USA TSTST 2016 Solutions

United States of America — TST Selection Test

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§0 Problems

1. Let $A = A(x, y)$ and $B = B(x, y)$ be two-variable polynomials with real coefficients. Suppose that $A(x, y)/B(x, y)$ is a polynomial in x for infinitely many values of y, and a polynomial in y for infinitely many values of x. Prove that B divides A, meaning there exists a third polynomial C with real coefficients such that $A = B \cdot C$.

2. Let ABC be a scalene triangle with orthocenter H and circumcenter O and denote by M, N the midpoints of $\overline{AH}$, $\overline{BC}$. Suppose the circle γ with diameter $\overline{AH}$ meets the circumcircle of ABC at $G \neq A$, and meets line $\overline{AN}$ at $Q \neq A$. The tangent to γ at G meets line OM at P. Show that the circumcircles of $\triangle GNQ$ and $\triangle MBC$ intersect on $\overline{PN}$.

3. Decide whether or not there exists a nonconstant polynomial $Q(x)$ with integer coefficients with the following property: for every positive integer $n > 2$, the numbers
$$Q(0),\ Q(1), Q(2),\ \dots,\ Q(n-1)$$
produce at most $0.499n$ distinct residues when taken modulo n.

4. Prove that if n and k are positive integers satisfying $\varphi^k(n) = 1$, then $n \le 3^k$. (Here φ^k denotes k applications of the Euler phi function.)

5. In the coordinate plane are finitely many *walls*, which are disjoint line segments, none of which are parallel to either axis. A bulldozer starts at an arbitrary point and moves in the $+x$ direction. Every time it hits a wall, it turns at a right angle to its path, away from the wall, and continues moving. (Thus the bulldozer always moves parallel to the axes.)

   Prove that it is impossible for the bulldozer to hit both sides of every wall.

6. Let ABC be a triangle with incenter I, and whose incircle is tangent to $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ at D, E, F, respectively. Let K be the foot of the altitude from D to $\overline{EF}$. Suppose that the circumcircle of $\triangle AIB$ meets the incircle at two distinct points C_1 and C_2, while the circumcircle of $\triangle AIC$ meets the incircle at two distinct points B_1 and B_2. Prove that the radical axis of the circumcircles of $\triangle BB_1B_2$ and $\triangle CC_1C_2$ passes through the midpoint M of $\overline{DK}$.

§1 Solutions to Day 1

§1.1 TSTST 2016/1, proposed by Victor Wang

Available online at https://aops.com/community/p6575197.

Problem statement

Let $A = A(x, y)$ and $B = B(x, y)$ be two-variable polynomials with real coefficients. Suppose that $A(x, y)/B(x, y)$ is a polynomial in x for infinitely many values of y, and a polynomial in y for infinitely many values of x. Prove that B divides A, meaning there exists a third polynomial C with real coefficients such that $A = B \cdot C$.

This is essentially an application of the division algorithm, but the details require significant care.

First, we claim that A/B can be written as a polynomial in x whose coefficients are rational functions in y. To see this, use the division algorithm to get

$$A = Q \cdot B + R \qquad Q, R \in (\mathbb{R}(y))[x]$$

where Q and R are polynomials in x whose coefficients are rational functions in y, and moreover $\deg_x B > \deg_x R$.

Now, we claim that $R \equiv 0$. Indeed, we have by hypothesis that for infinitely many values of y_0 that $B(x, y_0)$ divides $A(x, y_0)$, which means $B(x, y_0) \mid R(x, y_0)$ as polynomials in $\mathbb{R}[x]$. Now, we have $\deg_x B(x, y_0) > \deg_x R(x, y_0)$ outside of finitely many values of y_0 (but not all of them!); this means for infinitely many y_0 we have $R(x, y_0) \equiv 0$. So each coefficient of x^i (in $\mathbb{R}(y)$) has infinitely many roots, hence is a zero polynomial.

Consequently, we are able to write $A/B = F(x, y)/M(y)$ where $F \in \mathbb{R}[x, y]$ and $M \in \mathbb{R}[y]$ are each polynomials. Repeating the same argument now gives

$$\frac{A}{B} = \frac{F(x, y)}{M(y)} = \frac{G(x, y)}{N(x)}.$$

Now, by unique factorization of polynomials in $\mathbb{R}[x, y]$, we can discuss GCD's. So, we tacitly assume $\gcd(F, M) = \gcd(G, N) = (1)$. Also, we obviously have $\gcd(M, N) = (1)$. But $F \cdot N = G \cdot M$, so $M \mid F \cdot N$, thus we conclude M is the constant polynomial. This implies the result.

> **Remark.** This fact does not generalize to arbitrary functions that are separately polynomial: see e.g. http://aops.com/community/c6h523650p2978180.

§1.2 TSTST 2016/2, proposed by Evan Chen

Available online at https://aops.com/community/p6575204.

Problem statement

Let ABC be a scalene triangle with orthocenter H and circumcenter O and denote by M, N the midpoints of $\overline{AH}$, $\overline{BC}$. Suppose the circle γ with diameter $\overline{AH}$ meets the circumcircle of ABC at $G \neq A$, and meets line $\overline{AN}$ at $Q \neq A$. The tangent to γ at G meets line OM at P. Show that the circumcircles of $\triangle GNQ$ and $\triangle MBC$ intersect on $\overline{PN}$.

We present two solutions, one using essentially only power of a point, and the other more involved.

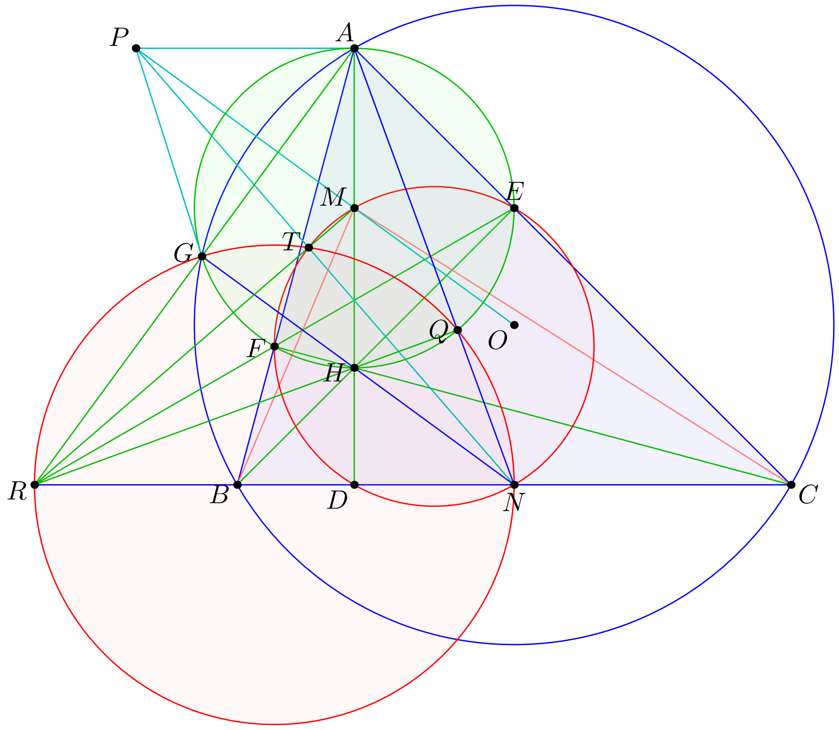
¶ First solution (found by contestants). Denote by $\triangle DEF$ the orthic triangle. Observe $\overline{PA}$ and $\overline{PG}$ are tangents to γ, since $\overline{OM}$ is the perpendicular bisector of $\overline{AG}$. Also note that $\overline{AG}$, $\overline{EF}$, $\overline{BC}$ are concurrent at some point R by radical axis on (ABC), γ, $(BFEC)$.

Now, consider circles $(PAGM)$, $(MFDNE)$, and (MBC). We already saw the point R satisfies

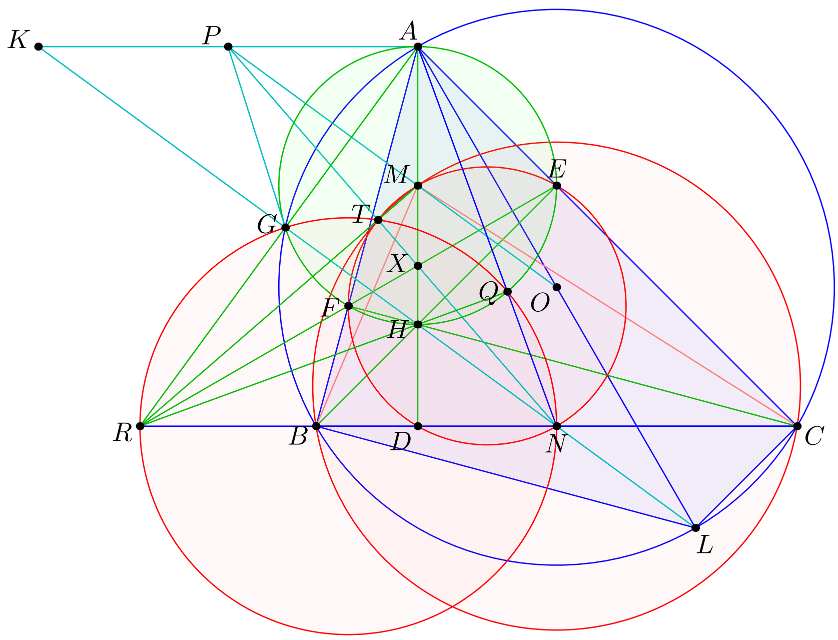
$$RA \cdot RG = RE \cdot RF = RB \cdot RC$$

and hence has equal powers to all three circles; but since the circles at M already, they must actually be coaxial. Assume they meet again at $T \in \overline{RM}$, say. Then $\angle PTM$ and $\angle MTN$ are both right angles, hence T lies on $\overline{PN}$.

Finally H is the orthocenter of $\triangle ARN$, and thus the circle with diameter $\overline{RN}$ passes through G, Q, N.



¶ Alternate solution (by proposer). Let L be diametrically opposite A on the circumcircle. Denote by $\triangle DEF$ the orthic triangle. Let $X = \overline{AH} \cap \overline{EF}$. Finally, let T be the second intersection of $(MFDNE)$ and (MBC).



We begin with a few easy observations. First, points H, G, N, L are collinear and $\angle AGL = 90^\circ$. Also, Q is the foot from H to $\overline{AN}$. Consequently, lines AG, EF, HQ, BC, TM concur at a point R (radical axis). Moreover, we already know $\angle MTN = 90^\circ$. This implies T lies on the circle with diameter $\overline{RN}$, which is exactly the circumcircle of $\triangle GQN$.

Note by Brokard's Theorem on $AFHE$, the point X is the orthocenter of $\triangle MBC$. But $\angle MTN = 90^\circ$ already, and N is the midpoint of $\overline{BC}$. Consequently, points T, X, N are collinear.

Finally, we claim P, X, N are collinear, which solves the problem. Note $P = \overline{GG} \cap \overline{AA}$. Set $K = \overline{HNL} \cap \overline{AP}$. Then by noting

$$-1 = (D, X; A, H) \overset{N}{=} (\infty, \overline{NX} \cap \overline{AK}; A, K)$$

we see that $\overline{NX}$ bisects segment $\overline{AK}$, as desired. (A more projective finish is to show that $\overline{PXN}$ is the polar of R to γ).

> **Remark.** The original problem proposal reads as follows:
>
> > Let ABC be a triangle with orthocenter H and circumcenter O and denote by M, N the midpoints of $\overline{AH}$, $\overline{BC}$. Suppose ray OM meets the line parallel to $\overline{BC}$ through A at P. Prove that the line through the circumcenter of $\triangle MBC$ and the midpoint of $\overline{OH}$ is parallel to $\overline{NP}$.
>
> The points G and Q were added to the picture later to prevent the problem from being immediate by coordinates.

§1.3 TSTST 2016/3, proposed by Yang Liu

Available online at https://aops.com/community/p6575217.

Problem statement

Decide whether or not there exists a nonconstant polynomial $Q(x)$ with integer coefficients with the following property: for every positive integer $n > 2$, the numbers

$$Q(0),\ Q(1), Q(2),\ \dots,\ Q(n-1)$$

produce at most $0.499n$ distinct residues when taken modulo n.

We claim that
$$Q(x) = 420(x^2-1)^2$$
works. Clearly, it suffices to prove the result when $n = 4$ and when n is an odd prime p. The case $n = 4$ is trivial, so assume now $n = p$ is an odd prime.

First, we prove the following easy claim.

Claim — For any odd prime p, there are at least $\frac{1}{2}(p-3)$ values of a for which $\left(\frac{1-a^2}{p}\right) = +1$.

Proof. Note that if $k \neq 0$ and $k^2 \neq -1$, then $a = \frac{1-k^2}{k^2+1}$ works. □

Remark. The above identity comes from starting with the equation $1 - a^2 = b^2$, and writing it as $\left(\frac{1}{b}\right)^2 - \left(\frac{a}{b}\right)^2 = 1$. Then solve $\frac{1}{b} - \frac{a}{b} = k$ and $\frac{1}{b} + \frac{a}{b} = 1/k$ for a.

Let $F(x) = (x^2-1)^2$. The range of F modulo p is contained within the $\frac{1}{2}(p+1)$ quadratic residues modulo p. On the other hand, if for some t neither of $1 \pm t$ is a quadratic residue, then t^2 is omitted from the range of F as well. Call such a value of t *useful*, and let N be the number of useful residues. We aim to show $N \geq \frac{1}{4}p - 2$.

We compute a lower bound on the number N of useful t by writing

$$\begin{aligned}
N &= \frac{1}{4}\left(\sum_t \left[\left(1 - \left(\frac{1-t}{p}\right)\right)\left(1 - \left(\frac{1+t}{p}\right)\right)\right] - \left(1 - \left(\frac{2}{p}\right)\right) - \left(1 - \left(\frac{-2}{p}\right)\right)\right) \\
&\geq \frac{1}{4}\sum_t \left[\left(1 - \left(\frac{1-t}{p}\right)\right)\left(1 - \left(\frac{1+t}{p}\right)\right)\right] - 1 \\
&= \frac{1}{4}\left(p + \sum_t \left(\frac{1-t^2}{p}\right)\right) - 1 \\
&\geq \frac{1}{4}\left(p + (+1)\cdot \tfrac{1}{2}(p-3) + 0 \cdot 2 + (-1)\cdot\left((p-2) - \tfrac{1}{2}(p-3)\right)\right) - 1 \\
&\geq \frac{1}{4}(p-5).
\end{aligned}$$

Thus, the range of F has size at most
$$\frac{1}{2}(p+1) - \frac{1}{2}N \leq \frac{3}{8}(p+3).$$
This is less than $0.499p$ for any $p \geq 11$.

Remark. In fact, the computation above is essentially an equality. There are only two points where terms are dropped: one, when $p \equiv 3 \pmod 4$ there are no $k^2 = -1$ in the lemma, and secondly, the terms $1 - (2/p)$ and $1 - (-2/p)$ are dropped in the initial estimate for N. With suitable modifications, one can show that in fact, the range of F is exactly equal to

$$\frac{1}{2}(p+1) - \frac{1}{2}N = \begin{cases} \frac{1}{8}(3p+5) & p \equiv 1 \pmod 8 \\ \frac{1}{8}(3p+7) & p \equiv 3 \pmod 8 \\ \frac{1}{8}(3p+9) & p \equiv 5 \pmod 8 \\ \frac{1}{8}(3p+3) & p \equiv 7 \pmod 8. \end{cases}$$

§2 Solutions to Day 2

§2.1 TSTST 2016/4, proposed by Linus Hamilton

Available online at https://aops.com/community/p6580534.

Problem statement

Prove that if n and k are positive integers satisfying $\varphi^k(n) = 1$, then $n \le 3^k$. (Here φ^k denotes k applications of the Euler phi function.)

The main observation is that the exponent of 2 decreases by at most 1 with each application of φ. This will give us the desired estimate.

Define the *weight* function w on positive integers as follows: it satisfies

$$\begin{aligned} w(ab) &= w(a) + w(b); \\ w(2) &= 1; \quad \text{and} \\ w(p) &= w(p-1) \quad \text{for any prime } p > 2. \end{aligned}$$

By induction, we see that $w(n)$ counts the powers of 2 that are produced as φ is repeatedly applied to n. In particular, $k \ge w(n)$.

From $w(2) = 1$, it suffices to prove that $w(p) \ge \log_3 p$ for every $p > 2$. We use strong induction and note that

$$w(p) = w(2) + w\left(\frac{p-1}{2}\right) \ge 1 + \log_3(p-1) - \log_3 2 \ge \log_3 p$$

for any $p > 2$. This solves the problem.

Remark. One can motivate this solution through small cases $2^x 3^y$ like $2^x 17^w$, $2^x 3^y 7^z$, $2^x 11^t$.

Moreover, the stronger bound

$$n \le 2 \cdot 3^{k-1}$$

is true and best possible.

§2.2 TSTST 2016/5, proposed by Linus Hamilton, Cynthia Stoner

Available online at https://aops.com/community/p6580545.

Problem statement

In the coordinate plane are finitely many *walls*, which are disjoint line segments, none of which are parallel to either axis. A bulldozer starts at an arbitrary point and moves in the $+x$ direction. Every time it hits a wall, it turns at a right angle to its path, away from the wall, and continues moving. (Thus the bulldozer always moves parallel to the axes.)

Prove that it is impossible for the bulldozer to hit both sides of every wall.

We say a wall v is *above* another wall w if some point on v is directly above a point on w. (This relation is anti-symmetric, as walls do not intersect).

The critical claim is as follows:

Claim — There exists a lowest wall, i.e. a wall not above any other walls.

Proof. Assume not. Then we get a directed cycle of some length $n \ge 3$: it's possible to construct a series of points P_i, Q_i, for $i = 1, \dots, n$ (indices modulo n), such that the point Q_i is directly above P_{i+1} for each i, the segment $\overline{Q_i P_{i+1}}$ does not intersect any wall in its interior, and finally each segment $\overline{P_i Q_i}$ is contained inside a wall. This gives us a broken line on $2n$ vertices which is not self-intersecting.

Now consider the leftmost vertical segment $\overline{Q_i P_{i+1}}$ and the rightmost vertical segment $\overline{Q_j P_{j+1}}$. The broken line gives a path from P_{i+1} to Q_j, as well as a path from P_{j+1} to Q_i. These clearly must intersect, contradiction. □

Remark. This claim is Iran TST 2010.

Thus if the bulldozer eventually moves upwards indefinitely, it may never hit the bottom side of the lowest wall. Similarly, if the bulldozer eventually moves downwards indefinitely, it may never hit the upper side of the highest wall.

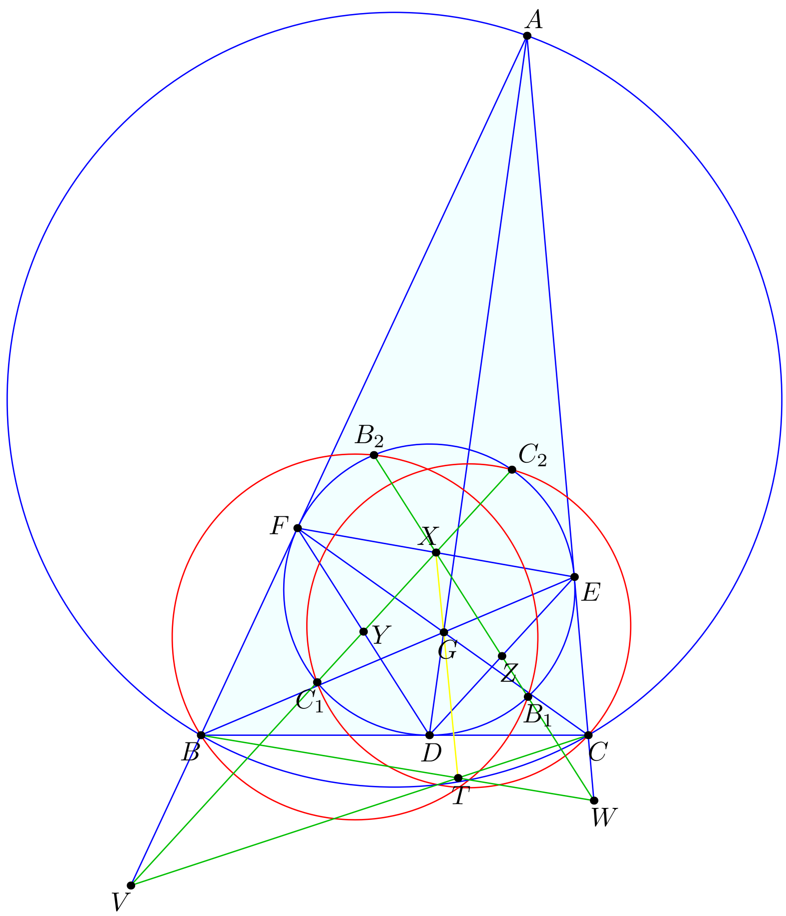
§2.3 TSTST 2016/6, proposed by Danielle Wang

Available online at https://aops.com/community/p6580553.

Problem statement

Let ABC be a triangle with incenter I, and whose incircle is tangent to $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ at D, E, F, respectively. Let K be the foot of the altitude from D to $\overline{EF}$. Suppose that the circumcircle of $\triangle AIB$ meets the incircle at two distinct points C_1 and C_2, while the circumcircle of $\triangle AIC$ meets the incircle at two distinct points B_1 and B_2. Prove that the radical axis of the circumcircles of $\triangle BB_1B_2$ and $\triangle CC_1C_2$ passes through the midpoint M of $\overline{DK}$.

¶ First solution (Allen Liu). Let X, Y, Z be midpoints of EF, FD, DE, and let G be the Gergonne point. By radical axis on $(AEIF)$, (DEF), (AIC) we see that B_1, X, B_2 are collinear. Likewise, B_1, Z, B_2 are collinear, so lines B_1B_2 and XZ coincide. Similarly, lines C_1C_2 and XY coincide. In particular lines B_1B_2 and C_1C_2 meet at X.



Note G is the symmedian point of DEF, so it is well-known that XG passes through the midpoint of DK. So we just have to prove G lies on the radical axis.

First, note that $\triangle DEF$ is the cevian triangle of the Gergonne point G. Set $V = \overline{XY} \cap \overline{AB}$, $W = \overline{XZ} \cap \overline{AC}$, and $T = \overline{BW} \cap \overline{CV}$.

We begin with the following completely projective claim.

Claim — The points X, G, T are collinear.

Proof. It suffices to view $\triangle XYZ$ as any cevian triangle of $\triangle DEF$ (which is likewise any cevian triangle of $\triangle ABC$). Then

- By Cevian Nest on $\triangle ABC$, it follows that $\overline{AX}$, $\overline{BY}$, $\overline{CZ}$ are concurrent.
- Hence $\triangle BYV$ and $\triangle CZW$ are perspective.
- Hence $\triangle BZW$ and $\triangle CYV$ are perspective too.
- Hence we deduce by Desargues theorem that T, X, and $\overline{BZ} \cap \overline{CY}$ are collinear.
- Finally, the Cevian Nest theorem applied on $\triangle GBC$ (which has cevian triangles $\triangle DFE$, $\triangle XZY$) we deduce G, X, and $\overline{BZ} \cap \overline{CY}$, proving the claim.

One could also proceed by using barycentric coordinates on $\triangle DEF$. $\square$

Remark (Eric Shen)**.** The first four bullets can be replaced by non-projective means: one can check that $\overline{BZ} \cap \overline{CY}$ is the radical center of (BIC), (BB_1B_2), (CC_1C_2) and therefore it lies on line $\overline{XT}$.

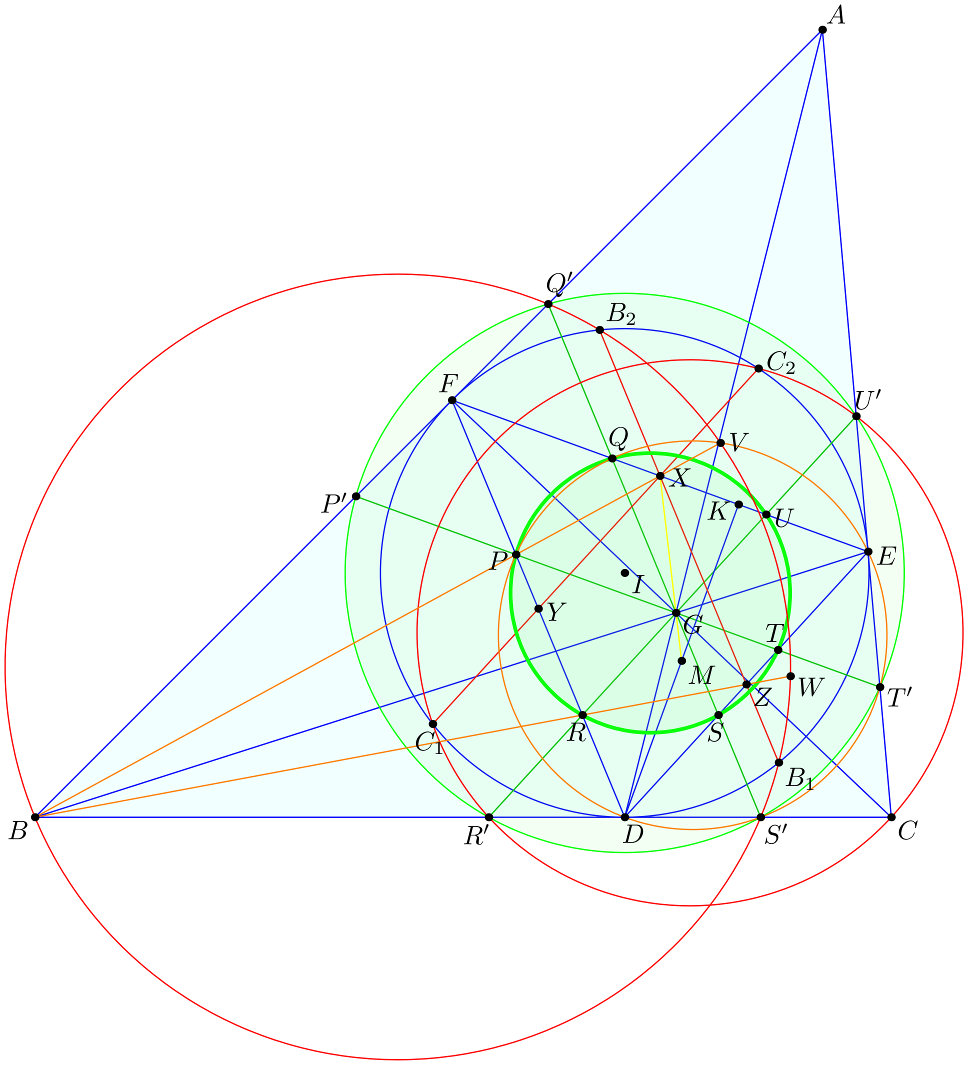
Now, we contend point V is the radical center (CC_1C_2), (ABC) and (DEF). To see this, let $V' = \overline{ED} \cap \overline{AB}$; then $(FV'; AB)$ is harmonic, and V is the midpoint of $\overline{FV'}$, and thus $VA \cdot VB = VF^2 = VC_1 \cdot VC_2$.

So in fact $\overline{CV}$ is the radical axis of (ABC) and (CC_1C_2).

Similarly, $\overline{BW}$ is the radical axis of (ABC) and (BB_1B_2). Thus T is the radical center of (ABC), (BB_1B_2), (CC_1C_2).

This completes the proof, as now $\overline{XT}$ is the desired radical axis.

¶ Second solution (Evan Chen). As before, we just have to prove G lies on the radical axis.



Construct parallelograms $GPFQ$, $GRDS$, $GTUE$ such that $P, R \in DF$, $S, T \in DE$, $Q, U \in EF$. As FG bisects PQ and is isogonal to FZ, we find $PQED$, hence $PQRU$, is cyclic. Repeating the same logic and noticing PR, ST, QU not concurrent, all six points $PQRSTU$ are cyclic. Moreover, since PQ bisects GF, we see that a dilation with factor 2 at G sends PQ to $P', Q' \in AB$, say, with F the midpoint of $P'Q'$. Define $R', S' \in BC$ similarly now and $T', U' \in CA$.

Note that $EQPDS'$ is in cyclic too, as $\measuredangle DS'Q = \measuredangle DRS = \measuredangle DEF$. By homothety through B, points B, P, X are collinear; assume they meet $(EQPDS')$ again at V. Thus $EVQPDS'$ is cyclic, and now

$$\measuredangle BVS' = \measuredangle PVS' = \measuredangle PQS = \measuredangle PTS = \measuredangle FED = \measuredangle XEZ = \measuredangle XVZ$$

hence V lies on $(BQ'S')$.

Since $FB \parallel QP$, we get $EVFB$ is cyclic too, so $XV \cdot XB = XE \cdot XF$ now; thus X lies on the radical axis of $(BS'Q')$ and (DEF). By the same argument with $W \in BZ$, we get Z lies on the radical axis too. Thus the radical axis of $(BS'Q')$ and (DEF) must be line XZ, which coincides with B_1B_2; so $(BB_1B_2) = (BS'Q')$.

Analogously, $(CC_1C_2) = (CR'U')$. Since $G = Q'S' \cap R'U'$, we need only prove that $Q'R'S'U'$ is cyclic. But $QRSU$ is cyclic, so we are done.

The circle $(PQRSTU)$ is called the *Lemoine circle* of ABC.