TSTST 2011 Solution Notes

Lincoln, Nebraska

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§1 Solutions to Day 1

§1.1 Solution to TSTST 1

Find all real-valued functions f defined on pairs of real numbers, having the following property: for all real numbers a, b, c, the median of $f(a,b), f(b,c), f(c,a)$ equals the median of a, b, c.

(The *median* of three real numbers, not necessarily distinct, is the number that is in the middle when the three numbers are arranged in nondecreasing order.)

The following solution is joint with Andrew He.

We prove the following main claim, from which repeated applications can deduce the problem.

Claim — Let $a < b < c$ be arbitrary. On $\{a,b,c\}^2$, f takes one of the following two forms, where the column indicates the x-value and the row indicates the y-value.

$$\begin{array}{c|ccc} f & a & b & c \\ \hline a & a & b & \ge c \\ b & \le a & b & \ge c \\ c & \le a & b & c \end{array} \qquad \text{or} \qquad \begin{array}{c|ccc} f & a & b & c \\ \hline a & a & \le a & \le a \\ b & b & b & b \\ c & \ge c & \ge c & c \end{array}$$

Proof. First, we of course have $f(x,x) = x$ for all x. Now:

- By considering the assertion for (a,a,c) and (a,c,c) we see that one of $f(a,c)$ and $f(c,a)$ is $\ge c$ and the other is $\le a$.
- Hence, by considering (a,b,c) we find that one of $f(a,b)$ and $f(b,c)$ must be b, and similarly for $f(b,a)$ and $f(c,b)$.
- Now, WLOG $f(b,a) = b$; we prove we get the first case.
- By considering (a,a,b) we deduce $f(a,b) \le a$, so $f(b,c) = b$ and then $f(c,b) \ge c$.
- Finally, considering (c,b,a) once again in conjunction with the first bullet, we arrive at the conclusion that $f(a,c) \le a$; similarly $f(c,a) \ge c$. □

From this it's easy to obtain that $f(x,y) \equiv x$ or $f(x,y) \equiv y$ are the only solutions.

§1.2 Solution to TSTST 2

Two circles ω_1 and ω_2 intersect at points A and B. Line ℓ is tangent to ω_1 at P and to ω_2 at Q so that A is closer to ℓ than B. Let X and Y be points on major arcs $\widehat{PA}$ (on ω_1) and AQ (on ω_2), respectively, such that $AX/PX = AY/QY = c$. Extend segments PA and QA through A to R and S, respectively, such that $AR = AS = c \cdot PQ$. Given that the circumcenter of triangle ARS lies on line XY, prove that $\angle XPA = \angle AQY$.

We begin as follows:

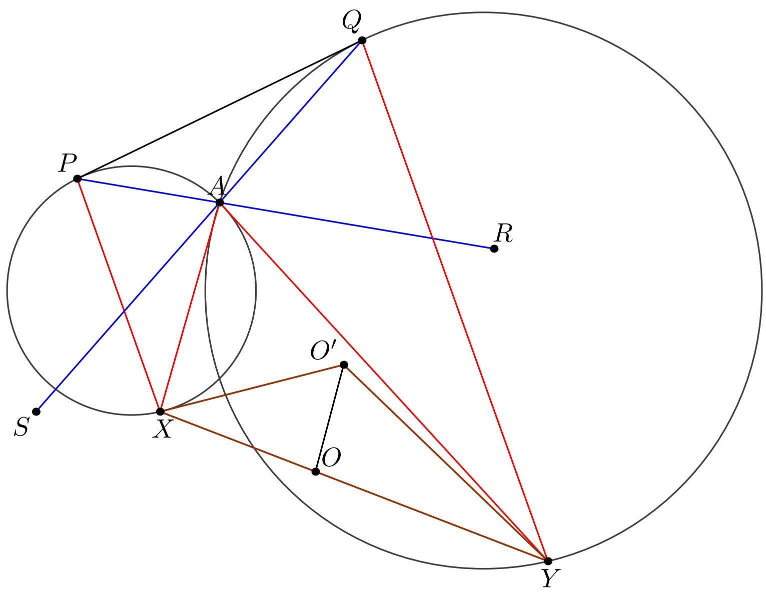
> **Claim** — There is a spiral similarity centered at X mapping AR to PQ. Similarly there is a spiral similarity centered at Y mapping SA to PQ.

Proof. Since $\measuredangle XAR = \measuredangle XAP = \measuredangle XPQ$, and $AR/AX = PQ/PX$ is given. □

Now the composition of the two spiral similarities

$$AR \overset{X}{\longmapsto} PQ \overset{Y}{\longmapsto} SA$$

must be a rotation, since $AR = AS$. The center of this rotation must coincide with the circumcenter O of $\triangle ARS$, which is known to lie on line XY.



Thus, we may let O' be the image under the rotation at X, so that

$$\triangle XPA \overset{+}{\sim} \triangle XO'O, \qquad \triangle YQA \overset{+}{\sim} \triangle YO'O.$$

Because

$$\frac{XO}{XO'} = \frac{XA}{XP} = c\frac{YQ}{YA} = \frac{YO}{YO'}$$

it follows $\overline{O'O}$ bisects $\angle XO'Y$. Finally, we have

$$\measuredangle XPA = \measuredangle XO'O = \measuredangle OO'Y = \measuredangle AQY.$$

Remark. Indeed, this also shows $\overline{XP} \parallel \overline{YQ}$; so the positive homothety from ω_1 to ω_2 maps P to Q and X to Y.

§1.3 Solution to TSTST 3

Prove that there exists a real constant c such that for any pair (x, y) of real numbers, there exist relatively prime integers m and n satisfying the relation

$$\sqrt{(x-m)^2 + (y-n)^2} < c \log(x^2 + y^2 + 2).$$

This is actually the same problem as USAMO 2014/6. Surprise!

§2 Solutions to Day 2

§2.1 Solution to TSTST 4

Acute triangle ABC is inscribed in circle ω. Let H and O denote its orthocenter and circumcenter, respectively. Let M and N be the midpoints of sides AB and AC, respectively. Rays MH and NH meet ω at P and Q, respectively. Lines MN and PQ meet at R. Prove that $\overline{OA} \perp \overline{RA}$.

Let MH and NH meet the nine-point circle again at P' and Q', respectively. Recall that H is the center of the homothety between the circumcircle and the nine-point circle. From this we can see that P and Q are the images of this homothety, meaning that

$$HQ = 2HQ' \quad \text{and} \quad HP = 2HP'.$$

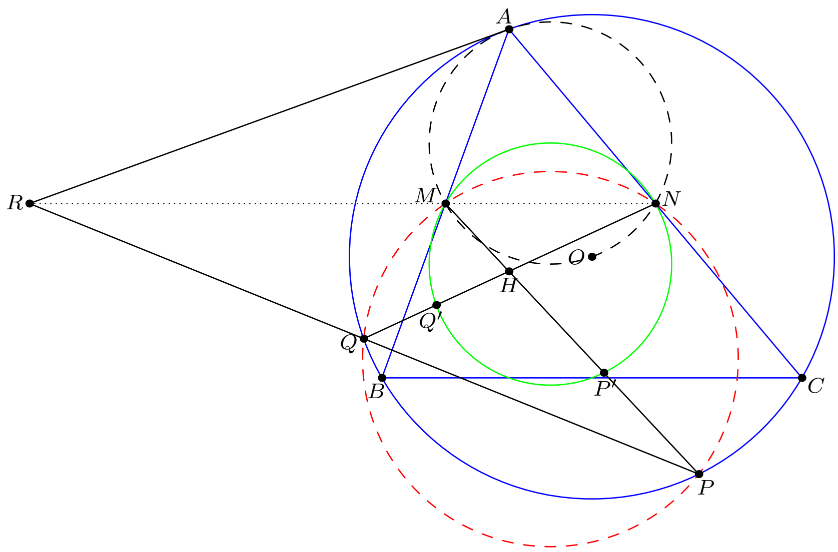
Since M, P', Q', N are cyclic, Power of a Point gives us

$$MH \cdot HP' = HN \cdot HQ'.$$

Multiplying both sides by two, we thus derive

$$HM \cdot HP = HN \cdot HQ.$$

It follows that the points M, N, P, Q are concyclic.



Let ω_1, ω_2, ω_3 denote the circumcircles of $MNPQ$, AMN, and ABC, respectively. The radical axis of ω_1 and ω_2 is line MN, while the radical axis of ω_1 and ω_3 is line PQ. Hence the line R lies on the radical axis of ω_2 and ω_3.

But we claim that ω_2 and ω_3 are internally tangent at A. This follows by noting the homothety at A with ratio 2 sends M to B and N to C. Hence the radical axis of ω_2 and ω_3 is a line tangent to both circles at A.

Hence $\overline{RA}$ is tangent to ω_3. Therefore, $\overline{RA} \perp \overline{OA}$.

§2.2 Solution to TSTST 5

At a certain orphanage, every pair of orphans are either friends or enemies. For every three of an orphan's friends, an even number of pairs of them are enemies. Prove that it's possible to assign each orphan two parents such that every pair of friends shares exactly one parent, but no pair of enemies does, and no three parents are in a love triangle (where each pair of them has a child).

Of course, we consider the graph with vertices as children and edges as friendships. Consider all the maximal cliques in the graph (i.e. repeatedly remove maximal cliques until no edges remain; thus all edges are in some clique).

Claim — Every vertex is in at most two maximal cliques.

Proof. Indeed, consider a vertex v adjacent to w_1 and w_2, but with w_1 not adjacent to w_2. Then by condition, any third vertex u must be adjacent to exactly one of w_1 and w_2. Moreover, given vertices u and u' adjacent to w_1, vertices u and u' are adjacent too. This proves the claim. $\square$

Now, for every maximal clique we assign a particular parent to all vertices in that clique, adding in additional distinct parents if there are any deficient children. This satisfies the friendship/enemy condition. Moreover, one can readily check that there are no love triangles: given children a, b, c such that a and b share a parent while a and c share another parent, according to the claim b and c can't share a third parent. This completes the problem.

Remark. This solution is highly motivated for the following reason: by experimenting with small cases, one quickly finds that given some vertices which form a clique, one *must* assign some particular parent to all vertices in that clique. That is, the requirements of the problem are sufficiently rigid that there is no room for freedom on our part, so we know *a priori* that an assignment based on cliques (as above) must work. From there we know exactly what to prove, and everything else follows through.

Ironically, the condition that there be no love triangle actually makes the problem easier, because it tells us exactly what to do!

§2.3 Solution to TSTST 6

Let a, b, c be real numbers in the interval $[0, 1]$ with $a+b, b+c, c+a \geq 1$. Prove that

$$1 \leq (1-a)^2 + (1-b)^2 + (1-c)^2 + \frac{2\sqrt{2}abc}{\sqrt{a^2+b^2+c^2}}.$$

The following approach is due to Ashwin Sah.

We will prove the inequality for any a, b, c the sides of a possibly degenerate triangle (which is implied by the condition), ignoring the particular constant 1. Homogenizing, we instead prove the problem in the following form:

Claim — We have

$$k^2 \leq (k-a)^2 + (k-b)^2 + (k-c)^2 + \frac{2\sqrt{2}abc}{\sqrt{a^2+b^2+c^2}}$$

for any a, b, c, k with (a, b, c) the sides of a possibly degenerate triangle.

Proof. For any particular (a, b, c) this is a quadratic in k of the form $2k^2 - 2(a+b+c)k + C \geq 0$; thus we will verify it holds for $k = \frac{1}{2}(a+b+c)$.

Letting $x = \frac{1}{2}(b+c-a)$ as is usual, the problem rearranges to In that case, the problem amounts to

$$(x+y+z)^2 \leq x^2 + y^2 + z^2 + \frac{2(x+y)(y+z)(z+x)}{\sqrt{x^2+y^2+z^2+xy+yz+zx}}$$

or equivalently

$$x^2 + y^2 + z^2 + xy + yz + zx \leq \left(\frac{(x+y)(y+z)(z+x)}{xy+yz+zx}\right)^2.$$

To show this, one may let $t = xy + yz + zx$, then using $(x+y)(x+z) = x^2 + B$ this becomes

$$t^2(x^2+y^2+z^2+t) \leq (x^2+t)(y^2+t)(z^2+t)$$

which is obvious upon expansion. □

Remark. The inequality holds actually for all real numbers a, b, c, with very disgusting proofs.

§3 Solutions to Day 3

§3.1 Solution to TSTST 7

Let ABC be a triangle. Its excircles touch sides BC, CA, AB at D, E, F. Prove that the perimeter of triangle ABC is at most twice that of triangle DEF.

Solution by August Chen: It turns out that it is enough to take the orthogonal projection of EF onto side BC (which has length $a - (s-a)(\cos B + \cos C)$) and sum cyclically:

$$\begin{aligned} -s + \sum_{\text{cyc}} EF &\ge -s + \sum_{\text{cyc}} \left[a - (s-a)\left(\cos B + \cos C\right)\right] \\ &= s - \sum_{\text{cyc}} a \cos A = \sum_{\text{cyc}} a \left(\frac{1}{2} - \cos A\right) \\ &= R \sum_{\text{cyc}} \sin A (1 - 2\cos A) \\ &= R \sum_{\text{cyc}} \left(\sin A - \sin 2A\right). \end{aligned}$$

Thus we're done upon noting that

$$\sin 2B + \sin 2C = \sin(B+C)\cos(B-C) = \sin A \cos(B-C) \le \sin A.$$

(Alternatively, one can avoid trigonometry by substituting $\cos A = \frac{b^2+c^2-a^2}{2bc}$ and doing some routine but long calculation.)

§3.2 Solution to TSTST 8

Let $x_0, x_1, \dots, x_{n_0-1}$ be integers, and let $d_1, d_2, \dots, d_k$ be positive integers with $n_0 = d_1 > d_2 > \dots > d_k$ and $\gcd(d_1, d_2, \dots, d_k) = 1$. For every integer $n \ge n_0$, define

$$x_n = \left\lfloor \frac{x_{n-d_1} + x_{n-d_2} + \dots + x_{n-d_k}}{k} \right\rfloor .$$

Show that the sequence (x_n) is eventually constant.

Note that if the initial terms are contained in some interval $[A, B]$ then they will remain in that interval. Thus the sequence is eventually periodic. Discard initial terms and let the period be T; we will consider all indices modulo T from now on.

Let M be the maximal term in the sequence (which makes sense since the sequence is periodic). Note that if $x_n = M$, we must have $x_{n-d_i} = M$ for all i as well. By taking a linear combination $\sum c_i d_i \equiv 1 \pmod T$ (possibly be Bezout's theorem, since $\gcd_i(d_i) = 1$), we conclude $x_{n-1} = M$, as desired.

§3.3 Solution to TSTST 9

Let n be a positive integer. Suppose we are given $2^n + 1$ distinct sets, each containing finitely many objects. Place each set into one of two categories, the red sets and the blue sets, so that there is at least one set in each category. We define the *symmetric difference* of two sets as the set of objects belonging to exactly one of the two sets. Prove that there are at least 2^n different sets which can be obtained as the symmetric difference of a red set and a blue set.

We can interpret the problem as working with binary strings of length $\ell \ge n + 1$, with ℓ the number of elements across all sets.

Let F be a field of cardinality 2^ℓ, hence $F \cong \mathbb{F}_2^{\oplus \ell}$.

Then, we can think of red/blue as elements of F, so we have some $B \subseteq F$, and an $R \subseteq F$. We wish to prove that $|B + R| \ge 2^n$. Want $|B + R| \ge 2^n$.

Equivalently, any element of a set X with $|X| = 2^n - 1$ should omit some element of $|B + R|$. To prove this: we know $|B| + |R| = 2^n + 1$, and define

$$P(b, r) = \prod_{x \in X} (b + r - x).$$

Consider $b^{|B|-1} r^{|R|-1}$. The coefficient of is $\binom{2^n-1}{|B|-1}$, which is odd (say by Lucas theorem), so the nullstellensatz applies.