TSTST 2012 Solution Notes

Lincoln, Nebraska

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March 28, 2021

Contents

§1 Solutions to Day 1

§1.1 Solution to TSTST 1

Determine all infinite strings of letters with the following properties:

(a) Each letter is either T or S,

(b) If position i and j both have the letter T, then position $i+j$ has the letter S,

(c) There are infinitely many integers k such that position $2k-1$ has the kth T.

We wish to find all infinite sequences $a_1, a_2, \dots$ of positive integers satisfying the following properties:

(a) $a_1 < a_2 < a_3 < \cdots$,

(b) there are no positive integers i, j, k, not necessarily distinct, such that $a_i + a_j = a_k$,

(c) there are infinitely many k such that $a_k = 2k-1$.

If $a_k = 2k-1$ for some $k > 1$, let $A_k = \{a_1, a_2, \dots, a_k\}$. By (b) and symmetry, we have

$$2k-1 \ge \frac{|A_k - A_k| - 1}{2} + |A_k| \ge \frac{2|A_k| - 2}{2} + |A_k| = 2k-1.$$

But in order for $|A_k - A_k| = 2|A_k| - 1$, we must have A_k an arithmetic progression, whence $a_n = 2n-1$ for all n by taking k arbitrarily large.

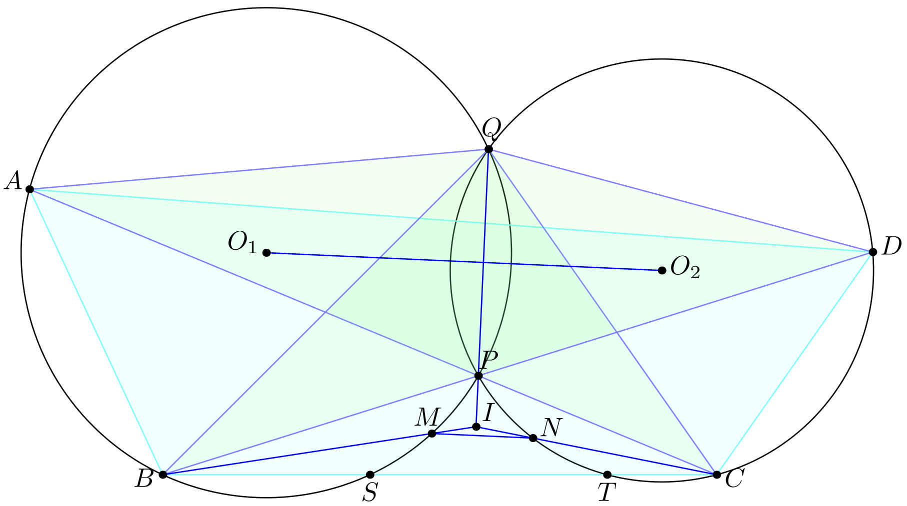
§1.2 Solution to TSTST 2

Let $ABCD$ be a quadrilateral with $AC = BD$. Diagonals AC and BD meet at P. Let ω_1 and O_1 denote the circumcircle and circumcenter of triangle ABP. Let ω_2 and O_2 denote the circumcircle and circumcenter of triangle CDP. Segment BC meets ω_1 and ω_2 again at S and T (other than B and C), respectively. Let M and N be the midpoints of minor arcs $\widehat{SP}$ (not including B) and $\widehat{TP}$ (not including C). Prove that $\overline{MN} \parallel \overline{O_1O_2}$.

Let Q be the second intersection point of ω_1, ω_2. Suffice to show $\overline{QP} \perp \overline{MN}$. Now Q is the center of a spiral *congruence* which sends $\overline{AC} \mapsto \overline{BD}$. So $\triangle QAB$ and $\triangle QCD$ are similar isosceles. Now,

$$\measuredangle QPA = \measuredangle QBA = \measuredangle DCQ = \measuredangle DPQ$$

and so $\overline{QP}$ is bisects $\angle BPC$.



Now, let $I = \overline{BM} \cap \overline{CN} \cap \overline{PQ}$ be the incenter of $\triangle PBC$. Then $IM \cdot IB = IP \cdot IQ = IN \cdot IC$, so $BMNC$ is cyclic, meaning $\overline{MN}$ is antiparallel to $\overline{BC}$ through $\angle BIC$. Since $\overline{QPI}$ passes through the circumcenter of $\triangle BIC$, it follows now $\overline{QPI} \perp \overline{MN}$ as desired.

§1.3 Solution to TSTST 3

Let $\mathbb{N}$ be the set of positive integers. Let $f\colon \mathbb{N} \to \mathbb{N}$ be a function satisfying the following two conditions:

(a) $f(m)$ and $f(n)$ are relatively prime whenever m and n are relatively prime.

(b) $n \le f(n) \le n + 2012$ for all n.

Prove that for any natural number n and any prime p, if p divides $f(n)$ then p divides n.

First short solution, by Jeffrey Kwan Let p_0, p_1, p_2, ... denote the sequence of all prime numbers, in any order. Pick *any* primes q_i such that

$$q_0 \mid f(p_0), \quad q_1 \mid f(p_1), \quad q_2 \mid f(p_2), \quad \text{etc.}$$

This is possible since each f value above exceeds 1. Also, since by hypothesis the $f(p_i)$ are pairwise coprime, the primes q_i are all pairwise distinct.

> **Claim** — We must have $q_i = p_i$ for each i. (Therefore, $f(p_i)$ is a power of p_i for each i.)

Proof. Assume to the contrary that $q_0 \ne p_0$. By changing labels if necessary, assume $\min(p_1, p_2, \dots, p_{2012}) > 2012$. Then by Chinese remainder theorem we can choose an integer m such that

$$\begin{aligned} m + i &\equiv 0 \pmod{q_i} \\ m &\not\equiv 0 \pmod{p_i} \end{aligned}$$

for $0 \le i \le 2012$. But now $f(m)$ should be coprime to all $f(p_i)$, ergo coprime to $q_0 q_1 \dots q_{2012}$, violating $m \le f(m) \le m + 2012$. $\square$

All that is left to do is note that whenever $p \nmid n$, we have $\gcd(f(p), f(n)) = 1$, hence $p \nmid f(n)$. This is the contrapositive of the problem statement.

Second solution with a grid Fix n and p, and assume for contradiction $p \nmid n$.

> **Claim** — There exists a large integer N with $f(N) = N$, that also satisfies $N \equiv 1 \pmod{n}$ and $N \equiv 0 \pmod{p}$.

Proof. We'll need to pick both N and an ancillary integer M. Here is how: pick $2012 \cdot 2013$ distinct primes $q_{i,j} > n + p + 2013$ for every $i = 1, \dots, 2012$ and $j = 0, \dots, 2012$, and use it to fill in the following table:

$$\begin{array}{c|cccc} & N+1 & N+2 & \dots & N+2012 \\ \hline M & q_{0,1} & q_{0,2} & \dots & q_{0,2012} \\ M+1 & q_{1,1} & q_{1,2} & \dots & q_{1,2012} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ M+2012 & q_{2012,1} & q_{2012,2} & \dots & q_{2012,2012} \end{array}.$$

By the Chinese Remainder Theorem, we can construct N such that $N + 1 \equiv 0 \pmod{q_{i,1}}$ for every i, and similarly for $N + 2$, and so on. Moreover, we can also tack on the extra conditions $N \equiv 0 \pmod{p}$ and $N \equiv 1 \pmod{n}$ we wanted.

Notice that N cannot be divisible by any of the $q_{i,j}$'s, since the $q_{i,j}$'s are greater than 2012.

After we've chosen N, we can pick M such that $M \equiv 0 \pmod{q_{0,j}}$ for every j, and similarly $M+1 \equiv 0 \pmod{q_{1,j}}$, et cetera. Moreover, we can tack on the condition $M \equiv 1 \pmod{N}$, which ensures $\gcd(M, N) = 1$.

What does this do? We claim that $f(N) = N$ now. Indeed $f(M)$ and $f(N)$ are relatively prime; but look at the table! The table tells us that $f(M)$ must have a common factor with each of $N+1, \dots, N+2012$. So the only possibility is that $f(N) = N$. $\square$

Now we're basically done. Since $N \equiv 1 \pmod{n}$, we have $\gcd(N, n) = 1$ and hence $1 = \gcd(f(N), f(n)) = \gcd(N, f(n))$. But $p \mid N$ and $p \mid f(n)$, contradiction.

§2 Solutions to Day 2

§2.1 Solution to TSTST 4

In scalene triangle ABC, let the feet of the perpendiculars from A to $\overline{BC}$, B to $\overline{CA}$, C to $\overline{AB}$ be A_1, B_1, C_1, respectively. Denote by A_2 the intersection of lines BC and B_1C_1. Define B_2 and C_2 analogously. Let D, E, F be the respective midpoints of sides $\overline{BC}$, $\overline{CA}$, $\overline{AB}$. Show that the perpendiculars from D to $\overline{AA_2}$, E to $\overline{BB_2}$ and F to $\overline{CC_2}$ are concurrent.

We claim that they pass through the orthocenter H. Indeed, consider the circle with diameter $\overline{BC}$, which circumscribes quadrilateral BCB_1C_1 and has center D. Then by Brokard theorem, $\overline{AA_2}$ is the polar of line H. Thus $\overline{DH} \perp \overline{AA_2}$.

§2.2 Solution to TSTST 5

A rational number x is given. Prove that there exists a sequence $x_0, x_1, x_2, \dots$ of rational numbers with the following properties:

(a) $x_0 = x$;

(b) for every $n \ge 1$, either $x_n = 2x_{n-1}$ or $x_n = 2x_{n-1} + \frac{1}{n}$;

(c) x_n is an integer for some n.

Think of the sequence as a process over time. We'll show that:

Claim — At any given time t, if the denominator of x_t is some odd prime power $q = p^e$, then we can delete a factor of p from the denominator, while only adding powers of two to the denominator.

(Thus we can just delete off all the odd primes one by one and then double appropriately many times.)

Proof. The idea is to add only fractions of the form $(2^k q)^{-1}$.

Indeed, let n be large, and suppose $t < 2^{r+1}q < 2^{r+2}q < \dots < 2^{r+m}q < n$. For some binary variables $\varepsilon_i \in \{0,1\}$ we can have

$$x_n = 2^{n-t}x_t + c_1 \cdot \frac{\varepsilon_1}{q} + c_2 \cdot \frac{\varepsilon_2}{q} \dots + c_s \cdot \frac{\varepsilon_m}{q}$$

where c_i is some power of 2 (to be exact, $c_i = \frac{2^{n-2^{r+i}q}}{2^{r+1}}$, but the exact value doesn't matter).

If m is large enough the set $\{0, c_1\} + \{0, c_2\} + \dots + \{0, c_m\}$ spans everything modulo p. (Actually, Cauchy-Davenport implies $m = p$ is enough, but one can also just use Pigeonhole to notice some residue appears more than p times, for $m = O(p^2)$.) Thus we can eliminate one factor of p from the denominator, as desired. $\square$

§2.3 Solution to TSTST 6

Positive real numbers x, y, z satisfy $xyz + xy + yz + zx = x + y + z + 1$. Prove that

$$\frac{1}{3}\left(\sqrt{\frac{1+x^2}{1+x}} + \sqrt{\frac{1+y^2}{1+y}} + \sqrt{\frac{1+z^2}{1+z}}\right) \le \left(\frac{x+y+z}{3}\right)^{5/8}.$$

The key is the identity

$$\begin{aligned}
\frac{x^2+1}{x+1} &= \frac{(x^2+1)(y+1)(z+1)}{(x+1)(y+1)(z+1)} \\
&= \frac{x(xyz+xy+xz)+x^2+yz+y+z+1}{2(1+x+y+z)} \\
&= \frac{x(x+y+z+1-yz)+x^2+yz+y+z+1}{2(1+x+y+z)} \\
&= \frac{(x+y)(x+z)+x^2+(x-xyz+y+z+1)}{2(1+x+y+z)} \\
&= \frac{2(x+y)(x+z)}{2(1+x+y+z)} \\
&= \frac{(x+y)(x+z)}{1+x+y+z}.
\end{aligned}$$

> **Remark.** The "trick" can be rephrased as $(x^2+1)(y+1)(z+1) = 2(x+y)(x+z)$.

After this, straight Cauchy in the obvious way will do it (reducing everything to an inequality in $s = x + y + z$). One writes

$$\begin{aligned}
\left(\sum_{\text{cyc}} \frac{\sqrt{(x+y)(x+z)}}{\sqrt{1+s}}\right)^2 &\le \frac{\left(\sum_{\text{cyc}} x+y\right)\left(\sum_{\text{cyc}} x+z\right)}{1+s} \\
&= \frac{4s^2}{1+s}
\end{aligned}$$

and so it suffices to check that $\frac{4s^2}{1+s} \le 9(s/3)^{5/4}$, which is true because

$$(s/3)^5 \cdot 9^4 \cdot (1+s)^4 - (4s^2)^4 = s^5(s-3)^2(27s^2+14s+3) \ge 0.$$

§3 Solutions to Day 3

§3.1 Solution to TSTST 7

Triangle ABC is inscribed in circle Ω. The interior angle bisector of angle A intersects side BC and Ω at D and L (other than A), respectively. Let M be the midpoint of side BC. The circumcircle of triangle ADM intersects sides AB and AC again at Q and P (other than A), respectively. Let N be the midpoint of segment PQ, and let H be the foot of the perpendicular from L to line ND. Prove that line ML is tangent to the circumcircle of triangle HMN.

By angle chasing, equivalent to show $\overline{MN} \parallel \overline{AD}$, so discard the point H. We now present a three solutions.

First solution using vectors We first contend that:

Claim — We have $QB = PC$.

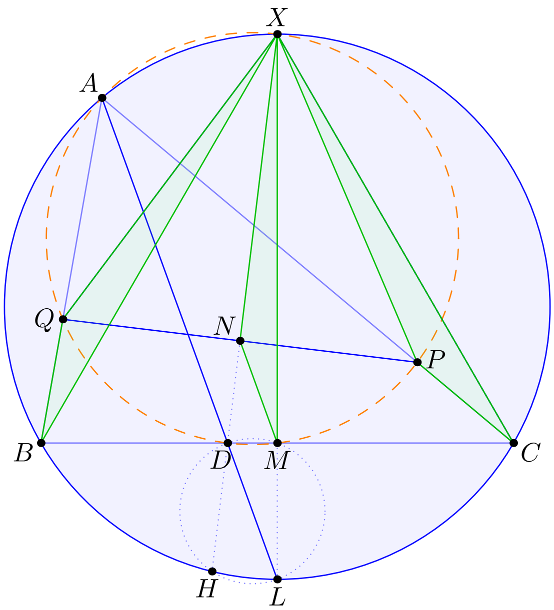
Proof. Power of a Point gives $BM \cdot BD = AB \cdot QB$. Then use the angle bisector theorem. □

Now notice that the vector

$$\overrightarrow{MN} = \frac{1}{2}\left(\overrightarrow{BQ} + \overrightarrow{CP}\right)$$

which must be parallel to the angle bisector since $\overrightarrow{BQ}$ and $\overrightarrow{CP}$ have the same magnitude.

Second solution using spiral similarity let X be the arc midpoint of BAC. Then $ADMX$ is cyclic with diameter $\overline{AM}$, and hence X is the Miquel point X of $QBPC$ is the midpoint of arc BAC. Moreover $\overline{XND}$ collinear (as $XP = XQ$, $DP = DQ$) on (APQ).



Then $\triangle XNM \sim \triangle XPC$ spirally, and

$$\measuredangle XMN = \measuredangle XCP = \measuredangle XCA = \measuredangle XLA$$

thus done.

Third solution using barycentrics (mine) Once reduced to $\overline{MN} \parallel \overline{AB}$, straight bary will also work. By power of a point one obtains

$$\begin{aligned} P &= \left(a^2 : 0 : 2b(b+c) - a^2\right) \\ Q &= \left(a^2 : 2c(b+c) - a^2 : 0\right) \\ \implies N &= \left(a^2(b+c) : 2bc(b+c) - ba^2 : 2bc(b+c) - ca^2\right). \end{aligned}$$

Now the point at infinity along $\overline{AD}$ is $(-(b+c) : b : c)$ and so we need only verify

$$\det \begin{bmatrix} a^2(b+c) & 2bc(b+c) - ba^2 & 2bc(b+c) - ca^2 \\ 0 & 1 & 1 \\ -(b+c) & b & c \end{bmatrix} = 0$$

which follows since the first row is $-a^2$ times the third row plus $2bc(b+c)$ times the second row.

§3.2 Solution to TSTST 8

Let n be a positive integer. Consider a triangular array of nonnegative integers as follows:

$$\begin{array}{lccccccccc}
\text{Row 1:} & & & & & a_{0,1} & & & & \\
\text{Row 2:} & & & & a_{0,2} & & a_{1,2} & & & \\
\vdots & & & \iddots & & \vdots & & \ddots & & \\
\text{Row } n-1\text{:} & & a_{0,n-1} & & a_{1,n-1} & & \dots & & a_{n-2,n-1} & \\
\text{Row } n\text{:} & a_{0,n} & & a_{1,n} & & a_{2,n} & & \dots & & a_{n-1,n}
\end{array}$$

Call such a triangular array *stable* if for every $0 \le i < j < k \le n$ we have

$$a_{i,j} + a_{j,k} \le a_{i,k} \le a_{i,j} + a_{j,k} + 1.$$

For $s_1, \dots, s_n$ any nondecreasing sequence of nonnegative integers, prove that there exists a unique stable triangular array such that the sum of all of the entries in row k is equal to s_k.

Firstly, here are illustrative examples showing the arrays for $(s_1, s_2, s_3, s_4) = (2, 5, 9, x)$ where $9 \le x \le 14$. (The array has been left justified.)

$$\begin{bmatrix} 2 & \swarrow & & \\ 4 & 1 & \swarrow & \\ 5 & 3 & 1 & \swarrow \\ 5 & 3 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & \swarrow & & \\ 4 & 1 & \swarrow & \\ 5 & 3 & 1 & \swarrow \\ \mathbf{6} & 3 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & \swarrow & & \\ 4 & 1 & \swarrow & \\ 5 & 3 & 1 & \swarrow \\ 6 & 3 & \mathbf{2} & 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & \swarrow & & \\ 4 & 1 & \swarrow & \\ 5 & 3 & 1 & \swarrow \\ 6 & \mathbf{4} & 2 & 0 \end{bmatrix} \begin{bmatrix} 2 & \swarrow & & \\ 4 & 1 & \swarrow & \\ 5 & 3 & 1 & \swarrow \\ 6 & 4 & 2 & \mathbf{1} \end{bmatrix} \begin{bmatrix} 2 & \swarrow & & \\ 4 & 1 & \swarrow & \\ 5 & 3 & 1 & \swarrow \\ \mathbf{7} & 4 & 2 & 1 \end{bmatrix}$$

Now we outline the proof. By induction on n, we may assume the first $n-1$ rows are fixed. Now, let $N = s_n$ vary. Now, we prove our result by (another) induction on $N \ge s_{n-1}$.

The base case $N = s_{n-1}$ is done by copying the $n-1$st row and adding a zero at the end. This is also unique, since $a_{i,n} \ge a_{i-1,n} + a_{n-1,n}$ for all $i = 0, \dots, n-2$, whence $\sum a_{i,n} \ge s_{n-1}$ follows.

Now the inductive step is based on the following lemma, which illustrates the idea of a "unique increasable entry".

Lemma

Fix a stable array Construct a tournament on the n entries of the last row as follows: for $i < j$,

- $a_{i,n} \to a_{j,n}$ if $a_{i,n} = a_{i,j} + a_{j,n}$, and
- $a_{j,n} \to a_{i,n}$ if $a_{i,n} = a_{i,j} + a_{j,n} + 1$.

Then this tournament is transitive. Also, except for $N = s_{n-1}$, a 0 entry is never a source.

Intuitively, $a_{i,n} \to a_{j,n}$ if $a_{i,n}$ blocks $a_{j,n}$ from increasing. For instance, in the example

$$\begin{bmatrix} 2 & \swarrow & & \\ 4 & 1 & \swarrow & \\ 5 & 3 & 1 & \swarrow \\ \mathbf{6} & 3 & 1 & 0 \end{bmatrix}$$

the tournament is $1 \to 3 \to 0 \to 6$.

Proof of lemma. Let $0 \le i < j < k < n$ be indices. Let $x = a_{i,n}$, $y = a_{j,n}$, $z = a_{k,n}$, $p = a_{i,j}$, $s = a_{i,k}$, $q = a_{j,k}$. Picture:

$$\begin{bmatrix} p & \swarrow & \\ s & q & \swarrow \\ x & y & z \end{bmatrix}$$

If $x \to y \to z \to x$ happens, that means $x = y + p$, $y = q + z$, $x = s + z + 1$, which gives $s = p + q - 1$, contradiction. Similarly if $x \leftarrow y \leftarrow z \leftarrow x$ then $x = y + p + 1$, $y = q + z + 1$, $x = s + z$, which gives $s = p + q + 2$, also contradiction. $\square$

Now this allows us to perform our induction. Indeed, to show existence from N to $N+1$ we take a source of the tournament above and increase it. Conversely, to show uniqueness for N, note that we can take the (nonzero) sink of the tournament and decrement it, which gives $N - 1$; our uniqueness inductive hypothesis now finishes.

Remark. Colin Tang found a nice proof of uniqueness:

$$s_k + \sum_{i=1}^{k-1} a_{0,i} \le k a_{0,k} \le s_k + \sum_{i=1} (a_{0,i} + 1)$$

and similarly for other entries.

§3.3 Solution to TSTST 9

Given a set S of n variables, a binary operation $\times$ on S is called *simple* if it satisfies $(x \times y) \times z = x \times (y \times z)$ for all $x, y, z \in S$ and $x \times y \in \{x, y\}$ for all $x, y \in S$. Given a simple operation $\times$ on S, any string of elements in S can be reduced to a single element, such as $xyz \to x \times (y \times z)$. A string of variables in S is called *full* if it contains each variable in S at least once, and two strings are *equivalent* if they evaluate to the same variable regardless of which simple $\times$ is chosen. For example xxx, xx, and x are equivalent, but these are only full if $n = 1$. Suppose T is a set of full strings such that any full string is equivalent to exactly one element of T. Determine the number of elements of T.

The answer is $(n!)^2$. In fact it is possible to essentially find all $\times$: one assigns a real number to each variable in S. Then $x \times y$ takes the larger of $\{x, y\}$, and in the event of a tie picks either "left" or "right", where the choice of side is fixed among elements of each size.

First solution (Steven Hao) The main trick is the two lemmas, which are not hard to show (and are motivated by our conjecture).

$$xx = x$$
$$xyxzx = xyzx.$$

Consequently, define a **double rainbow** to be the concatenation of two full strings of length n, of which there are $(n!)^2$. We claim that these form equivalence classes for T.

To see that any string s is equivalent to a double rainbow, note that $s = ss$, and hence using the second identity above repeatedly lets us reduce ss to a double rainbow.

To see two distinct double rainbows R_1 and R_2 aren't equivalent, one can use the construction mentioned in the beginning. Specifically, take two variables a and b which do not appear in the same order in R_1 and R_2. Then it's not hard to see that $abab$, $abba$, $baab$, $baba$ are pairwise non-equivalent by choosing "left" or "right" appropriately. Now construct $\times$ on the whole set by having a and b be the largest variables, so the rest of the variables don't matter in the evaluation of the string.

Second solution outline (Ankan Bhattacharya) We outline a proof of the characterization claimed earlier, which will also give the answer $(n!)^2$. We say $a \sim b$ if $ab \neq ba$. Also, say $a > b$ if $ab = ba = a$. The following are proved by finite casework, using the fact that $\{ab, bc, ca\}$ always has exactly two distinct elements for any different a, b, c.

- If $a > b$ and $b > c$ then $a > c$.
- If $a \sim b$ and $b \sim c$ then $ab = a$ if and only if $bc = b$.
- If $a \sim b$ and $b \sim c$ then $a \sim c$.
- If $a \sim b$ and $a > c$ then $b > c$.
- If $a \sim b$ and $c > a$ then $c > b$.

This gives us the total ordering on the elements and the equivalence classes by $\sim$. In this we way can check the claimed operations are the only ones.

We can then (as in the first solution) verify that every full string is equivalent to a unique double rainbow — but this time we prove it by simply considering all possible $\times$, because we have classified them all.