of generality, assume $\overline{AD} \nparallel \overline{BC}$, as this case holds by continuity. Let (BCY) meet $\overline{AB}$ again at W, let (ADX) meet $\overline{CD}$ again at Z, and let $\overline{WZ}$ meet (ADX) and (BCY) again at R and S.



Note that $(WXYZ)$ is cyclic since

$$\measuredangle XWY + \measuredangle YZX = \measuredangle YWB + \measuredangle XZD = \measuredangle YCB + \measuredangle XAD = 0^\circ$$

and $(PQRS)$ is cyclic since

$$\measuredangle PQS = \measuredangle YQS = \measuredangle YWS = \measuredangle PXZ = \measuredangle PRZ = \measuredangle SRP.$$

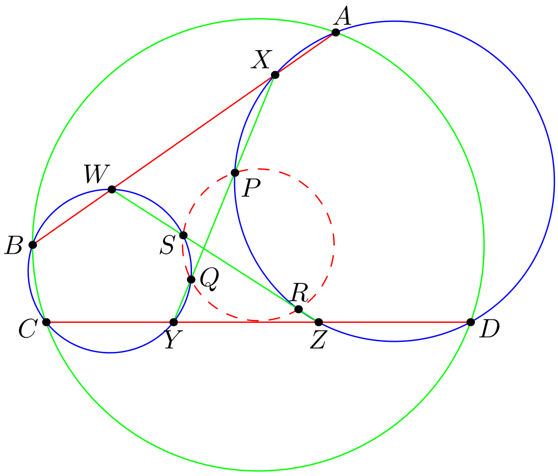
Additionally, $\overline{AD} \parallel \overline{PR}$ since

$$\measuredangle DAX + \measuredangle AXP + \measuredangle XPR = \measuredangle YWX + \measuredangle WXY + \measuredangle XYW = 0^\circ,$$

and $\overline{BC} \parallel \overline{SQ}$ similarly.

Lastly, $(ABCD)$ and $(PQRS)$ are concentric; if not, using the radical axis theorem twice shows that their radical axis must be parallel to both $\overline{AD}$ and $\overline{BC}$, contradiction.

¶ Seventh solution using Cayley-Bacharach (author). Define points W, Z, R, S as in the previous solution.



The quartics $(ADXZ) \cup (BCWY)$ and $\overline{XY} \cup \overline{WZ} \cup (ABCD)$ meet at the 16 points

$$A, B, C, D, W, X, Y, Z, P, Q, R, S, I, I, J, J,$$

where I and J are the circular points at infinity. Since $\overline{AB} \cup \overline{CD} \cup (PQR)$ contains the 13 points

$$A, B, C, D, P, Q, R, W, X, Y, Z, I, J,$$

it must contain S, I, and J as well, by quartic Cayley-Bacharach. Thus, $(PQRS)$ is cyclic and intersects $(ABCD)$ at I, I, J, and J, implying that the two circles are concentric, as desired.

> **Remark** (Author comments)**.** Holden says he came up with this problem via the Cayley-Bacharach solution, by trying to get two quartics to intersect.

§1.2 TSTST 2021/2, proposed by Merlijn Staps

Available online at https://aops.com/community/p23586635.

Problem statement

Let $a_1 < a_2 < a_3 < a_4 < \cdots$ be an infinite sequence of real numbers in the interval $(0,1)$. Show that there exists a number that occurs exactly once in the sequence

$$\frac{a_1}{1}, \frac{a_2}{2}, \frac{a_3}{3}, \frac{a_4}{4}, \dots .$$

We present three solutions.

¶ Solution 1 (Merlijn Staps). We argue by contradiction, so suppose that for each λ for which the set $S_\lambda = \{k : a_k/k = \lambda\}$ is non-empty, it contains at least two elements. Note that S_λ is always a finite set because $a_k = k\lambda$ implies $k < 1/\lambda$.

Write m_λ and M_λ for the smallest and largest element of S_λ, respectively, and define $T_\lambda = \{m_\lambda, m_\lambda+1, \dots, M_\lambda\}$ as the smallest set of consecutive positive integers that contains S_λ. Then all T_λ are sets of at least two consecutive positive integers, and moreover the T_λ cover $\mathbb{N}$. Additionally, each positive integer is covered finitely many times because there are only finitely many possible values of m_λ smaller than any fixed integer.

Recall that if three intervals have a point in common then one of them is contained in the union of the other two. Thus, if any positive integer is covered more than twice by the sets T_λ, we may throw out one set while maintaining the property that the T_λ cover $\mathbb{N}$. By using the fact that each positive integer is covered finitely many times, we can apply this process so that each positive integer is eventually covered at most twice.

Let Λ denote the set of the λ-values for which T_λ remains in our collection of sets; then $\bigcup_{\lambda\in\Lambda} T_\lambda = \mathbb{N}$ and each positive integer is contained in at most two sets T_λ.

We now obtain

$$\sum_{\lambda\in\Lambda}\sum_{k\in T_\lambda}(a_{k+1}-a_k) \le 2\sum_{k\ge 1}(a_{k+1}-a_k) \le 2.$$

On the other hand, because $a_{m_\lambda} = \lambda m_\lambda$ and $a_{M_\lambda} = \lambda M_\lambda$, we have

$$\begin{aligned} 2\sum_{k\in T_\lambda}(a_{k+1}-a_k) &\ge 2\sum_{m_\lambda \le k < M_\lambda}(a_{k+1}-a_k) = 2(a_{M_\lambda}-a_{m_\lambda}) = 2(M_\lambda - m_\lambda)\lambda \\ &= 2(M_\lambda - m_\lambda)\cdot\frac{a_{m_\lambda}}{m_\lambda} \ge (M_\lambda - m_\lambda + 1)\cdot \frac{a_1}{m_\lambda} \ge a_1 \cdot \sum_{k\in T_\lambda}\frac{1}{k}. \end{aligned}$$

Combining this with our first estimate, and using the fact that the T_λ cover $\mathbb{N}$, we obtain

$$4 \ge 2\sum_{\lambda\in\Lambda}\sum_{k\in T_\lambda}(a_{k+1}-a_k) \ge a_1\sum_{\lambda\in\Lambda}\sum_{k\in T_\lambda}\frac{1}{k} \ge a_1\sum_{k\ge 1}\frac{1}{k},$$

contradicting the fact that the harmonic series diverges.

¶ Solution 2 (Sanjana Das). Assume for the sake of contradiction that no number appears exactly once in the sequence. For every $i < j$ with $a_i/i = a_j/j$, draw an edge between i and j, so every i has an edge (and being connected by an edge is a transitive property). Call i *good* if it has an edge with some $j > i$.

First, each i has finite degree – otherwise

$$\frac{a_{x_1}}{x_1} = \frac{a_{x_2}}{x_2} = \cdots$$

for an infinite increasing sequence of positive integers x_i, but then the a_{x_i} are unbounded.

Now we use the following process to build a sequence of indices whose a_i we can lower-bound:

- Start at $x_1 = 1$, which is good.
- If we're currently at good index x_i, then let s_i be the largest positive integer such that x_i has an edge to $x_i + s_i$. (This exists because the degrees are finite.)
- Let t_i be the smallest positive integer for which $x_i + s_i + t_i$ is good, and let this be x_{i+1}. This exists because if all numbers $k \le x \le 2k$ are bad, they must each connect to some number less than k (if two connect to each other, the smaller one is good), but then two connect to the same number, and therefore to each other – this is the idea we will use later to bound the t_i as well.

Then $x_i = 1 + s_1 + t_1 + \cdots + s_{i-1} + t_{i-1}$, and we have

$$a_{x_{i+1}} > a_{x_i+s_i} = \frac{x_i + s_i}{x_i} a_{x_i} = \frac{1 + (s_1 + \cdots + s_{i-1} + s_i) + (t_1 + \cdots + t_{i-1})}{1 + (s_1 + \cdots + s_{i-1}) + (t_1 + \cdots + t_{i-1})} a_{x_i}.$$

This means

$$c_n := \frac{a_{x_n}}{a_1} > \prod_{i=1}^{n-1} \frac{1 + (s_1 + \cdots + s_{i-1} + s_i) + (t_1 + \cdots + t_{i-1})}{1 + (s_1 + \cdots + s_{i-1}) + (t_1 + \cdots + t_{i-1})}.$$

Lemma

$t_1 + \cdots + t_n \le s_1 + \cdots + s_n$ for each n.

Proof. Consider $1 \le i \le n$. Note that for every i, the $t_i - 1$ integers strictly between $x_i + s_i$ and $x_i + s_i + t_i$ are all bad, so each such index x must have an edge to some $y < x$.

First we claim that if $x \in (x_i + s_i, x_i + s_i + t_i)$, then x cannot have an edge to x_j for any $j \le i$. This is because $x > x_i + s_i \ge x_j + s_j$, contradicting the fact that $x_j + s_j$ is the largest neighbor of x_j.

This also means x doesn't have an edge to $x_j + s_j$ for any $j \le i$, since if it did, it would have an edge to x_j.

Second, no two bad values of x can have an edge, since then the smaller one is good. This also means no two bad x can have an edge to the same y.

Then each of the $\sum(t_i - 1)$ values in the intervals $(x_i + s_i, x_i + s_i + t_i)$ for $1 \le i \le n$ must have an edge to an unique y in one of the intervals $(x_i, x_i + s_i)$ (not necessarily with the same i). Therefore

$$\sum(t_i - 1) \le \sum(s_i - 1) \implies \sum t_i \le \sum s_i. \qquad \square$$

Now note that if $a > b$, then $\frac{a+x}{b+x} = 1 + \frac{a-b}{b+x}$ is decreasing in x. This means

$$c_n > \prod_{i=1}^{n-1} \frac{1+2s_1+\dots+2s_{i-1}+s_i}{1+2s_1+\dots+2s_{i-1}} > \prod_{i=1}^{n-1} \frac{1+2s_1+\dots+2s_{i-1}+2s_i}{1+2s_1+\dots+2s_{i-1}+s_i},$$

By multiplying both products, we have a telescoping product, which results in

$$c_n^2 \geq 1 + 2s_1 + \dots + 2s_n + 2s_{n+1}.$$

The right hand side is unbounded since the s_i are positive integers, while $c_n = a_{x_n}/a_1 < 1/a_1$ is bounded, contradiction.

¶ Solution 3 (Gopal Goel). Suppose for sake of contradiction that the problem is false. Call an index i a *pin* if

$$\frac{a_j}{j} = \frac{a_i}{i} \implies j \geq i.$$

Lemma

There exists k such that if we have $\frac{a_i}{i} = \frac{a_j}{j}$ with $j > i \geq k$, then $j \leq 1.1i$.

Proof. Note that for any i, there are only finitely many j with $\frac{a_j}{j} = \frac{a_i}{i}$, otherwise $a_j = \frac{ja_i}{i}$ is unbounded. Thus it suffices to find k for which $j \leq 1.1i$ when $j > i \geq k$.

Suppose no such k exists. Then, take a pair $j_1 > i_1$ such that $\frac{a_{j_1}}{j_1} = \frac{a_{i_1}}{i_1}$ and $j_1 > 1.1i_1$, or $a_{j_1} > 1.1a_{i_1}$. Now, since $k = j_1$ can't work, there exists a pair $j_2 > i_2 \geq i_1$ such that $\frac{a_{j_2}}{j_2} = \frac{a_{i_2}}{i_2}$ and $j_2 > 1.1i_2$, or $a_{j_2} > 1.1a_{i_2}$. Continuing in this fashion, we see that

$$a_{j_\ell} > 1.1a_{i_\ell} > 1.1a_{j_{\ell-1}},$$

so we have that $a_{j_\ell} > 1.1^\ell a_{i_1}$. Taking $\ell > \log_{1.1}(1/a_1)$ gives the desired contradiction. □

Lemma

For $N > k^2$, there are at most $0.8N$ pins in $[\sqrt{N}, N)$.

Proof. By the first lemma, we see that the number of pins in $[\sqrt{N}, \frac{N}{1.1})$ is at most the number of non-pins in $[\sqrt{N}, N)$. Therefore, if the number of pins in $[\sqrt{N}, N)$ is p, then we have

$$p - N\left(1 - \frac{1}{1.1}\right) \leq N - p,$$

so $p \leq 0.8N$, as desired. □

We say that i is the pin of j if it is the smallest index such that $\frac{a_i}{i} = \frac{a_j}{j}$. The pin of j is always a pin.

Given an index i, let $f(i)$ denote the largest index less than i that is not a pin (we leave the function undefined when no such index exists, as we are only interested in the behavior for large i). Then f is weakly increasing and unbounded by the first lemma. Let N_0 be a positive integer such that $f(\sqrt{N_0}) > k$.

Take any $N > N_0$ such that N is not a pin. Let $b_0 = N$, and b_1 be the pin of b_0. Recursively define $b_{2i} = f(b_{2i-1})$, and b_{2i+1} to be the pin of b_{2i}.

Let ℓ be the largest odd index such that $b_\ell \geq \sqrt{N}$. We first show that $b_\ell \leq 100\sqrt{N}$. Since $N > N_0$, we have $b_{\ell+1} > k$. By the choice of ℓ we have $b_{\ell+2} < \sqrt{N}$, so

$$b_{\ell+1} < 1.1 b_{\ell+2} < 1.1\sqrt{N}$$

by the first lemma. We see that all the indices from $b_{\ell+1}+1$ to b_ℓ must be pins, so we have at least $b_\ell - 1.1\sqrt{N}$ pins in $[\sqrt{N}, b_\ell)$. Combined with the second lemma, this shows that $b_\ell \leq 100\sqrt{N}$.

Now, we have that $a_{b_{2i}} = \frac{b_{2i}}{b_{2i+1}} a_{b_{2i+1}}$ and $a_{b_{2i+1}} > a_{b_{2i+2}}$, so combining gives us

$$\frac{a_{b_0}}{a_{b_\ell}} > \frac{b_0}{b_1}\frac{b_2}{b_3}\cdots\frac{b_{\ell-1}}{b_\ell}.$$

Note that there are at least

$$(b_1 - b_2) + (b_3 - b_4) + \cdots + (b_{\ell-2} - b_{\ell-1})$$

pins in $[\sqrt{N}, N)$, so by the second lemma, that sum is at most $0.8N$. Thus,

$$\begin{aligned}(b_0 - b_1) + (b_2 - b_3) + \cdots + (b_{\ell-1} - b_\ell) &= b_0 - [(b_1 - b_2) + \cdots + (b_{\ell-2} - b_{\ell-1})] - b_\ell \\ &\geq 0.2N - 100\sqrt{N}.\end{aligned}$$

Then

$$\begin{aligned}\frac{b_0}{b_1}\frac{b_2}{b_3}\cdots\frac{b_{\ell-1}}{b_\ell} &\geq 1 + \frac{b_0 - b_1}{b_1} + \cdots + \frac{b_{\ell-1} - b_\ell}{b_\ell} \\ &> 1 + \frac{b_0 - b_1}{b_0} + \cdots + \frac{b_{\ell-1} - b_\ell}{b_0} \\ &\geq 1 + \frac{0.2N - 100\sqrt{N}}{N},\end{aligned}$$

which is at least 1.01 if N_0 is large enough. Thus, we see that

$$a_N > 1.01 a_{b_\ell} \geq 1.01 a_{\lfloor\sqrt{N}\rfloor}$$

if $N > N_0$ is not a pin. Since there are arbitrarily large non-pins, this implies that the sequence (a_n) is unbounded, which is the desired contradiction.

§1.3 TSTST 2021/3, proposed by Merlijn Staps

Available online at https://aops.com/community/p23586679.

Problem statement

Find all positive integers $k > 1$ for which there exists a positive integer n such that $\binom{n}{k}$ is divisible by n, and $\binom{n}{m}$ is not divisible by n for $2 \le m < k$.

Such an n exists for any k.

First, suppose k is prime. We choose $n = (k-1)!$. For $m < k$, it follows from $m! \mid n$ that

$$\begin{aligned} (n-1)(n-2)\cdots(n-m+1) &\equiv (-1)(-2)\cdots(-m+1) \\ &\equiv (-1)^{m-1}(m-1)! \\ &\not\equiv 0 \pmod{m!}. \end{aligned}$$

We see that $\binom{n}{m}$ is not a multiple of m. For $m = k$, note that $\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$. Because $k \nmid n$, we must have $k \mid \binom{n-1}{k-1}$, and it follows that $n \mid \binom{n}{k}$.

Now suppose k is composite. We will choose n to satisfy a number of congruence relations. For each prime $p \le k$, let

$$t_p = \nu_p(\operatorname{lcm}(1, 2, \dots, k-1)) = \max(\nu_p(1), \nu_p(2), \dots, \nu_p(k-1))$$

and choose $k_p \in \{1, 2, \dots, k-1\}$ as large as possible such that $\nu_p(k_p) = t_p$. We now require

$$\begin{aligned} n &\equiv 0 \pmod{p^{t_p+1}} && \text{if } p \nmid k; && (1) \\ \nu_p(n-k_p) &= t_p + \nu_p(k) && \text{if } p \mid k. && (2) \end{aligned}$$

for all $p \le k$. From the Chinese Remainder Theorem, we know that an n exists that satisfies (1) and (2) (indeed, a sufficient condition for (2) is the congruence $n \equiv k_p + p^{t_p+\nu_p(k)} \pmod{p^{t_p+\nu_p(k)+1}}$). We show that this n has the required property.

We first will compute $\nu_p(n-i)$ for primes $p < k$ and $1 \le i < k$.

- If $p \nmid k$, then we have $\nu_p(i), \nu_p(n-i) \le t_p$ and $\nu_p(n) > t_p$, so $\nu_p(n-i) = \nu_p(i)$;
- If $p \mid k$ and $i \neq k_p$, then we have $\nu_p(i), \nu_p(n-i) \le t_p$ and $\nu_p(n) \ge t_p$, so again $\nu_p(n-i) = \nu_p(i)$;
- If $p \mid k$ and $i = k_p$, then we have $\nu_p(n-i) = \nu_p(i) + \nu_p(k)$ by (2).

We conclude that $\nu_p(n-i) = \nu_p(i)$ always holds, except when $i = k_p$, when we have $\nu_p(n-i) = \nu_p(i) + \nu_p(k)$ (this formula holds irrespective of whether $p \mid k$ or $p \nmid k$).

We can now show that $\binom{n}{k}$ is divisible by n, which amounts to showing that $k!$ divides $(n-1)(n-2)\cdots(n-k+1)$. Indeed, for each prime $p \le k$ we have

$$\begin{aligned} \nu_p((n-1)(n-2)\dots(n-k+1)) &= \nu_p(n-k_p) + \sum_{i<k, i \neq k_p} \nu_p(n-i) \\ &= \nu_p(k_p) + \nu_p(k) + \sum_{i<k, i\neq k_p} \nu_p(i) \end{aligned}$$

$$= \sum_{i=1}^{k} \nu_p(i) = \nu_p(k!),$$

so it follows that $(n-1)(n-2)\cdots(n-k+1)$ is a multiple of $k!$.

Finally, let $1 < m < k$. We will show that n does *not* divide $\binom{n}{m}$, which amounts to showing that $m!$ does not divide $(n-1)(n-2)\cdots(n-m+1)$. First, suppose that m has a prime divisor q that does not divide k. Then we have

$$\begin{aligned}
\nu_q\left((n-1)(n-2)\dots(n-m+1)\right) &= \sum_{i=1}^{m-1} \nu_q(n-i) \\
&= \sum_{i=1}^{m-1} \nu_q(i) \\
&= \nu_q((m-1)!) < \nu_q(m!),
\end{aligned}$$

as desired. Therefore, suppose that m is only divisible by primes that divide k. If there is such a prime p with $\nu_p(m) > \nu_p(k)$, then it follows that

$$\begin{aligned}
\nu_p\left((n-1)(n-2)\dots(n-m+1)\right) &= \nu_p(k) + \sum_{i=1}^{m-1} \nu_p(i) \\
&< \nu_p(m) + \sum_{i=1}^{m-1} \nu_p(i) \\
&= \nu_p(m!),
\end{aligned}$$

so $m!$ cannot divide $(n-1)(n-2)\dots(n-m+1)$. On the other hand, suppose that $\nu_p(m) \le \nu_p(k)$ for all $p \mid k$, which would mean that $m \mid k$ and hence $m \le \frac{k}{2}$. Consider a prime p dividing m. We have $k_p \ge \frac{k}{2}$, because otherwise $2k_p$ could have been used instead of k_p. It follows that $m \le \frac{k}{2} \le k_p$. Therefore, we obtain

$$\begin{aligned}
\nu_p\left((n-1)(n-2)\dots(n-m+1)\right) &= \sum_{i=1}^{m-1} \nu_p(n-i) \\
&= \sum_{i=1}^{m-1} \nu_p(i) \\
&= \nu_p((m-1)!) < \nu_p(m!),
\end{aligned}$$

showing that $(n-1)(n-2)\cdots(n-m+1)$ is not divisible by $m!$. This shows that $\binom{n}{m}$ is not divisible by n for $m < k$, and hence n does have the required property.

§2 Solutions to Day 2

§2.1 TSTST 2021/4, proposed by Holden Mui

Available online at https://aops.com/community/p23864177.

Problem statement

Let a and b be positive integers. Suppose that there are infinitely many pairs of positive integers (m,n) for which m^2+an+b and n^2+am+b are both perfect squares. Prove that a divides $2b$.

Treating a and b as fixed, we are given that there are infinitely many quadrpules (m,n,r,s) which satisfy the system

$$m^2+an+b=(m+r)^2$$
$$n^2+am+b=(n+s)^2$$

We say that (r,s) is *exceptional* if there exists infinitely many (m,n) that satisfy.

Claim — If (r,s) is exceptional, then either

- $0<r<a/2$, and $0<s<\frac{1}{4}a^2$; or
- $0<s<a/2$, and $0<r<\frac{1}{4}a^2$; or
- $r^2+s^2\le 2b$.

In particular, finitely many pairs (r,s) can be exceptional.

Proof. Sum the two equations to get:

$$r^2+s^2-2b=(a-2r)m+(a-2s)n. \qquad (\dagger)$$

If $0<r<a/2$, then the idea is to use the bound $an+b\ge 2m+1$ to get $m\le \frac{an+b-1}{2}$. Consequently,

$$(n+s)^2=n^2+am+b\le n^2+a\cdot\frac{an+b-1}{2}+b$$

For this to hold for infinitely many integers n, we need $2s\le\frac{a^2}{2}$, by comparing coefficients.

A similar case occurs when $0<s<a/2$.

If $\min(r,s)>a/2$, then $(\dagger)$ forces $r^2+s^2\le 2b$, giving the last case. $\square$

Hence, there exists some particular pair (r,s) for which there are infinitely many solutions (m,n). Simplifying the system gives

$$an=2rm+r^2-b$$
$$2sn=am+b-s^2$$

Since the system is linear, for there to be infinitely many solutions (m,n) the system must be dependent. This gives

$$\frac{a}{2s}=\frac{2r}{a}=\frac{r^2-b}{b-s^2}$$

so $a = 2\sqrt{rs}$ and $b = \frac{s^2\sqrt{r}+r^2\sqrt{s}}{\sqrt{r}+\sqrt{s}}$. Since rs must be square, we can reparametrize as $r = kx^2$, $s = ky^2$, and $\gcd(x, y) = 1$. This gives

$$\begin{aligned} a &= 2kxy \\ b &= k^2xy(x^2 - xy + y^2). \end{aligned}$$

Thus, $a \mid 2b$, as desired.

§2.2 TSTST 2021/5, proposed by Vincent Huang

Available online at https://aops.com/community/p23864182.

Problem statement

Let T be a tree on n vertices with exactly k leaves. Suppose that there exists a subset of at least $\frac{n+k-1}{2}$ vertices of T, no two of which are adjacent. Show that the longest path in T contains an even number of edges.

The longest path in T must go between two leaves. The solutions presented here will solve the problem by showing that in the unique 2-coloring of T, all leaves are the same color.

¶ Solution 1 (Ankan Bhattacharya, Jeffery Li).

Lemma

If S is an independent set of T, then

$$\sum_{v \in S} \deg(v) \le n-1.$$

Equality holds if and only if S is one of the two components of the unique 2-coloring of T.

Proof. Each edge of T is incident to at most one vertex of S, so we obtain the inequality by counting how many vertices of S each edge is incident to. For equality to hold, each edge is incident to exactly one vertex of S, which implies the 2-coloring. □

We are given that there exists an independent set of at least $\frac{n+k-1}{2}$ vertices. By greedily choosing vertices of smallest degree, the sum of the degrees of these vertices is at least

$$k + 2 \cdot \frac{n-k-1}{2} = n-1.$$

Thus equality holds everywhere, which implies that the independent set contains every leaf and is one of the components of the 2-coloring.

¶ Solution 2 (Andrew Gu).

Lemma

The vertices of T can be partitioned into $k-1$ paths (i.e. the induced subgraph on each set of vertices is a path) such that all edges of T which are not part of a path are incident to an endpoint of a path.

Proof. Repeatedly trim the tree by taking a leaf and removing the longest path containing that leaf such that the remaining graph is still a tree. □

Now given a path of a vertices, at most $\frac{a+1}{2}$ of those vertices can be in an independent set of T. By the lemma, T can be partitioned into $k-1$ paths of $a_1, \dots, a_{k-1}$ vertices, so the maximum size of an independent set of T is

$$\sum \frac{a_i + 1}{2} = \frac{n + k - 1}{2}.$$

For equality to hold, each path in the partition must have an odd number of vertices, and has a unique 2-coloring in red and blue where the endpoints are red. The unique independent set of T of size $\frac{n+k-1}{2}$ is then the set of red vertices. By the lemma, the edges of T which are not part of a path connect an endpoint of a path (which is colored red) to another vertex (which must be blue, because the red vertices are independent). Thus the coloring of the paths extends to the unique 2-coloring of T. The leaves of T are endpoints of paths, so they are all red.

§2.3 TSTST 2021/6, proposed by Nikolai Beluhov

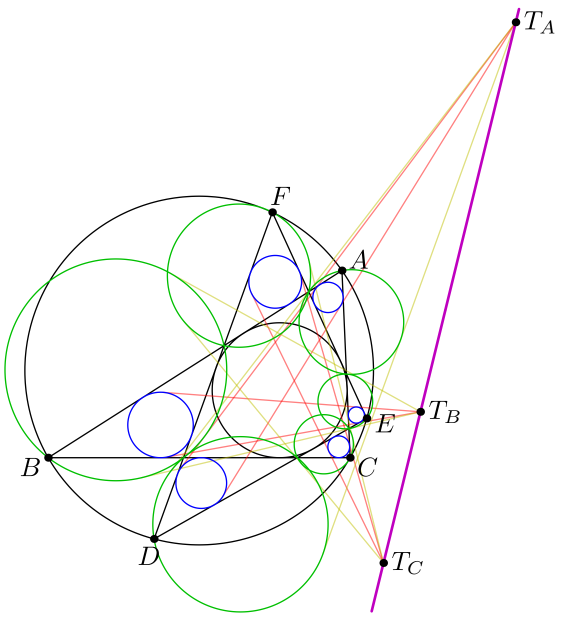
Available online at https://aops.com/community/p23864189.

Problem statement

Triangles ABC and DEF share circumcircle Ω and incircle ω so that points A, F, B, D, C, and E occur in this order along Ω. Let Δ_A be the triangle formed by lines AB, AC, and EF, and define triangles $\Delta_B, \Delta_C, \dots, \Delta_F$ similarly. Furthermore, let Ω_A and ω_A be the circumcircle and incircle of triangle Δ_A, respectively, and define circles $\Omega_B, \omega_B, \dots, \Omega_F, \omega_F$ similarly.

(a) Prove that the two common external tangents to circles Ω_A and Ω_D and the two common external tangents to circles ω_A and ω_D are either concurrent or pairwise parallel.

(b) Suppose that these four lines meet at point T_A, and define points T_B and T_C similarly. Prove that points T_A, T_B, and T_C are collinear.



Let I and r be the center and radius of ω, and let O and R be the center and radius of Ω. Let O_A and I_A be the circumcenter and incenter of triangle Δ_A, and define O_B, $I_B, \dots, I_F$ similarly. Let ω touch EF at A_1, and define $B_1, C_1, \dots, F_1$ similarly.

¶ Part (a). All solutions to part (a) will prove the stronger claim that

$$(\Omega_A \cup \omega_A) \sim (\Omega_D \cup \omega_D).$$

The four lines will concur at the homothetic center of these figures (possibly at infinity).

Solution 1 (author) Let the second tangent to ω parallel to EF meet lines AB and AC at P and Q, respectively, and let the second tangent to ω parallel to BC meet lines DE and DF at R and S, respectively. Furthermore, let ω touch PQ and RS at U and V, respectively.

Let h be inversion with respect to ω. Then h maps A, B, and C onto the midpoints of the sides of triangle $D_1E_1F_1$. So h maps k onto the Euler circle of triangle $D_1E_1F_1$.

Similarly, h maps k onto the Euler circle of triangle $A_1B_1C_1$. Therefore, triangles $A_1B_1C_1$ and $D_1E_1F_1$ share a common nine-point circle γ. Let K be its center; its radius equals $\frac{1}{2}r$.

Let H be the reflection of I in K. Then H is the common orthocenter of triangles $A_1B_1C_1$ and $D_1E_1F_1$.

Let γ_U of center K_U and radius $\frac{1}{2}r$ be the Euler circle of triangle UE_1F_1, and let γ_V of center K_V and radius $\frac{1}{2}r$ be the Euler circle of triangle VB_1C_1.

Let H_U be the orthocenter of triangle UE_1F_1. Since quadrilateral $D_1E_1F_1U$ is cyclic, vectors $\overrightarrow{HH_U}$ and $\overrightarrow{D_1U}$ are equal. Consequently, $\overrightarrow{KK_U} = \frac{1}{2}\overrightarrow{D_1U}$. Similarly, $\overrightarrow{KK_V} = \frac{1}{2}\overrightarrow{A_1V}$.

Since both of A_1U and D_1V are diameters in ω, vectors $\overrightarrow{D_1U}$ and $\overrightarrow{A_1V}$ are equal. Therefore, K_U and K_V coincide, and so do γ_U and γ_V.

As above, h maps γ_U onto the circumcircle of triangle APQ and γ_V onto the circumcircle of triangle DRS. Therefore, triangles APQ and DRS are inscribed inside the same circle Ω_{AD}.

Since EF and PQ are parallel, triangles Δ_A and APQ are homothetic, and so are figures $\Omega_A \cup \omega_A$ and $\Omega_{AD} \cup \omega$. Consequently, we have

$$(\Omega_A \cup \omega_A) \sim (\Omega_{AD} \cup \omega) \sim (\Omega_D \cup \omega_D),$$

which solves part (a).

Solution 2 (Michael Ren) As in the previous solution, let the second tangent to ω parallel to EF meet AB and AC at P and Q, respectively. Let (APQ) meet Ω again at D', so that D' is the Miquel point of $\{AB, AC, BC, PQ\}$. Since the quadrilateral formed by these lines has incircle ω, it is classical that $D'I$ bisects $\angle PD'C$ and $BD'Q$ (e.g. by DDIT).

Let ℓ be the tangent to Ω at D' and $D'I$ meet Ω again at M. We have

$$\measuredangle(\ell, D'B) = \measuredangle D'CB = \measuredangle D'QP = \measuredangle(D'Q, EF).$$

Therefore $D'I$ also bisects the angle between ℓ and the line parallel to EF through D'. This means that M is one of the arc midpoints of EF. Additionally, D' lies on arc BC not containing A, so $D' = D$.

Similarly, letting the second tangent to ω parallel to BC meet DE and DF again at R and S, we have $ADRS$ cyclic.

Lemma

There exists a circle Ω_{AD} tangent to Ω_A and Ω_D at A and D, respectively.

Proof. (This step is due to Ankan Bhattacharya.) It is equivalent to have $\measuredangle OAO_A = \measuredangle O_DDO$. Taking isogonals with respect to the shared angle of $\triangle ABC$ and Δ_A, we see that

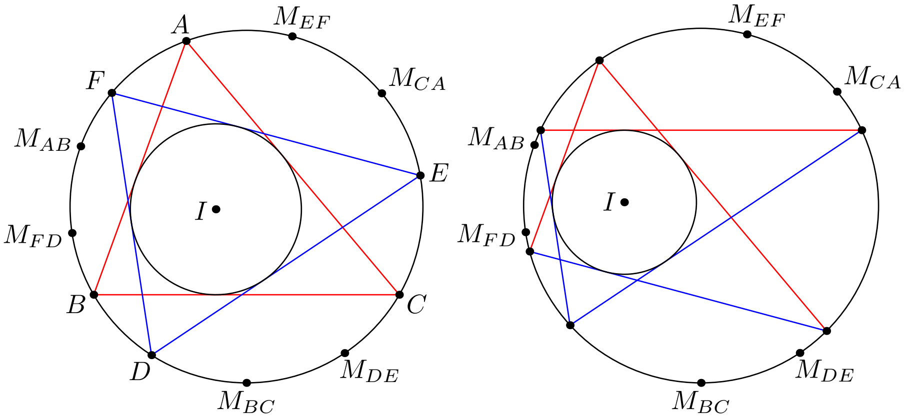
$$\measuredangle OAO_A = \measuredangle(\perp EF, \perp BC) = \measuredangle(EF, BC).$$

(Here, $\perp EF$ means the direction perpendicular to EF.) By symmetry, this is equal to $\measuredangle O_DDO$. $\square$

The circle $(ADPQ)$ passes through A and D, and is tangent to Ω_A by homothety. Therefore it coincides with Ω_{AD}, as does $(ADRS)$. Like the previous solution, we finish with

$$(\Omega_A \cup \omega_A) \sim (\Omega_{AD} \cup \omega) \sim (\Omega_D \cup \omega_D).$$

Solution 3 (Andrew Gu) Construct triangles homothetic to Δ_A and Δ_D (with positive ratio) which have the same circumcircle; it suffices to show that these copies have the same incircle as well. Let O be the center of this common circumcircle, which we take to be the origin, and M_{XY} denote the point on the circle such that the tangent at that point is parallel to line XY (from the two possible choices, we make the choice that corresponds to the arc midpoint on Ω, e.g. M_{AB} should correspond to the arc midpoint on the internal angle bisector of ACB). The left diagram below shows the original triangles ABC and DEF, while the right diagram shows the triangles homothetic to Δ_A and Δ_D.



Using the fact that the incenter is the orthocenter of the arc midpoints, the incenter of Δ_A in this reference frame is $M_{AB} + M_{AC} - M_{EF}$ and the incenter of Δ_D in this reference frame is $M_{DE} + M_{DF} - M_{BC}$. Since ABC and DEF share a common incenter, we have

$$M_{AB} + M_{BC} + M_{CA} = M_{DE} + M_{EF} + M_{FD}.$$

Thus the copies of Δ_A and Δ_D have the same incenter, and therefore the same incircle as well (Euler's theorem determines the inradius).

¶ Part (b). We present several solutions for this part of the problem. Solutions 3 and 4 require solving part (a) first, while the others do not. Solutions 1, 4, and 5 define T_A solely as the exsimilicenter of ω_A and ω_D, whereas solutions 2 and 3 define T_A solely as the exsimilicenter of Ω_A and Ω_D.

Solution 1 (author) By Monge's theorem applied to ω, ω_A, and ω_D, points A, D, and T_A are collinear. Therefore, $T_A = AD \cap I_A I_D$.

Let p be pole-and-polar correspondence with respect to ω. Then p maps A onto line E_1F_1 and D onto line B_1C_1. Consequently, p maps line AD onto $X_A = B_1C_1 \cap E_1F_1$.

Furthermore, p maps the line that bisects the angle formed by lines AB and EF and does not contain I onto the midpoint of segment A_1F_1. Similarly, p maps the line that bisects the angle formed by lines AC and EF and does not contain I onto the midpoint of segment A_1E_1. So p maps I_A onto the midline of triangle $A_1E_1F_1$ opposite A_1. Similarly, p maps I_D onto the midline of triangle $D_1B_1C_1$ opposite D_1. Consequently, p maps line

$I_A I_D$ onto the intersection point Y_A of this pair of midlines, and p maps T_A onto line $X_A Y_A$.

As in the solution to part (a), let H be the common orthocenter of triangles $A_1B_1C_1$ and $D_1E_1F_1$. Let H_A be the foot of the altitude from A_1 in triangle $A_1B_1C_1$ and let H_D be the foot of the altitude from D_1 in triangle $D_1E_1F_1$. Furthermore, let $L_A = HA_1 \cap E_1F_1$ and $L_D = HD_1 \cap B_1C_1$.

Since the reflection of H in line B_1C_1 lies on ω, $A_1H \cdot HH_A$ equals half the power of H with respect to ω. Similarly, $D_1H \cdot HH_D$ equals half the power of H with respect to ω.

Then $A_1H \cdot HH_A = D_1H \cdot HH_D$ and $A_1HH_D \sim D_1HH_A$. Since $\angle HH_DL_A = 90^\circ = \angle HH_AL_D$, figures $A_1HH_DL_A$ and $D_1HH_AL_D$ are similar as well. Consequently,

$$\frac{HL_A}{L_AA_1} = \frac{HL_D}{L_DD_1} = s$$

as a signed ratio.

Let the line through A_1 parallel to E_1F_1 and the line through D_1 parallel to B_1C_1 meet at Z_A. Then $HX_A/X_AZ_A = s$ and Y_A is the midpoint of segment X_AZ_A. Therefore, H lies on line X_AY_A. This means that T_A lies on the polar of H with respect to ω, and by symmetry so do T_B and T_C.

Solution 2 (author) As in the first solution to part (a), let h be inversion with respect to ω, let γ of center K be the common Euler circle of triangles $A_1B_1C_1$ and $D_1E_1F_1$, and let H be their common orthocenter.

Again as in the solution to part (a), h maps Ω_A onto the nine-point circle γ_A of triangle $A_1E_1F_1$ and Ω_D onto the nine-point circle γ_D of triangle $D_1B_1C_1$.

Let K_A and K_D be the centers of γ_A and γ_D, respectively, and let ℓ_A be the perpendicular bisector of segment K_AK_D. Since γ_A and γ_D are congruent (both of them are of radius $\frac{1}{2}r$), they are reflections of each other across ℓ_A.

Inversion h maps the two common external tangents of Ω_A and Ω_D onto the two circles α and β through I that are tangent to both of γ_A and γ_D in the same way. (That is, either internally to both or externally to both.) Consequently, α and β are reflections of each other in ℓ_A and so their second point of intersection S_A, which h maps T_A onto, is the reflection of I in ℓ_A.

Define ℓ_B, ℓ_C, S_B, and S_C similarly. Then S_B is the reflection of I in ℓ_B and S_C is the reflection of I in ℓ_C.

As in the solution to part (a), $\overrightarrow{KK_A} = \frac{1}{2}\overrightarrow{D_1A_1}$ and $\overrightarrow{KK_D} = \frac{1}{2}\overrightarrow{A_1D_1}$. Consequently, K is the midpoint of segment K_AK_D and so K lies on ℓ_A. Similarly, K lies on ℓ_B and ℓ_C.

Therefore, all four points I, S_A, S_B, and S_C lie on the circle of center K that contains I. (This is also the circle on diameter IH.) Since points S_A, S_B, and S_C are concyclic with I, their images T_A, T_B, and T_C under h are collinear, and the solution is complete.

Solution 3 (Ankan Bhattacharya) From either of the first two solutions to part (a), we know that there is a circle Ω_{AD} passing through A and D which is (internally) tangent to Ω_A and Ω_D. By Monge's theorem applied to Ω_A, Ω_D, and Ω_{AD}, it follows that A, D, and T_A are collinear.

The inversion at T_A swapping Ω_A with Ω_D also swaps A with D because T_A lies on AD and A is not homologous to D. Let Ω_A and Ω_D meet Ω again at L_A and L_D. Since ADL_AL_D is cyclic, the same inversion at T_A also swaps L_A and L_D, so $T_A = AD \cap L_AL_D$.

By Taiwan TST 2014, L_A and L_D are the tangency points of the A-mixtilinear and D-mixtilinear incircles, respectively, with Ω. Therefore $AL_A \cap DL_D$ is the exsimilicenter X of Ω and ω. Then T_A, T_B, and T_C all lie on the polar of X with respect to Ω.

Solution 4 (Andrew Gu) We show that T_A lies on the radical axis of the point circle at I and Ω, which will solve the problem. Let I_A and I_D be the centers of ω_A and ω_D respectively. By the Monge's theorem applied to ω, ω_A, and ω_D, points A, D, and T_A are collinear. Additionally, these other triples are collinear: $(A, I_A, I), (D, I_D, I), (I_A, I_D, T)$. By Menelaus's theorem, we have

$$\frac{T_A D}{T_A A} = \frac{I_A I}{I_A A} \cdot \frac{I_D D}{I_D I}.$$

If s is the length of the side opposite A in Δ_A, then we compute

$$\begin{aligned} \frac{I_A I}{I_A A} &= \frac{s/\cos(A/2)}{r_A/\sin(A/2)} \\ &= \frac{2R_A \sin(A)\sin(A/2)}{\cos(A/2)} \\ &= \frac{4R_A \sin^2(A/2)}{r_A} \\ &= \frac{4R_A r^2}{r_A AI^2}. \end{aligned}$$

From part (a), we know that $\frac{R_A}{r_A} = \frac{R_D}{r_D}$. Therefore, doing a similar calculation for $\frac{I_D D}{I_D I}$, we get

$$\begin{aligned} \frac{T_A D}{T_A A} &= \frac{I_A I}{I_A A} \cdot \frac{I_D D}{I_D I} \\ &= \frac{4R_A r^2}{r_A AI^2} \cdot \frac{r_D DI^2}{4R_D r^2} \\ &= \frac{DI^2}{AI^2}. \end{aligned}$$

Thus T_A is the point where the tangent to (AID) at I meets AD and $T_A I^2 = T_A A \cdot T_A D$. This shows what we claimed at the start.

Solution 5 (Ankit Bisain) As in the previous solution, it suffices to show that $\frac{I_A I}{AI_A} \cdot \frac{DI_D}{I_D I} = \frac{DI^2}{AI^2}$. Let AI and DI meet Ω again at M and N, respectively. Let ℓ be the line parallel to BC and tangent to ω but different from BC. Then

$$\frac{DI_D}{I_D I} = \frac{d(D, BC)}{d(BC, \ell)} = \frac{DB \cdot DC/2R}{2r} = \frac{MI^2 - MD^2}{4Rr}.$$

Since $IDM \sim IAN$, we have

$$\frac{DI_D}{I_D I} \cdot \frac{I_A I}{AI_A} = \frac{MI^2 - MD^2}{NI^2 - NA^2} = \frac{DI^2}{AI^2},$$

as desired.

> **Remark** (Author comments on generalization of part (b) with a circumscribed hexagram). Let triangles ABC and DEF be circumscribed about the same circle ω so that they form a hexagram. However, we do not require anymore that they are inscribed in the same circle.
>
> Define circles Ω_A, $\omega_A, \dots, \omega_F$ as in the problem. Let T_A^{Circ} be the intersection point of the two common external tangents to circles Ω_A and Ω_D, and define points T_B^{Circ} and T_C^{Circ} similarly. Also let T_A^{In} be the intersection point of the two common external tangents to circles ω_A and ω_D, and define points T_B^{In} and T_C^{In} similarly.
>
> Then points T_A^{Circ}, T_B^{Circ}, and T_C^{Circ} are collinear and points T_A^{In}, T_B^{In}, and T_C^{In} are also collinear.

The second solution to part (b) of the problem works also for the circumcircles part of the generalisation. To see that segments K_AK_D, K_BK_E, and K_CK_F still have a common midpoint, let M be the centroid of points A, B, C, D, E, and F. Then the midpoint of segment K_AK_D divides segment OM externally in ratio $3 : 1$, and so do the other two midpoints as well.

For the incircles part of the generalisation, we start out as in the first solution to part (b) of the problem, and eventually we reduce everything to the following:

Let points A_1, B_1, C_1, D_1, E_1, and F_1 lie on circle ω. Let lines B_1C_1 and E_1F_1 meet at point X_A, let the line through A_1 parallel to B_1C_1 and the line through D_1 parallel to E_1F_1 meet at point Z_A, and define points X_B, Z_B, X_C, and Z_C similarly. Then lines X_AZ_A, X_BZ_B, and X_CZ_C are concurrent.

Take ω as the unit circle and assign complex numbers u, v, w, x, y, and z to points A_1, F_1, B_1, D_1, C_1, and E_1, respectively, so that when we permute u, v, w, x, y, and z cyclically the configuration remains unchanged. Then by standard complex bash formulas we obtain that each two out of our three lines meet at φ/ψ, where

$$\varphi = \sum_{\text{Cyc}} u^2vw(wx - wy + xy)(y - z)$$

and

$$\psi = -u^2w^2y^2 - v^2x^2z^2 - 4uvwxyz + \sum_{\text{Cyc}} u^2(vwxy - vwxz + vwyz - vxyz + wxyz).$$

(But the calculations were too difficult for me to do by hand, so I used SymPy.)

Remark (Author comments on generalization of part (b) with an inscribed hexagram)**.** Let triangles ABC and DEF be inscribed inside the same circle Ω so that they form a hexagram. However, we do not require anymore that they are circumscribed about the same circle.

Define points T_A^{Circ}, T_B^{Circ}, $\dots$, T_C^{In} as in the previous remark. It looks like once again points T_A^{Circ}, T_B^{Circ}, and T_C^{Circ} are collinear and points T_A^{In}, T_B^{In}, and T_C^{In} are also collinear. However, I do not have proofs of these claims.

Remark (Further generalization from Andrew Gu)**.** Let ABC and DEF be triangles which share an inconic, or equivalently share a circumconic. Define points $T_A^{\text{Circ}}, T_B^{\text{Circ}}, \dots, T_C^{\text{In}}$ as in the previous remarks. Then it is conjectured that points T_A^{Circ}, T_B^{Circ}, and T_C^{Circ} are collinear and points T_A^{In}, T_B^{In}, and T_C^{In} are also collinear. (Note that extraversion may be required depending on the configuration of points, e.g. excircles instead of incircles.) Additionally, it appears that the insimilicenters of the circumcircles lie on a line perpendicular to the line through T_A^{Circ}, T_B^{Circ}, and T_C^{Circ}.

§3 Solutions to Day 3

§3.1 TSTST 2021/7, proposed by Ankit Bisain, Holden Mui

Available online at https://aops.com/community/p24130213.

Problem statement

Let M be a finite set of lattice points and n be a positive integer. A *mine-avoiding path* is a path of lattice points with length n, beginning at $(0,0)$ and ending at a point on the line $x+y=n$, that does not contain any point in M. Prove that if there exists a mine-avoiding path, then there exist at least $2^{n-|M|}$ mine-avoiding paths.

We present two approaches.

¶ Solution 1. We prove the statement by induction on n. We use $n=0$ as a base case, where the statement follows from $1 \geq 2^{-|M|}$. For the inductive step, let $n>0$. There exists at least one mine-avoiding path, which must pass through either $(0,1)$ or $(1,0)$. We consider two cases:

Case 1: there exist mine-avoiding paths starting at both $(1,0)$ and $(0,1)$.

By the inductive hypothesis, there are at least $2^{n-1-|M|}$ mine-avoiding paths starting from each of $(1,0)$ and $(0,1)$. Then the total number of mine-avoiding paths is at least $2^{n-1-|M|}+2^{n-1-|M|}=2^{n-|M|}$.

Case 2: only one of $(1,0)$ and $(0,1)$ is on a mine-avoiding path.

Without loss of generality, suppose no mine-avoiding path starts at $(0,1)$. Then some element of M must be of the form $(0,k)$ for $1 \leq k \leq n$ in order to block the path along the y-axis. This element can be ignored for any mine-avoiding path starting at $(1,0)$. By the inductive hypothesis, there are at least $2^{n-1-(|M|-1)}=2^{n-|M|}$ mine-avoiding paths.

This completes the induction step, which solves the problem.

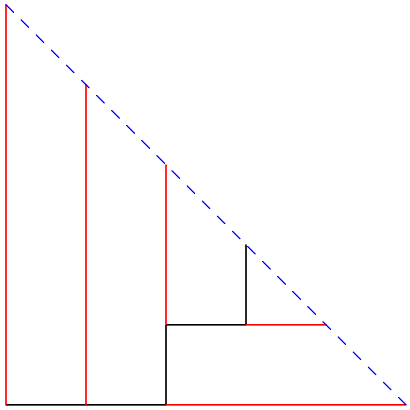
¶ Solution 2.

Lemma

If $|M|<n$, there is more than one mine-avoiding path.

Proof. Let $P_0, P_1, \dots, P_n$ be a mine-avoiding path. Set $P_i=(x_i,y_i)$. For $0 \leq i < n$, define a path Q_i as follows:

- Make the first $i+1$ points $P_0, P_1, \dots, P_i$.
- If $P_i \to P_{i+1}$ is one unit up, go right until $(n-y_i, y_i)$.
- If $P_i \to P_{i+1}$ is one unit right, go up until $(x_i, n-x_i)$.

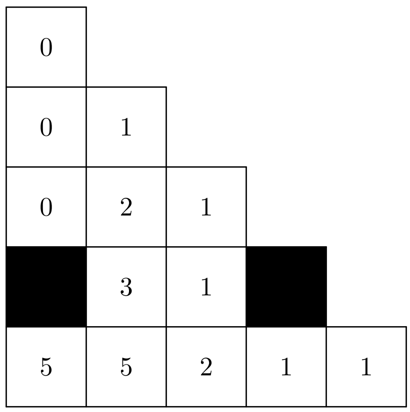


The diagram above is an example for $n = 5$ with the new segments formed by the Q_i in red, and the line $x + y = n$ in blue.

By definition, M has less than n points, none of which are in the original path. Since all Q_i only intersect in the original path, each mine is in at most one of $Q_0, Q_1, \dots, Q_{n-1}$. By the Pigeonhole Principle, one of the Q_i is mine-avoiding. $\square$

Now, consider the following process:

- Start at $(0, 0)$.
- If there is only one choice of next step that is part of a mine-avoiding path, make that choice.
- Repeat the above until at a point with two possible steps that are part of mine-avoiding paths.
- Add a mine to the choice of next step with more mine-avoiding paths through it. If both have the same number of mine-avoiding paths through them, add a mine arbitrarily.



For instance, consider the above diagram for $n = 4$. Lattice points are replaced with squares. Mines are black squares and each non-mine is labelled with the number of mine-avoiding paths passing through it. This process would start at $(0, 0)$, go to $(1, 0)$, then place a mine at $(1, 1)$.

This path increases the size of M by one, and reduces the number of mine-avoiding paths to a nonzero number at most half of the original. Repeat this process until there is only one path left. By our lemma, the number of mines must be at least n by the end of the process, so the process was iterated at least $n - |M|$ times. By the halving property, there must have been at least $2^{n-|M|}$ mine-avoiding paths before the process, as desired.

§3.2 TSTST 2021/8, proposed by Fedir Yudin

Available online at https://aops.com/community/p24130228.

Problem statement

Let ABC be a scalene triangle. Points A_1, B_1 and C_1 are chosen on segments BC, CA, and AB, respectively, such that $\triangle A_1B_1C_1$ and $\triangle ABC$ are similar. Let A_2 be the unique point on line B_1C_1 such that $AA_2 = A_1A_2$. Points B_2 and C_2 are defined similarly. Prove that $\triangle A_2B_2C_2$ and $\triangle ABC$ are similar.

We give three solutions.

¶ Solution 1 (author). We'll use the following lemma.

Lemma

Suppose that $PQRS$ is a convex quadrilateral with $\angle P = \angle R$. Then there is a point T on QS such that $\angle QPT = \angle SRP$, $\angle TRQ = \angle RPS$, and $PT = RT$.

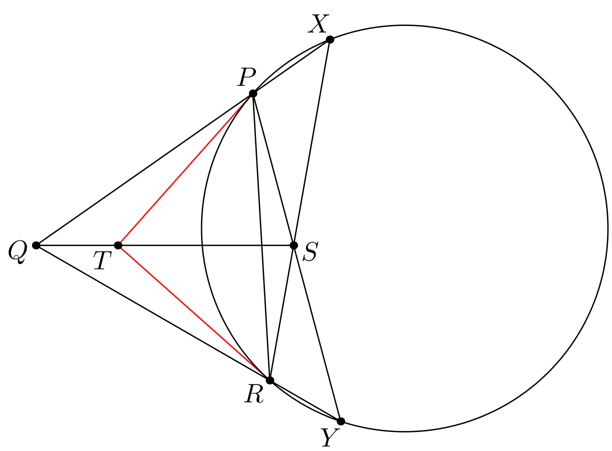
Before proving the lemma, we will show how it solves the problem. The lemma applied for the quadrilateral $AB_1A_1C_1$ with $\angle A = \angle A_1$ shows that $\angle B_1A_1A_2 = \angle C_1AA_1$. This implies that the point A_2 in $\triangle A_1B_1C_1$ corresponds to the point A_1 in $\triangle ABC$. Then $\triangle A_2B_2C_2 \sim \triangle A_1B_1C_1 \sim \triangle ABC$, as desired.

Additionally, $PT = RT$ is a corollary of the angle conditions because

$$\measuredangle PRT = \measuredangle SRQ - \measuredangle TRQ - \measuredangle SRP = \measuredangle QPS - \measuredangle RPS - \measuredangle QPT = \measuredangle TPR.$$

Therefore we only need to prove the angle conditions.

Proof 1 of lemma Denote $X = PQ \cap RS$ and $Y = PS \cap RQ$. Note that $\angle XPY = \angle XRY$, so $PRXY$ is cyclic. Let T be the point of intersection of tangents to this circle at P and R. By Pascal's theorem for the degenerate hexagon $PPXRRY$, we have $T \in QS$ (alternatively, Q, S, and T are collinear on the pole of $PR \cap XY$ with respect to the circle). Also, $\measuredangle QPT = \measuredangle XRP = \measuredangle SRP$ and similarly $\measuredangle TRQ = \measuredangle RPY = \measuredangle RPS$, so we're done.



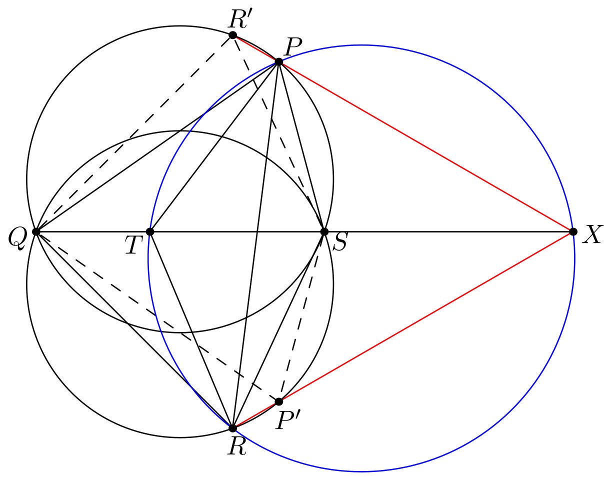
Proof 2 of lemma Let P' and R' be the reflections of P and R in QS. Note that PR' and RP' intersect at a point X on QS. Let T be the second intersection of the circumcircle of $\triangle PRX$ with QS. Note that

$$\begin{aligned}
\measuredangle PXT &= \measuredangle R'PQ + \measuredangle PQS \\
&= \measuredangle R'SQ + \measuredangle PQS \\
&= \measuredangle QSR + \measuredangle PQS \\
&= \measuredangle(PQ, SR) \\
&= \measuredangle QPR + \measuredangle PRS.
\end{aligned}$$

This means that

$$\begin{aligned}
\measuredangle QPT &= \measuredangle QPR - \measuredangle TPR \\
&= \measuredangle QPR - \measuredangle TXR \\
&= \measuredangle QPR - \measuredangle PXT \\
&= \measuredangle QPR - \measuredangle QPR - \measuredangle PRS \\
&= \measuredangle SRP.
\end{aligned}$$

Similarly, $\measuredangle QRT = \measuredangle SPR$, so we're done.



Proof 3 of lemma Let T be the point on QS such that $\angle QPT = \angle SRP$. Then we have

$$\frac{QT}{TS} = \frac{\sin QPT \cdot PT/\sin PQT}{\sin TPS \cdot PT/\sin TSP} = \frac{PQ/\sin PRQ}{PS/\sin SRP} = \frac{R(\triangle PQR)}{R(\triangle PRS)},$$

which is symmetric in P and R, so we're done.

¶ Solution 2 (Ankan Bhattacharya). We prove the main claim $\frac{B_1A_2}{A_2C_1} = \frac{BA_1}{A_1C}$.

Let $\triangle A_0B_0C_0$ be the medial triangle of $\triangle ABC$. In addition, let A_1' be the reflection of A_1 over $\overline{B_1C_1}$, and let X be the point satisfying $\triangle XBC \overset{-}{\sim} \triangle AB_1C_1$, so that we have a compound similarity

$$\triangle ABC \sqcup X \overset{-}{\sim} \triangle A_1'B_1C_1 \sqcup A.$$

Finally, let O_A be the circumcenter of $\triangle A_1'B_1C_1$, and let A_2^* be the point on $\overline{B_1C_1}$ satisfying $\frac{B_1A_2^*}{A_2^*C_1} = \frac{BA_1}{A_1C}$.

Recall that O is the Miquel point of $\triangle A_1B_1C_1$, as well as its orthocenter.

Claim — $\overline{AA_1'} \parallel \overline{BC}$.

Proof. We need to verify that the foot from A_1 to $\overline{B_1C_1}$ lies on the A-midline. This follows from the fact that O is both the Miquel point and the orthocenter. □

Claim — $\overline{AX} \parallel \overline{B_1C_1}$.

Proof. From the compound similarity,

$$\measuredangle(\overline{BC}, \overline{AX}) = \measuredangle(\overline{AA_1'}, \overline{B_1C_1}).$$

As $\overline{AA_1'} \parallel \overline{BC}$, we obtain $\overline{AX} \parallel \overline{B_1C_1}$. □

Claim — $\overline{AX} \perp \overline{A_1O}$.

Proof. Because O is the orthocenter of $\triangle A_1B_1C_1$. □

Claim — $\overline{AA_1'} \perp \overline{A_2^*O_A}$.

Proof. Follows from $\overline{AX} \perp \overline{A_1O}$ after the similarity

$$\triangle ABC \sqcup X \overset{-}{\sim} \triangle A_1'B_1C_1 \sqcup A.$$ □

Claim — $AA_2^* = A_1'A_2$.

Proof. Since $\measuredangle C_1AB_1 = \measuredangle C_1A_1'B_1$, $AO_A = A_1'O_A$, so $\overline{AA_1'} \perp \overline{A_2^*O_A}$ implies $AA_2^* = A_1'A_2^*$. □

Finally, $A_1'A_2^* = A_1A_2^*$ by reflections, so $AA_2^* = A_1A_2^*$, and $A_2^* = A_2$.

§3.3 TSTST 2021/9, proposed by Victor Wang

Available online at https://aops.com/community/p24130243.

Problem statement

Let $q = p^r$ for a prime number p and positive integer r. Let $\zeta = e^{\frac{2\pi i}{q}}$. Find the least positive integer n such that

$$\sum_{\substack{1 \le k \le q \\ \gcd(k,p)=1}} \frac{1}{(1-\zeta^k)^n}$$

is not an integer. (The sum is over all $1 \le k \le q$ with p not dividing k.)

Let S_q denote the set of primitive qth roots of unity (thus, the sum in question is a sum over S_q).

¶ Solution 1 (author). Let $\zeta_p = e^{2\pi i/p}$ be a fixed primitive pth root of unity. Observe that the given sum is an integer for all $n \le 0$ (e.g. because the sum is an integer symmetric polynomial in the primitive qth roots of unity). By expanding polynomials in the basis $(1-x)^k$, it follows that if the sum in the problem statement is an integer for all $n < n_0$, then

$$\sum_{\omega \in S_q} \frac{f(\omega)}{(1-\omega)^n} \in \mathbb{Z}$$

for all $n < n_0$ and $f \in \mathbb{Z}[x]$, whereas for $n = n_0$ there is some $f \in \mathbb{Z}[x]$ for which the sum is not an integer (e.g. $f = 1$).

Let $z_q = r\phi(q) - q/p = p^{r-1}[r(p-1)-1]$. We claim that the answer is $n = z_q + 1$. We prove this by induction on r. First is the base case $r = 1$.

Lemma

There exist polynomials $u, v \in \mathbb{Z}[x]$ such that $(1-\omega)^{p-1}/p = u(\omega)$ and $p/(1-\omega)^{p-1} = v(\omega)$ for all $\omega \in S_p$.

(What we are saying is that p is $(1-\omega)^{p-1}$ times a *unit* (invertible algebraic integer), namely $v(\omega)$.)

Proof. Note that $p = (1-\omega)\cdots(1-\omega^{p-1})$. Thus we can write

$$\frac{p}{(1-\omega)^{p-1}} = \frac{1-\omega}{1-\omega} \cdot \frac{1-\omega^2}{1-\omega} \cdots \frac{1-\omega^{p-1}}{1-\omega}$$

and take

$$v(x) = \prod_{k=1}^{p-1} \frac{1-x^k}{1-x}.$$

Similarly, the polynomial u is

$$u(x) = \prod_{k=1}^{p-1} \frac{1-x^{k\ell_k}}{1-x^k}$$

where ℓ_k is a multiplicative inverse of k modulo p. □

Now, the main idea: given $g \in \mathbb{Z}[x]$, observe that

$$S = \sum_{\omega \in S_p} (1-\omega) g(\omega)$$

is divisible by $1-\zeta_p^k$ (i.e. it is $1-\zeta_p^k$ times an algebraic integer) for every k coprime to p. By symmetric sums, S is an integer; since S^{p-1} is divisible by $(1-\zeta_p)\cdots(1-\zeta_p^{p-1}) = p$, the integer S must itself be divisible by p. (Alternatively, since $h(x) := (1-x)g(x)$ vanishes at $x=1$, one can interpret S using a roots of unity filter: $S = p \cdot h([x^0]+[x^p]+\cdots) \equiv 0 \pmod p$.) Now write

$$\mathbb{Z} \ni \frac{S}{p} = \sum_{\omega \in S_p} \frac{(1-\omega)^{p-1}}{p} \frac{g(\omega)}{(1-\omega)^{p-2}} = \sum_{\omega \in S_p} u(\omega) \frac{g(\omega)}{(1-\omega)^{p-2}}.$$

Taking $g = v \cdot (1-x)^k$ for $k \geq 0$, we see that the sum in the problem statement is an integer for any $n \leq p-2$.

Finally, we have

$$\sum_{\omega \in S_p} \frac{u(\omega)}{(1-\omega)^{p-1}} = \sum_{\omega \in S_p} \frac{1}{p} = \frac{p-1}{p} \notin \mathbb{Z},$$

so the sum is not an integer for $n = p-1$.

Now let $r \geq 2$ and assume the induction hypothesis for $r-1$.

Lemma

There exist polynomials $U, V \in \mathbb{Z}[x]$ such that $(1-\omega)^p/(1-\omega^p) = U(\omega)$ and $(1-\omega^p)/(1-\omega)^p = V(\omega)$ for all $\omega \in S_q$. (Again, these are units.)

Proof. Similarly to the previous lemma, we write $1-\omega^p = (1-\omega\zeta_p^0)\cdots(1-\omega\zeta_p^{p-1})$. The polynomials U and V are

$$U(x) = \prod_{k=1}^{p-1} \frac{1-x^{(kq/p+1)\ell_k}}{1-x^{kq/p+1}}$$

$$V(x) = \prod_{k=1}^{p-1} \frac{1-x^{kq/p+1}}{1-x}$$

where ℓ_k is a multiplicative inverse of $kq/p+1$ modulo q. □

Corollary

If $\omega \in S_q$, then $(1-\omega)^{\phi(q)}/p$ is a unit.

Proof. Induct on r. For $r=1$, this is the first lemma. For the inductive step, we are given that $(1-\omega^p)^{\phi(q/p)}/p$ is a unit. By the second lemma, $(1-\omega)^{\phi(q)}/(1-\omega^p)^{\phi(q/p)}$ is also a unit. Multiplying these together yields another unit. □

Thus we have polynomials $A, B \in \mathbb{Z}[x]$ such that

$$\begin{aligned} A(\omega) &= \frac{p}{(1-\omega)^{\phi(q)}} V(\omega)^{z_{q/p}} \\ B(\omega) &= \frac{(1-\omega)^{\phi(q)}}{p} U(\omega)^{z_{q/p}} \end{aligned}$$

for all $\omega \in S_q$.

Given $g \in \mathbb{Z}[x]$, consider the pth roots of unity filter

$$S(x) := \sum_{k=0}^{p-1} g(\zeta_p^k x) = p \cdot h(x^p),$$

where $h \in \mathbb{Z}[x]$. Then

$$ph(\eta) = S(\omega) = \sum_{\omega^p = \eta} g(\omega)$$

for all $\eta \in S_{q/p}$, so

$$\begin{aligned} \frac{h(\eta)}{(1-\eta)^{z_{q/p}}} = \frac{S(\omega)}{p(1-\eta)^{z_{q/p}}} &= \sum_{\omega^p=\eta} \frac{(1-\omega)^{pz_{q/p}}}{(1-\omega^p)^{z_{q/p}}} \frac{g(\omega)}{p(1-\omega)^{pz_{q/p}}} \\ &= \sum_{\omega^p=\eta} U(\omega)^{z_{q/p}} \frac{(1-\omega)^{\phi(q)}}{p} \frac{g(\omega)}{(1-\omega)^{z_q}}. \end{aligned}$$

(Implicit in the last line is $z_q = \phi(q) + pz_{q/p}$.) Since $U(\omega)$ and $(1-\omega)^{\phi(q)}/p$ are units, we can let $g = A \cdot f$ for arbitrary $f \in \mathbb{Z}[x]$, so that the expression in the summation simplifies to $f(\omega)/(1-\omega)^{z_q}$. From this we conclude that for any $f \in \mathbb{Z}[x]$, there exists $h \in \mathbb{Z}[x]$ such that

$$\begin{aligned} \sum_{\omega \in S_q} \frac{f(\omega)}{(1-\omega)^{z_q}} &= \sum_{\eta \in S_{q/p}} \sum_{\omega^p=\eta} \frac{f(\omega)}{(1-\omega)^{z_q}} \\ &= \sum_{\eta \in S_{q/p}} \frac{h(\eta)}{(1-\eta)^{z_{q/p}}}. \end{aligned}$$

By the inductive hypothesis, this is always an integer.

In the other direction, for $\eta \in S_{q/p}$ we have

$$\begin{aligned} \sum_{\omega^p=\eta} \frac{B(\omega)}{(1-\omega)^{1+z_q}} &= \sum_{\omega^p=\eta} \frac{1}{p(1-\omega^p)^{z_{q/p}}(1-\omega)} \\ &= \frac{1}{p(1-\eta)^{z_{q/p}}} \sum_{\omega^p=\eta} \frac{1}{1-\omega} \\ &= \frac{1}{p(1-\eta)^{z_{q/p}}} \left[\frac{px^{p-1}}{x^p-\eta} \right]_{x=1} \\ &= \frac{1}{(1-\eta)^{1+z_{q/p}}}. \end{aligned}$$

Summing over all $\eta \in S_{q/p}$, we conclude by the inductive hypothesis that

$$\sum_{\omega \in S_q} \frac{B(\omega)}{(1-\omega)^{1+z_q}} = \sum_{\eta \in S_{q/p}} \frac{1}{(1-\eta)^{1+z_{q/p}}}$$

is not an integer. This completes the solution.

¶ Solution 2 (Nikolai Beluhov). Suppose that the complex numbers $\frac{1}{1-\omega}$ for $\omega \in S_q$ are the roots of

$$P(x) = x^d - c_1 x^{d-1} + c_2 x^{d-2} - \cdots \pm c_d,$$

so that c_k is their k-th elementary symmetric polynomial and $d = \phi(q) = (p-1)p^{r-1}$. Additionally denote

$$S_n = \sum_{\omega \in S_q} \frac{1}{(1-\omega)^n}.$$

Then, by Newton's identities,

$$\begin{aligned} S_1 &= c_1, \\ S_2 &= c_1 S_1 - 2c_2, \\ S_3 &= c_1 S_2 - c_2 S_1 + 3c_3, \end{aligned}$$

and so on. The general pattern when $n \le d$ is

$$S_n = \left[\sum_{j=1}^{n-1} (-1)^{j+1} c_j S_{n-j} \right] + (-1)^{n+1} n c_n.$$

After that, when $n > d$, the pattern changes to

$$S_n = \sum_{j=1}^{d} (-1)^{j+1} c_j S_{n-j}.$$

Lemma

All of the c_i are integers except for c_d. Furthermore, c_d is $1/p$ times an integer.

Proof. The qth cyclotomic polynomial is

$$\Phi_q(x) = 1 + x^{p^{r-1}} + x^{2p^{r-1}} + \cdots + x^{(p-1)p^{r-1}}.$$

The polynomial

$$Q(x) = 1 + (1+x)^{p^{r-1}} + (1+x)^{2p^{r-1}} + \cdots + (1+x)^{(p-1)p^{r-1}}$$

has roots $\omega - 1$ for $\omega \in S_q$, so it is equal to $p(-x)^d P(-1/x)$ by comparing constant coefficients. Comparing the remaining coefficients, we find that c_n is $1/p$ times the x^n coefficient of Q.

Since $(x+y)^p \equiv x^p + y^p \pmod p$, we conclude that, modulo p,

$$\begin{aligned} Q(x) &\equiv 1 + \left(1 + x^{p^{r-1}}\right) + \left(1 + x^{p^{r-1}}\right)^2 + \cdots + \left(1 + x^{p^{r-1}}\right)^{p-1} \\ &\equiv \left[\left(1 + x^{p^{r-1}}\right)^p - 1 \right] / x^{p^{r-1}}. \end{aligned}$$

Since $\binom{p}{j}$ is a multiple of p when $0 < j < p$, it follows that all coefficients of $Q(x)$ are multiples of p save for the leading one. Therefore, c_n is an integer when $n < d$, while c_d is $1/p$ times an integer. $\square$

By the recurrences above, S_n is an integer for $n < d$. When $r = 1$, we get that dc_d is not an integer, so S_d is not an integer, either. Thus the answer for $r = 1$ is $n = p - 1$.

Suppose now that $r \ge 2$. Then dc_d does become an integer, so S_d is an integer as well.

Lemma

For all n with $1 \le n \le d$, we have $\nu_p(nc_n) \ge r-2$. Furthermore, the smallest n such that $\nu_p(nc_n) = r-2$ is $d - p^{r-1} + 1$.

Proof. The value of nc_n is $1/p$ times the coefficient of x^{n-1} in the derivative $Q'(x)$. This derivative is

$$p^{r-1}(1+x)^{p^{r-1}-1}\left[\sum_{k=1}^{p-1} k(1+x)^{(k-1)p^{r-1}}\right].$$

What we want to prove reduces to showing that all coefficients of the polynomial in the square brackets are multiples of p except for the leading one.

Using the same trick $(x+y)^p \equiv x^p + y^p \pmod p$ as before and also writing w for $x^{p^{r-1}}$, modulo p the polynomial in the square brackets becomes

$$1 + 2(1+w) + 3(1+w)^2 + \cdots + (p-1)(1+w)^{p-2}.$$

This is the derivative of

$$1 + (1+w) + (1+w)^2 + \cdots + (1+w)^{p-1} = [(1+w)^p - 1]/w$$

and so, since $\binom{p}{j}$ is a multiple of p when $0 < j < p$, we are done. □

Finally, we finish the problem with the following claim.

Claim — Let $m = d - p^{r-1}$. Then for all $k \ge 0$ and $1 \le j \le d$, we have

$$\begin{aligned} \nu_p(S_{kd+m+1}) &= r-2-k \\ \nu_p(S_{kd+m+j}) &\ge r-2-k. \end{aligned}$$

Proof. First, $S_1, S_2, \dots, S_m$ are all divisible by p^{r-1} by Newton's identities and the second lemma. Then $\nu_p(S_{m+1}) = r-2$ because

$$\nu_p((m+1)c_{m+1}) = r-2,$$

and all other terms in the recurrence relation are divisible by p^{r-1}. We can similarly check that $\nu_p(S_n) \ge r-2$ for $m+1 \le n \le d$. Newton's identities combined with the first lemma now imply the following for $n > d$:

- If $\nu_p(S_{n-j}) \ge \ell$ for all $1 \le j \le d$ and $\nu_p(S_{n-d}) \ge \ell+1$, then $\nu_p(S_n) \ge \ell$.
- If $\nu_p(S_{n-j}) \ge \ell$ for all $1 \le j \le d$ and $\nu_p(S_{n-d}) = \ell$, then $\nu_p(S_n) = \ell - 1$.

Together, these prove the claim by induction. □

By the claim, the smallest n for which $\nu_p(S_n) < 0$ (equivalent to S_n not being an integer, by the recurrences) is

$$n = (r-1)d + m + 1 = ((p-1)r-1)p^{r-1} + 1.$$

Remark. The original proposal was the following more general version:

> Let n be an integer with prime power factorization $q_1 \cdots q_m$. Let S_n denote the set of primitive nth roots of unity. Find all tuples of nonnegative integers $(z_1, \dots, z_m)$ such that
> $$\sum_{\omega \in S_n} \frac{f(\omega)}{(1-\omega^{n/q_1})^{z_1} \cdots (1-\omega^{n/q_m})^{z_m}} \in \mathbb{Z}$$
> for all polynomials $f \in \mathbb{Z}[x]$.

The maximal z_i are exponents in the prime ideal factorization of the different ideal of the cyclotomic extension $\mathbb{Q}(\zeta_n)/\mathbb{Q}$.

Remark. Let $F = (x^p - 1)/(x-1)$ be the minimal polynomial of $\zeta_p = e^{2\pi i/p}$ over $\mathbb{Q}$. A calculation of Euler shows that

$$(\mathbb{Z}[\zeta_p])^* := \{\alpha = g(\zeta_p) \in \mathbb{Q}[\zeta_p] : \sum_{\omega \in S_p} f(\omega)g(\omega) \in \mathbb{Z} \ \forall f \in \mathbb{Z}[x]\} = \frac{1}{F'(\zeta_p)} \cdot \mathbb{Z}[\zeta_p],$$

where

$$F'(\zeta_p) = \frac{p\zeta_p^{p-1} - [1 + \zeta_p + \cdots + \zeta_p^{p-1}]}{1-\zeta_p} = p(1-\zeta_p)^{-1}\zeta_p^{p-1}$$

is $(1-\zeta_p)^{[p-1]-1} = (1-\zeta_p)^{p-2}$ times a unit of $\mathbb{Z}[\zeta_p]$. Here, $(\mathbb{Z}[\zeta_p])^*$ is the dual lattice of $\mathbb{Z}[\zeta_p]$.

Remark. Let $K = \mathbb{Q}(\omega)$, so (p) factors as $(1-\omega)^{p-1}$ in the ring of integers $\mathcal{O}_K$ (which, for cyclotomic fields, can be shown to be $\mathbb{Z}[\omega]$). In particular, the *ramification index* e of $(1-\omega)$ over p is the exponent, $p-1$. Since $e = p-1$ is not divisible by p, we have so-called *tame ramification*. Now by the ramification theory of Dedekind's different ideal, the exponent z_1 that works when $n = p$ is $e - 1 = p - 2$.

Higher prime powers are more interesting because of wild ramification: p divides $\phi(p^r) = p^{r-1}(p-1)$ if and only if $r > 1$. (This is a similar phenomena to how Hensel's lemma for $x^2 - c$ is more interesting mod powers of 2 than mod odd prime powers.)

Remark. Let $F = (x^q - 1)/(x^{q/p} - 1)$ be the minimal polynomial of $\zeta_q = e^{2\pi i/q}$ over $\mathbb{Q}$. The aforementioned calculation of Euler shows that

$$(\mathbb{Z}[\zeta_q])^* := \{\alpha = g(\zeta_q) \in \mathbb{Q}[\zeta_q] : \sum_{\omega \in S_q} f(\omega)g(\omega) \in \mathbb{Z} \ \forall f \in \mathbb{Z}[x]\} = \frac{1}{F'(\zeta_q)} \cdot \mathbb{Z}[\zeta_q],$$

where the chain rule implies (using the computation from the prime case)

$$F'(\zeta_q) = [p(1-\zeta_p)^{-1}\zeta_p^{p-1}] \cdot \frac{q}{p}\zeta_q^{(q/p)-1} = q(1-\zeta_p)^{-1}\zeta_q^{-1}.$$

is $(1-\zeta_q)^{r\phi(q)-q/p} = (1-\zeta_q)^{z_q}$ times a unit of $\mathbb{Z}[\zeta_q]$.