TSTST 2013 Solution Notes

Lincoln, Nebraska

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§1 Solutions to Day 1

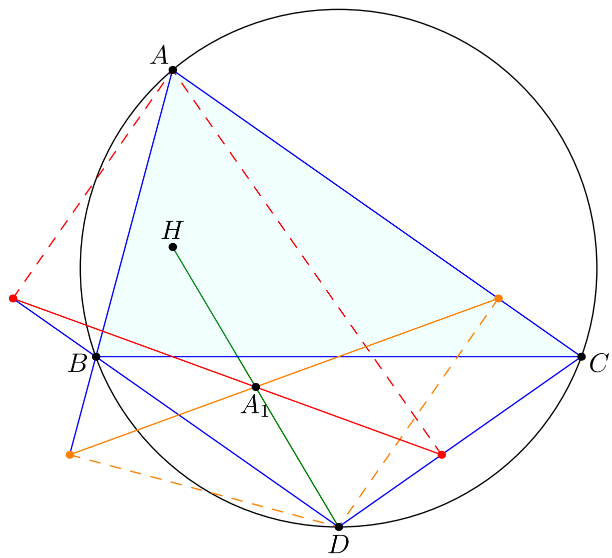
§1.1 Solution to TSTST 1

Let ABC be a triangle and D, E, F be the midpoints of arcs BC, CA, AB on the circumcircle. Line ℓ_a passes through the feet of the perpendiculars from A to $\overline{DB}$ and $\overline{DC}$. Line m_a passes through the feet of the perpendiculars from D to $\overline{AB}$ and $\overline{AC}$. Let A_1 denote the intersection of lines ℓ_a and m_a. Define points B_1 and C_1 similarly. Prove that triangles DEF and $A_1B_1C_1$ are similar to each other.

In fact, it is true for any points D, E, F on the circumcircle. More strongly we contend:

Claim — Point A_1 is the midpoint of $\overline{HD}$.

Proof. Lines m_a and ℓ_a are Simson lines, so they both pass through the point $(a+b+c+d)/2$ in complex coordinates. □



Hence $A_1B_1C_1$ is similar to DEF through a homothety at H with ratio $\frac{1}{2}$.

§1.2 Solution to TSTST 2

A finite sequence of integers $a_1, a_2, \dots, a_n$ is called *regular* if there exists a real number x satisfying

$$\lfloor kx \rfloor = a_k \quad \text{for } 1 \le k \le n.$$

Given a regular sequence $a_1, a_2, \dots, a_n$, for $1 \le k \le n$ we say that the term a_k is *forced* if the following condition is satisfied: the sequence

$$a_1,\ a_2,\ \dots,\ a_{k-1},\ b$$

is regular if and only if $b = a_k$.

Find the maximum possible number of forced terms in a regular sequence with 1000 terms.

The answer is 985. WLOG, by shifting $a_1 = 0$ (clearly a_1 isn't forced). Now, we construct regular sequences inductively using the following procedure. Start with the inequality

$$\tfrac{0}{1} \le x < \tfrac{1}{1}.$$

Then for each $k = 2, 3, \dots, 1000$ we perform the following procedure. If there is no fraction of the form $F = \frac{m}{k}$ in the interval $A \le x < B$, then a_k is forced, and the interval of possible x values does not change. Otherwise, a_k is not forced, and we pick a value of a_k and update the interval accordingly.

The theory of Farey sequences tells us that when we have a stage $\frac{a}{b} \le x < \frac{c}{d}$ then the next time we will find a fraction in that interval is exactly $\frac{a+c}{b+d}$ (at time $k = b + d$), and it will be the only such fraction.

So essentially, starting with $\frac{0}{1} \le x < \frac{1}{1}$ we repeatedly replace one of the endpoints of the intervals with the mediant, until one of the denominators exceeds 1000; we are trying to minimize the number of non-forced terms, which is the number of denominators that appear in this process. It is not hard to see that this optimum occurs by always replacing the smaller of the denominators, so that the sequence is

$$\begin{aligned}
\tfrac{0}{1} &\le x < \tfrac{1}{1} \\
\tfrac{0}{1} &\le x < \tfrac{1}{2} \\
\tfrac{1}{3} &\le x < \tfrac{1}{2} \\
\tfrac{1}{3} &\le x < \tfrac{2}{5} \\
\tfrac{3}{8} &\le x < \tfrac{2}{5} \\
\tfrac{3}{8} &\le x < \tfrac{5}{13}
\end{aligned}$$

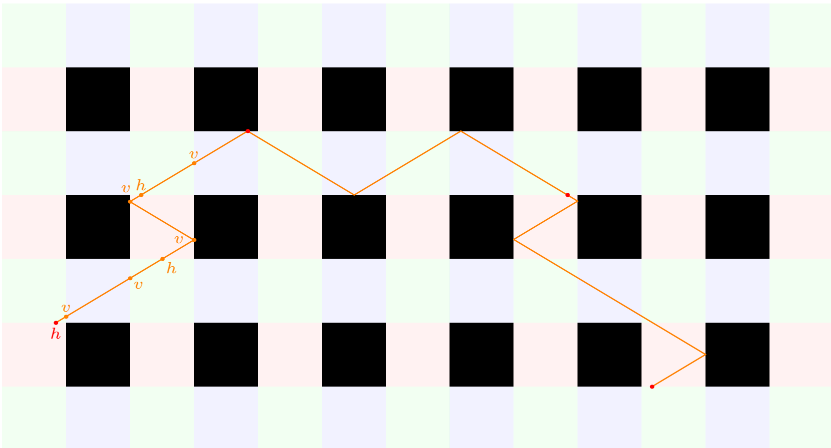
and so on; we see that the non-forced terms in this optimal configuration are exactly the Fibonacci numbers. There are 15 Fibonacci numbers less than 1000, hence the answer $1000 - 15 = 985$.

§1.3 Solution to TSTST 3

Divide the plane into an infinite square grid by drawing all the lines $x = m$ and $y = n$ for $m, n \in \mathbb{Z}$. Next, if a square's upper-right corner has both coordinates even, color it black; otherwise, color it white (in this way, exactly $1/4$ of the squares are black and no two black squares are adjacent). Let r and s be odd integers, and let (x, y) be a point in the interior of any white square such that $rx - sy$ is irrational. Shoot a laser out of this point with slope r/s; lasers pass through white squares and reflect off black squares. Prove that the path of this laser will from a closed loop.

Here is Sammy Luo's solution. Fix the speed of light at $\sqrt{r^2 + s^2}$ units per second. We prove periodicity every six seconds.

We re-color the white squares as red, blue, or green according as to whether they have a black square directly to the left/right, above/below, or neither, as shown below. Finally, we fix time zero to be a moment just before the laser passes a horizontal (WLOG) lattice line (not necessarily a wall). Shown below is an example for $(r, s) = (3, 5)$.



The main idea is to keep track of every time the laser passes a lattice line (again, not necessarily a wall). There are four possible types of events:

- A horizontal h event where the laser switches from red to green (or vice-versa);
- A horizontal h event where the laser rebounds off a wall, remaining in a blue square, but flips the x-component of its velocity;
- A vertical v event where the laser switches from blue to green (or vice-versa)
- A vertical v event where the laser rebounds off a wall, remaining in a red square, but flips the y-component of its velocity.

The first key observation is that:

Claim — In the first second, the laser will encounter exactly r horizontal events and s vertical events. In every second after that, the same sequence of $r + s$ events occurs.

Proof. Bouncing off a wall doesn't change this as opposed to if the laser had passed through the wall. □

We let the *key-word* be the sequence w of $r+s$ letters corresponding to the sequence. For example, the picture above denotes an example with keyword $w = hvvhvvhv$; so no matter what, every second, the laser will encounter eight lattice lines, which are horizontal and vertical in that order.

Claim — Color is periodic every 3 seconds.

Proof. The free group generated by h and v acts on the set $\{R, G, B\}$ of colors in an obvious way; consider this right action. First we consider the color of the square after each second. Note that with respect to color, each letter is an involution; so as far as color changes are concerned, it's enough to work with the reduced word w' obtained by modding out by $h^2 = 1$ and $v^2 = 1$. (For example, $w' = hv$ in our example.) In general, $w' = (hv)^k$ or $w' = (vh)^k$, for some odd integer k (since $k \equiv r \equiv s \equiv 1 \pmod 2$). Now we see that the action of hv on the set of colors is red $\mapsto$ blue $\mapsto$ green $\mapsto$ red, and similarly for vh (being the inverse). This implies that the color is periodic every three seconds. $\square$

Now in a 3-second period, consider the $3r$ horizontal events and $3s$ vertical events (both are odd). In order for the color to remain the same (as the only color changes are $R \leftrightarrow G$ for h and $B \leftrightarrow G$ for v) there must have been an even number of color swaps for each orientation. Therefore there was an odd number of wall collisions of each orientation. So, the laser is pointing in the opposite direction at the end of 3 seconds.

Finally, let x_t be the fractional part of the x coordinate after t seconds (the y-coordinate is always zero by our setup at these moments). Note that

$$x_{t+1} = \begin{cases} x_t & \text{even number of vertical wall collisions} \\ 1 - x_t & \text{odd numbers of vertical wall collisions} \end{cases}$$

Since over the there seconds there were an odd number of vertical collisions; it follows $x_3 = 1 - x_0$. Thus at the end of three seconds, the laser is in a symmetric position from the start; and in 6 seconds it will form a closed loop.

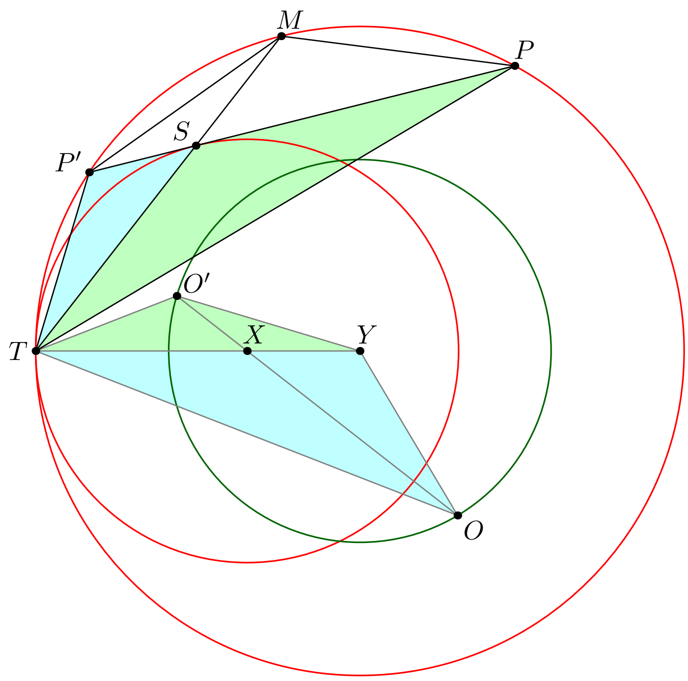
§2 Solutions to Day 2

§2.1 Solution to TSTST 4

Circle ω, centered at X, is internally tangent to circle Ω, centered at Y, at T. Let P and S be variable points on Ω and ω, respectively, such that line PS is tangent to ω (at S). Determine the locus of O – the circumcenter of triangle PST.

The answer is a circle centered at Y with radius $\sqrt{YX \cdot YT}$, minus the two points on line XY itself.

We let PS meet Ω again at P', and let O' be the circumcenter of $\triangle TPS'$. Note that O', X, O are collinear on the perpendicular bisector of line $\overline{TS}$ Finally, we let M denote the arc midpoint of PP' which lies on line TS (by homothety).



By three applications of Salmon theorem, we have the following spiral similarities all centered at T:

$$\triangle TSP \overset{+}{\sim} \triangle TO'Y$$
$$\triangle TP'S \overset{+}{\sim} \triangle TYO$$
$$\triangle TP'P \overset{+}{\sim} \triangle TO'O.$$

However, the shooting lemma also gives us two similarities:

$$\triangle TP'M \overset{+}{\sim} \triangle TSP$$
$$\triangle TMP \overset{+}{\sim} \triangle TP'S.$$

Putting everything together, we find that

$$TP'MP \overset{+}{\sim} TO'YO.$$

Then by shooting lemma, $YO' = YX \cdot YT$, so O indeed lies on the claimed circle.

As the line $\overline{O'O}$ may be any line through X other than line XY (one takes S to be the reflection of T across this line) one concludes the only two non-achievable points are the diametrically opposite ones on line XY of this circle (because this leads to the only degenerate situation where $S = T$).

§2.2 Solution to TSTST 5

Let p be a prime. Prove that in a complete graph with $1000p$ vertices whose edges are labelled with integers, one can find a cycle whose sum of labels is divisible by p.

Select $p-1$ disjoint triangles arbitrarily. If any of these triangles have 0 sum modulo p we are done. Otherwise, we may label the vertices u_i, x_i, and v_i (where $1 \le i \le p-1$) in such a way that $u_ix_i + x_iv_i \neq u_iv_i$.

Let $A_i = \{u_ix_i + x_iv_i, u_iv_i\}$. We can show that $|A_1 + A_2 + \dots + A_t| \ge \min\{p, t+1\}$ for each $1 \le t \le p-1$, by using induction on t alongside Cauchy-Davenport. So, $A_1 + A_2 + \dots + A_{p-1}$ spans all of $\mathbb{Z}_p$. All that's left to do is join the triangles together to form a cycle, and then delete either u_ix_i, x_iv_i or u_iv_i from each triangle in such a way that the final sum is 0 mod p.

§2.3 Solution to TSTST 6

Let $\mathbb{N}$ be the set of positive integers. Find all functions $f : \mathbb{N} \to \mathbb{N}$ that satisfy the equation

$$f^{abc-a}(abc) + f^{abc-b}(abc) + f^{abc-c}(abc) = a + b + c$$

for all $a, b, c \geq 2$. (Here f^k means f applied k times.)

The answer is $f(n) = n - 1$ for $n \geq 3$ with $f(1)$ and $f(2)$ arbitrary; check these work.

Lemma

We have $f^{t^2-t}(t^2) = t$ for all t.

Proof. We say $1 \leq k \leq 8$ is good if $f^{t^9-t^k}(t^9) = t^k$ for all t. First, we observe that

$$f^{t^9-t^3}(t^9) = t^3 \quad \text{and} \quad f^{t^3-t}(t^3) = t \implies f^{t^9-t}(t^9) = t.$$

so $k = 1$ and $k = 3$ are good. Then taking $(a, b, c) = (t, t^4, t^4)$, $(a, b, c) = (t^2, t^3, t^4)$ gives that $k = 4$ and $k = 2$ are good, respectively. The lemma follows from this $k = 1$ and $k = 2$ being good. $\square$

Now, letting $t = abc$ we combine

$$\begin{aligned} f^{t-a}(a) + f^{t-b}(b) + f^{t-c}(c) &= a + b + c \\ f^{t^2-ab}(t^2) + f^{t^2-t}(t^2) + f^{t^2-c}(t^2) &= ab + t + c \\ \implies \left[f^{t-a}(t) - a\right] + \left[f^{t-b}(t) - b\right] &= \left[f^{t-ab}(t) - ab\right] \end{aligned}$$

by subtracting and applying the lemma repeatedly. In other words, we have proven the second lemma:

Lemma

Let t be fixed, and define $g_t(n) = f^{t-n}(t) - n$ for $n < t$. If $a, b \geq 2$ and $ab \mid t$, $ab < t$, then $g_t(a) + g_t(b) = g_t(ab)$.

Now let $a, b \geq 2$ be arbitrary, and let $p > q > \max\{a, b\}$ be primes. Suppose $s = a^p b^q$ and $t = s^2$; then

$$pg_t(a) + qg_t(b) = g_t\left(a^p b^q\right) = g_t(s) = f^{s^2-s}(s) - s = 0.$$

Now

$$q \mid g_t(a) > -a \quad \text{and} \quad p \mid g_t(b) > -b \implies g_t(a) = g_t(b) = 0.$$

and so we conclude $f^{t-a}(t) = a$ and $f^{t-b}(t) = b$ for $a, b \geq 2$.

In particular, if $a = n$ and $b = n + 1$ then we deduce $f(n + 1) = n$ for all $n \geq 2$, as desired.

Remark. If you let $c = (ab)^2$ after the first lemma, you recover the 2-variable version!

§3 Solutions to Day 3

§3.1 Solution to TSTST 7

A country has n cities, labelled $1, 2, 3, \dots, n$. It wants to build exactly $n-1$ roads between certain pairs of cities so that every city is reachable from every other city via some sequence of roads. However, it is not permitted to put roads between pairs of cities that have labels differing by exactly 1, and it is also not permitted to put a road between cities 1 and n. Let T_n be the total number of possible ways to build these roads.

(a) For all odd n, prove that T_n is divisible by n.

(b) For all even n, prove that T_n is divisible by $n/2$.

You can just spin the tree!

Fixing n, the group $G = \mathbb{Z}/n\mathbb{Z}$ acts on the set of trees by rotation (where we imagine placing $1, 2, \dots, n$ along a circle).

Claim — For odd n, all trees have trivial stabilizer.

Proof. One way to see this is to look at the degree sequence. Suppose g^e fixes a tree T. Then so does g^k, for $k = \gcd(e, n)$. Then it follows that n/k divides $\sum_v \deg v = 2n - 2$. Since $\gcd(2n-2, n) = 1$ we must then have $k = n$. □

The proof for even n is identical except that $\gcd(2n-2, n) = 2$ and hence each tree either has stabilizer with size ≤ 2.

There is also a proof using linear algebra, using Kirchoff's tree formula. (Overkill.)

§3.2 Solution to TSTST 8

Define a function $f\colon \mathbb{N} \to \mathbb{N}$ by $f(1) = 1$, $f(n+1) = f(n) + 2^{f(n)}$ for every positive integer n. Prove that $f(1), f(2), \dots, f(3^{2013})$ leave distinct remainders when divided by 3^{2013}.

I'll prove by induction on $k \ge 1$ that any 3^k consecutive values of f produce distinct residues modulo 3^k. The base case $k = 1$ is easily checked (f is always odd, hence f cycles 1, 0, 2 mod 3).

For the inductive step, assume it's true up to k. Since $2^\bullet \pmod{3^{k+1}}$ cycles every $2 \cdot 3^k$, and f is always odd, it follows that

$$\begin{aligned} f(n+3^k) - f(n) &= 2^{f(n)} + 2^{f(n+1)} + \dots + 2^{f(n+3^k-1)} \pmod{3^{k+1}} \\ &\equiv 2^1 + 2^3 + \dots + 2^{2\cdot 3^k - 1} \pmod{3^{k+1}} \\ &= 2 \cdot \frac{4^{3^k} - 1}{4 - 1}. \end{aligned}$$

Hence

$$f(n+3^k) - f(n) \equiv C \pmod{3^{k+1}} \qquad \text{where} \qquad C = 2 \cdot \frac{4^{3^k} - 1}{4 - 1}$$

noting that C does not depend on n. Exponent lifting gives $\nu_3(C) = k$ hence $f(n)$, $f(n+3^k)$, $f(n+2\cdot 3^k)$ differ mod 3^{k+1} for all n, and the inductive hypothesis now solves the problem.

§3.3 Solution to TSTST 9

Let r be a rational number in the interval $[-1,1]$ and let $\theta = \cos^{-1} r$. Call a subset S of the plane good if S is unchanged upon rotation by θ around any point of S (in both clockwise and counterclockwise directions). Determine all values of r satisfying the following property: The midpoint of any two points in a good set also lies in the set.

The answer is that r has this property if and only if $r = \frac{4n-1}{4n}$ for some integer n.

Throughout the solution, we will let $r = \frac{a}{b}$ with $b > 0$ and $\gcd(a,b) = 1$. We also let

$$\omega = e^{i\theta} = \frac{a}{b} \pm \frac{\sqrt{b^2-a^2}}{b} i.$$

This means we may work with complex multiplication in the usual way; the rotation of z through center c is given by $z \mapsto \omega(z-c)+c$.

For most of our proof, we start by constructing a good set as follows.

- Start by letting $S_0 = \{0,1\}$.
- Let S_i consist of S_{i-1} plus all points that can be obtained by rotating a point of S_{i-1} through a different point of S_{i-1} (with scale factor ω).
- Let $S_\infty = \bigcup_{i \ge 0} S_i$.

The set S_∞ is the (minimal, by inclusion) good set containing 0 and 1. We are going to show that for most values of r, we have $\frac{1}{2} \notin S_\infty$.

Claim — If b is odd, then $\frac{1}{2} \notin S_\infty$.

Proof. Idea: denominators that appear are always odd.

Consider the ring

$$A = \mathbb{Z}_{\{b\}} = \left\{ \frac{s}{t} \mid s,t \in \mathbb{Z}, t \mid b^\infty \right\}$$

which consists of all rational numbers whose denominators divide b^∞. Then, $0, 1, \omega \in A[\sqrt{b^2-a^2}]$ and hence $S_\infty \subseteq A[\sqrt{b^2-a^2}]$ too. (This works even if $\sqrt{b^2-a^2} \in \mathbb{Z}$, in which case $S_\infty \subseteq A = A[\sqrt{b^2-a^2}]$.)

But $\frac{1}{2} \notin A[\sqrt{b^2-a^2}]$. $\square$

Claim — If b is even and $b - a \neq 1$, then $\frac{1}{2} \notin S_\infty$.

Proof. Idea: take modulo a prime dividing $b-a$.

Let $D = b^2 - a^2 \equiv 3 \pmod 4$. Let p be a prime divisor of $b-a$ with odd multiplicity. Because $\gcd(a,b) = 1$, we have $p \neq 2$ and $p \nmid b$.

Consider the ring

$$A = \mathbb{Z}_{(p)} = \left\{ \frac{s}{t} \mid s,t \in \mathbb{Z}, p \perp t \right\}$$

which consists of all rational numbers whose denominators are coprime to p. Then, $0, 1, \omega \in A[\sqrt{-D}]$ and hence $S_\infty \subseteq A[\sqrt{-D}]$ too.

Now, there is a well-defined "mod-p" ring homomorphism

$$\Psi \colon A[\sqrt{-D}] \to \mathbb{F}_p \quad \text{by} \quad x + y\sqrt{-D} \mapsto x \bmod p$$

which commutes with addition and multiplication (as $p \mid D$). Under this map,

$$\omega \mapsto \frac{a}{b} \bmod p = 1.$$

Consequently, the rotation $z \mapsto \omega(z-c)+c$ is just the identity map modulo p. In other words, the pre-image of any point in S_∞ under Ψ must be either $\Psi(0)=0$ or $\Psi(1)=1$.

However, $\Psi(1/2)=1/2$ is neither of these. So this point cannot be achieved. □

Claim — Suppose $a = 2n-1$ and $b = 2n$ for n an odd integer. Then $\frac{1}{2} \notin S_\infty$

Proof. Idea: ω is "algebraic integer" sans odd denominators.

This time, we define the ring

$$B = \mathbb{Z}_{(2)} = \left\{ \frac{s}{t} \mid s,t \in \mathbb{Z}, t \text{ odd} \right\}$$

of rational numbers with odd denominator. We carefully consider the ring $B[\omega]$ where

$$\omega = \frac{2n-1 \pm \sqrt{1-4n}}{2n}.$$

So $S_\infty \subseteq B[\omega]$ as $0,1,\omega \in B[\omega]$.

I claim that $B[\omega]$ is an integral extension of B; equivalently that ω is integral over B. Indeed, it is the ω root of the monic polynomial

$$(T-1)^2 + \frac{1}{n}(T-1) - \frac{1}{n} = 0$$

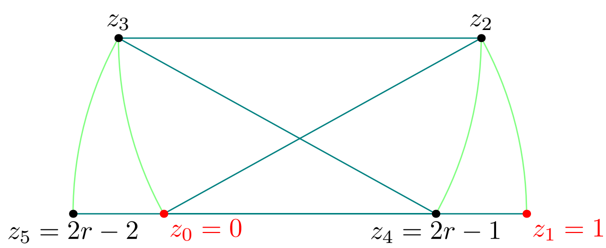
where $\frac{1}{n} \in B$ makes sense as n is odd.

On the other hand, $\frac{1}{2}$ is not integral over B so it is not an element of $B[\omega]$. □

It remains to show that if $r = \frac{4n-1}{4n}$, then goods sets satisfy the midpoint property. Again starting from the points $z_0 = 0$, $z_1 = 1$ construct the sequence

$$\begin{aligned} z_2 &= \omega(z_1 - z_0) + z_0 \\ z_3 &= \omega^{-1}(z_0 - z_2) + z_2 \\ z_4 &= \omega^{-1}(z_2 - z_3) + z_3 \\ z_5 &= \omega(z_3 - z_4) + z_4 \end{aligned}$$

as shown in the diagram below.



This construction shows that if we have the length-one segment $\{0,1\}$ then we can construct the length-one segment $\{2r-2, 2r-1\}$. In other words, we can shift the segment to the left by

$$1-(2r-1) = 2(1-r) = \frac{1}{2n}.$$

Repeating this construction n times gives the desired midpoint $\frac{1}{2}$.