USA TST 2025 Solutions

United States of America — Team Selection Test

Andrew Gu, Gopal Goel, and Luke Robitaille

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§1 Solutions to Day 1

§1.1 Solution to TST 1, by Anthony Wang

Let n be a positive integer. Ana and Banana play a game. Banana thinks of a function $f\colon \mathbb{Z} \to \mathbb{Z}$ and a prime number p. He tells Ana that f is nonconstant, $p < 100$, and $f(x+p) = f(x)$ for all integers x. Ana's goal is to determine the value of p. She writes down n integers $x_1, \dots, x_n$. After seeing this list, Banana writes down $f(x_1), \dots, f(x_n)$ in order. Ana wins if she can determine the value of p from this information. Find the smallest value of n for which Ana has a winning strategy.

The answer is $n = 83 + 89 - 1 = 171$.

In general, if Ana has to decide between periods from the set $\mathcal{P} := \{p_1 > p_2 > \dots > p_r\}$ of pairwise distinct relatively prime positive integers for $r \ge 3$, the answer is $p_2 + p_3 - 1$.

Bound

Suppose for the sake of contradiction that Ana has a winning sequence of integers $x_1, \dots, x_n$ with $n \le p_2 + p_3 - 2$. We will generate contradictions by providing two primes $p, q \in \mathcal{P}$ and assosciated nonconstant functions $f_p, f_q : \mathbb{Z} \to \mathbb{Z}$ with periods p and q respectively such that $f_p(x_i) = f_q(x_i)$ for all i.

Claim — There exists a prime $r \in \mathcal{P}$ such that for all primes $p \in \mathcal{P} \setminus \{r\}$, the set of integers $\{x_1, \dots, x_n\}$ forms a complete residue class modulo p (i.e. for all t, there exists i such that $x_i \equiv t \pmod p$).

Claim — Suppose for the sake of contradiction that such r didn't exist, and there were in fact two primes $p, q \in \mathcal{P}$ such that $\{x_1 \dots, x_n\}$ did not form a complete residue class modulo either p or q. Concretely, consider t, s such that there is no i with $x_i \equiv t \pmod p$ and not j with $x_j \equiv s \pmod q$.

Construct the functions $f_p, f_q : \mathbb{Z} \to \mathbb{Z}$ as

$$f_p(x) = \begin{cases} 0 & \text{if } x \not\equiv t \pmod p \\ 1 & \text{if } x \equiv t \pmod p \end{cases}$$

and

$$f_q(x) = \begin{cases} 0 & \text{if } x \not\equiv s \pmod q \\ 1 & \text{if } x \equiv s \pmod q \end{cases}.$$

We have $f_p(x_i) = f_q(x_i) = 0$ for all i, which is the desired contradiction.

Let r be the prime from the above claim. Let p, q be the largest two primes in $\mathcal{P} \setminus \{r\}$, so $n \le p + q - 2$. Construct the graph G_{pq} with vertex set $\{x_1, \dots, x_n\}$ and edge $x_i \sim x_j$ if $p \mid x_i - x_j$ or $q \mid x_i - x_j$. The following claim allows us to construct a pair of bad functions f_p, f_q.

Claim — The graph G_{pq} is disconnected.

Proof. Let G_p be the graph on vertex set $\{x_1, \dots, x_n\}$ with edge $x_i \sim x_j$ if $p \mid x_i - x_j$. Note that G_p is a collection of p disjoint cliques, one for each residuce class modulo p.

Prune the graph into G'_p, where each clique K_r is replaced by a path of edge-length $r-1$. Define G'_q similarly, and let G'_{pq} be the union of G'_p and G'_q.

Note that G_{pq} and G'_{pq} have the exact same connectivity properties. We have

$$|E(G'_{pq})| \le |E(G'_p)| + |E(G'_q)| = (n-p) + (n-q) \le n-2,$$

so G'_{pq} is disconnected, as desired. □

Suppose $A \sqcup B = \{x_1, \dots, x_n\}$ are sets of disjoint vertices in G_{pq}. Construct the functions $f_p, f_q : \mathbb{Z} \to \mathbb{Z}$ as

$$f_p(x) = \begin{cases} 0 & \text{if } x \equiv a \pmod p \text{ for some } a \in A \\ 1 & \text{if } x \equiv b \pmod p \text{ for some } b \in B \end{cases}$$

and

$$f_q(x) = \begin{cases} 0 & \text{if } x \equiv a \pmod q \text{ for some } a \in A \\ 1 & \text{if } x \equiv b \pmod q \text{ for some } b \in B \end{cases}.$$

These are well defined due to the fact that $p, q \nmid a-b$ for $a \in A$ and $b \in B$, and the fact that $A \sqcup B$ forms a complete residue class modulo p. Again, we have $f_p(x_i) = f_q(x_i)$ for all i, which is the desired contradiction.

Construction

Let $n = p_2 + p_3 - 1$. We claim that Ana has a winning strategy with the selection $x_i = p_1(i-1)$. Indeed, suppose that Banana writes down the values $y_1, \dots, y_n$ in order. We will show that Ana can always reconstruct p.

Claim — If $y_1 = \cdots = y_n$, then Ana can correctly guess $p = p_1$.

Proof. Suppose for the sake of contradiction that $p < p_1$. Then, since $x_1, \dots, x_n$ forms a complete residue class modulo p, f must be a constant function, which is the desired contradiction. □

We can now assume that $y_1, \dots, y_n$ are not all equal, which means $p \neq p_1$. Suppose for the sake of contradiction that there are two primes $q, r < p_1$ with assossiated nonconstant functions $f_p, f_q : \mathbb{Z} \to \mathbb{Z}$ with periods q and r respectively, such that $f_q(x_i) = f_r(x_i) = y_i$ for all i.

The following claim shows that $y_1, \dots, y_n$ must all be equal, which is the desired contradiction.

Claim — Let G be the graph on vertex set $\{0, \dots, q+r-2\}$ with edge $i \sim j$ if $|i-j| \in \{q, r\}$. The graph G is connected.

Proof. Note that G has $q+r-2$ edges, so it suffices to show that it has no cycles. Suppose for the sake of contradiction it had a cycle $c_1, \dots, c_k$ with $k \ge 3$ and indices taken mod k.

Suppose first that $c_{i+1} - c_i = q$. Then, $c_{i+2} - c_{i+1}$ cannot be $-q$ (else $c_i = c_{i+2}$), it cannot be r (else $c_{i+2} > q+r-2$), so $c_{i+2} - c_{i+1} \in \{q, -r\}$.

The same logic shows the following collated results:

$$c_{i+1} - c_i = q \implies c_{i+2} - c_{i+1} \in \{q, -r\}$$

$$\begin{aligned}
c_{i+1} - c_i = -q &\implies c_{i+2} - c_{i+1} \in \{-q, r\} \\
c_{i+1} - c_i = r &\implies c_{i+2} - c_{i+1} \in \{-q, r\} \\
c_{i+1} - c_i = -r &\implies c_{i+2} - c_{i+1} \in \{q, -r\}.
\end{aligned}$$

Thus, either all consecutive differences of vertices in the cycle are in $\{q, -r\}$, or all in $\{-q, r\}$. Assume the first case, proof is similar for second case.

Let a be the number of consecutive differences that are q, and b be the number that are $-r$. We see that $a + b = k$ and $qa - rb = 0$. The second condition implies that $a \geq r$ and $b \geq q$, so we have $k \geq q + r$, which is the desired contradiction since G has only $q + r - 1$ vertices. $\square$

§1.2 Solution to TST 2, by Holden Mui

Let $a_1, a_2, \dots$ and $b_1, b_2, \dots$ be sequences of real numbers for which $a_1 > b_1$ and

$$a_{n+1} = a_n^2 - 2b_n$$
$$b_{n+1} = b_n^2 - 2a_n$$

for all positive integers n. Prove that $a_1, a_2, \dots$ is eventually increasing (that is, there exists a positive integer N for which $a_k < a_{k+1}$ for all $k > N$).

Let r, s, and t be the complex roots of the polynomial $p(\lambda) = \lambda^3 - a_1\lambda^2 + b_1\lambda - 1$. By Vieta's formulas,

$$\begin{aligned} a_1 &= r + s + t \\ b_1 &= 1/r + 1/s + 1/t \\ 1 &= rst. \end{aligned}$$

Claim — For every positive integer n,

$$a_n = r^{2^{n-1}} + s^{2^{n-1}} + t^{2^{n-1}}$$

and

$$b_n = (1/r)^{2^{n-1}} + (1/s)^{2^{n-1}} + (1/t)^{2^{n-1}}.$$

Proof. The base case follows from Vieta's formulas above. For the inductive step, observe that $rst = 1$, so

$$\begin{aligned} a_{n+1} &= a_n^2 - 2b_n \\ &= \left(r^{2^{n-1}} + s^{2^{n-1}} + t^{2^{n-1}}\right)^2 - 2\left((1/r)^{2^{n-1}} + (1/s)^{2^{n-1}} + (1/t)^{2^{n-1}}\right) \\ &= \left(r^{2^{n-1}} + s^{2^{n-1}} + t^{2^{n-1}}\right)^2 - 2\left((st)^{2^{n-1}} + (tr)^{2^{n-1}} + (rs)^{2^{n-1}}\right) \\ &= r^{2^n} + s^{2^n} + t^{2^n} \end{aligned}$$

and similarly for b_{n+1}. □

Since $p(1) = b_1 - a_1 < 0$, p has a real root greater than 1; let r be the largest such root.

- If s and t are real, let $m = \max(|r|, |s|, |t|) > 1$ be the largest magnitude of the roots and $k \in \{1, 2, 3\}$ be the number of roots with that magnitude. Then asymptotically

  $$a_n = r^{2^{n-1}} + s^{2^{n-1}} + t^{2^{n-1}} \approx km^{2^{n-1}}$$

  which implies that $\{a_n\}$ is eventually increasing.

- If s and t are not real, they must be complex conjugates of each other, each with magnitude $\frac{1}{\sqrt{r}} < 1$. Therefore

  $$r^{2^{n-1}} - 2 < a_n < r^{2^{n-1}} + 2,$$

  so $\{a_n\}$ is eventually increasing.

§1.3 Solution to TST 3, by Ruben Carpenter

Let $A_1A_2\cdots A_{2025}$ be a convex 2025-gon, and let $A_i = A_{i+2025}$ for all integers i. Distinct points P and Q lie in its interior such that $\angle A_{i-1}A_iP = \angle QA_iA_{i+1}$ for all i. Define points P_i^j and Q_i^j for integers i and positive integers j as follows:

- For all i, $P_i^1 = Q_i^1 = A_i$.
- For all i and j, P_i^{j+1} and Q_i^{j+1} are the circumcenters of $PP_i^jP_{i+1}^j$ and $QQ_i^jQ_{i+1}^j$, respectively.

Let $\mathcal{P}$ and $\mathcal{Q}$ be the polygons $P_1^{2025}P_2^{2025}\cdots P_{2025}^{2025}$ and $Q_1^{2025}Q_2^{2025}\cdots Q_{2025}^{2025}$, respectively.

(a) Prove that $\mathcal{P}$ and $\mathcal{Q}$ are cyclic.

(b) Let O_P and O_Q be the circumcenters of $\mathcal{P}$ and $\mathcal{Q}$, respectively. Assuming that $O_P \neq O_Q$, show that O_PO_Q is parallel to PQ.

Let $n = 2025$. Let $\mathcal{P}_i$ and $\mathcal{Q}_i$ denote the polygons $P_1^i\cdots P_n^i$ and $Q_1^i\cdots Q_n^i$. In this notation, $\mathcal{P} = \mathcal{P}_n$, $\mathcal{Q} = \mathcal{Q}_n$, and $\mathcal{P}_1 = \mathcal{Q}_1 = A_1\cdots A_n$.

The angle condition for P and Q just says that they are isogonal conjugates in $\mathcal{P}_1$. We will first find some properties that do not depend on P having an isogonal conjugate.

Note that $P_{i-1}^{j+1}P_i^{j+1}$ is the perpendicular bisector of PP_{i+1}^j, so we can go backwards from $\mathcal{P}_j$ to $\mathcal{P}_{j-1}$ by reflecting over the sides. Use this to extend the points backwards to $\mathcal{P}_0$, i.e. define P_i^0 to be the reflection of P over $P_{i-1}^1P_i^1$.

Lemma 1.1

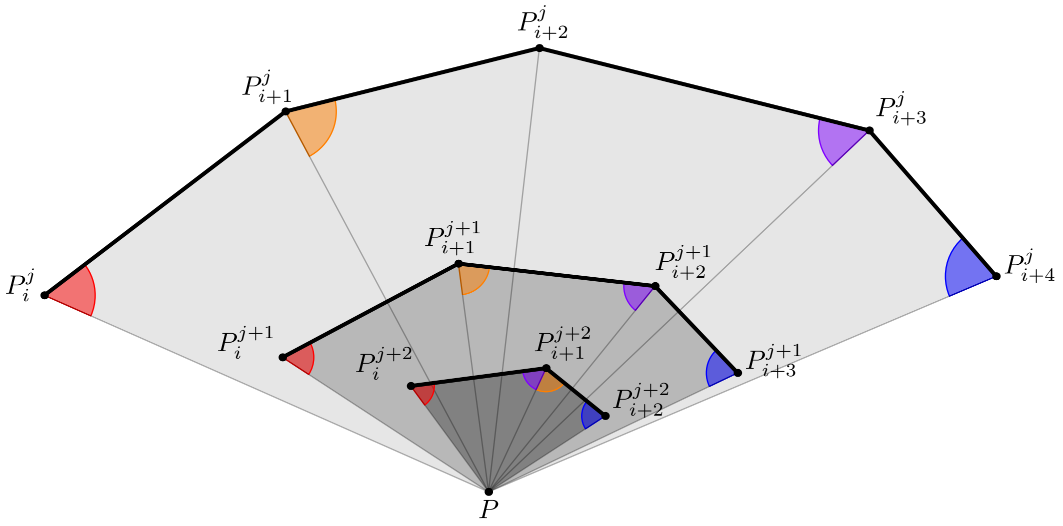
For integers $j \geq 0$ and i,

$$\measuredangle PP_i^jP_{i+1}^j = \measuredangle PP_i^{j+1}P_{i+1}^{j+1} \quad \text{and}$$
$$\measuredangle PP_{i+1}^jP_i^j = \measuredangle PP_i^{j+1}P_{i-1}^{j+1}.$$

Proof. This is a standard circumcenter fact in disguise but we will prove it here for completeness. If X is the antipode of P on $(PP_i^jP_{i+1}^j)$ (equivalently, the reflection of P over PP_i^{j+1}) and M is the midpoint of PP_{i+1}^j then

$$\measuredangle PP_i^jP_{i+1}^j = \measuredangle PXP_{i+1}^j = \measuredangle PP_i^{j+1}M = \measuredangle PP_i^{j+1}P_{i+1}^{j+1}.$$

The second equality is analogous. □



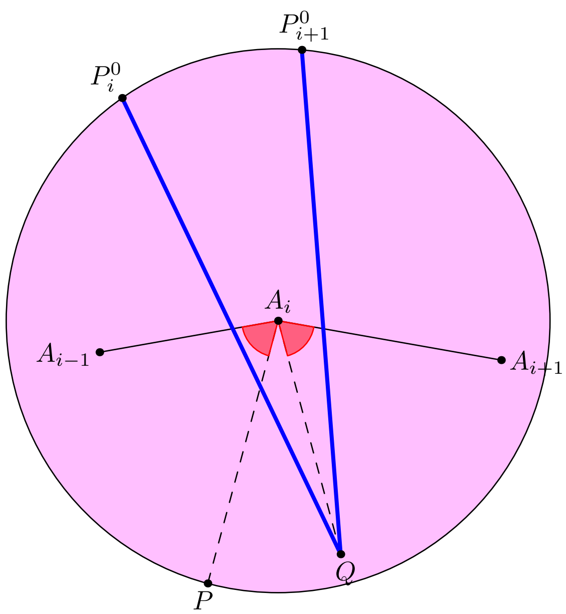
> **Lemma**
> The following similarity holds: $\mathcal{P}_n \cup P \overset{+}{\sim} \mathcal{P}_0 \cup P$.

Proof. By Lemma 1.1,

$$\begin{aligned} &\measuredangle PP_i^0P_{i+1}^0 = \measuredangle PP_i^1P_{i+1}^1 = \cdots = \measuredangle P_i^nP_{i+1}^n \quad \text{and} \\ &\measuredangle PP_{i+1}^0P_i^0 = \measuredangle PP_i^1P_{i-1}^1 = \cdots = \measuredangle P_{i-n+1}^nP_{i-n}^n = \measuredangle P_{i+1}^nP_i^n. \end{aligned}$$

Therefore $PP_i^nP_{i+1}^n \overset{+}{\sim} PP_i^0P_{i+1}^0$. Combining these similarities for all i shows that $\mathcal{P}_n \cup P \overset{+}{\sim} \mathcal{P}_0 \cup P$. □

To solve part (a), we just need to show that $\mathcal{P}_0$ and $\mathcal{Q}_0$ are cyclic. This is where we need the isogonal conjugate property, and we can generalize from analogous facts for the $n = 3$ case.



We have

$$\measuredangle PP_{i+1}^0P_i^0 = \measuredangle PA_iA_{i-1} = \measuredangle A_{i+1}A_iQ$$

so

$$90^\circ = \measuredangle(PP_{i+1}^0, A_iA_{i+1}) = \measuredangle(P_{i+1}^0P_i^0, AQ).$$

Since $A_iP_i^0 = A_iP_{i+1}^0$, it follows that A_iQ is the perpendicular bisector of $P_i^0P_{i+1}^0$ and $QP_i^0 = QP_{i+1}^0$. By applying this equality for all i, it follows that $\mathcal{P}_0$ is cyclic with circumcenter Q. This completes the solution for part (a).

For part (b), we will analyze the angle of rotation and scale factor for the similarity $\mathcal{P}_n \cup P \overset{+}{\sim} \mathcal{P}_0 \cup P$. Assume without loss of generality that $A_1A_2 \cdots A_n$ is labeled in clockwise order as in the earlier diagrams.

Claim — The spiral similarity at P sending $\mathcal{P}_0 \cup Q$ to $\mathcal{P}_n \cup O_P$ is the composition of a clockwise rotation by θ_P and a dilation with factor r_P, where

$$\theta_P = \frac{n\pi}{2} - \sum_{i=1}^{n} \angle PA_iA_{i-1} \quad \text{and} \quad r_P = \prod_{i=1}^{n} \frac{1}{2\sin\angle PA_iA_{i-1}}.$$

Analogously, the spiral similarity at Q sending $\mathcal{Q}_0 \cup P$ to $\mathcal{Q}_n \cup O_Q$ is the composition of a clockwise rotation by θ_Q and a dilation with factor r_Q, where

$$\theta_Q = \frac{n\pi}{2} - \sum_{i=1}^{n} \angle QA_iA_{i-1} \quad \text{and} \quad r_Q = \prod_{i=1}^{n} \frac{1}{2\sin\angle QA_iA_{i+1}}.$$

Proof. Note that

$$\angle PP_0^iPP_0^{i+1} = \frac{\pi}{2} - \angle PP_1^iP_0^i = \frac{\pi}{2} - \angle PP_i^1P_{i-1}^1 = \frac{\pi}{2} - \angle PA_iA_{i-1}$$

by circumcenter properties and Lemma 1.1. Summing over $0 \le i < n$ yields the claimed formula for θ_P, since the left hand side adds up to the rotation angle from PP_0^0 to PP_0^n.

By the law of sines,

$$\prod_{i=1}^{n} \frac{PP_i^{j+1}}{P_i^j} = \prod_{i=1}^{n} \frac{1}{2\sin\angle PP_{i+1}^jP_i^j} = \prod_{i=1}^{n} \frac{1}{2\sin\angle PA_iA_{i-1}}$$

where in the last equality we use Lemma 1.1 again. Multiply over $0 \le j < n$ and raise to the power $\frac{1}{n}$ to obtain

$$\left(\prod_{i=1}^{n} \frac{PP_i^n}{P_i^0}\right)^{\frac{1}{n}} = \prod_{i=1}^{n} \frac{1}{2\sin\angle PA_iA_{i-1}}.$$

This proves the formula for r_P because the left hand side gives the scale factor. The argument for Q is similar. Note that we reversed the angle in the formula for r_Q but not θ_Q because θ_Q depends on orientation. □

By the given angle conditions on P and Q we have $r_P = r_Q$. Meanwhile,

$$\theta_P + \theta_Q = n\pi - \sum_{i=1}^{n}(\angle PA_iA_{i-1} + \angle QA_iA_{i-1}) = n\pi - \sum_{i=1}^{n} \angle A_{i-1}A_iA_{i+1} = 2\pi.$$

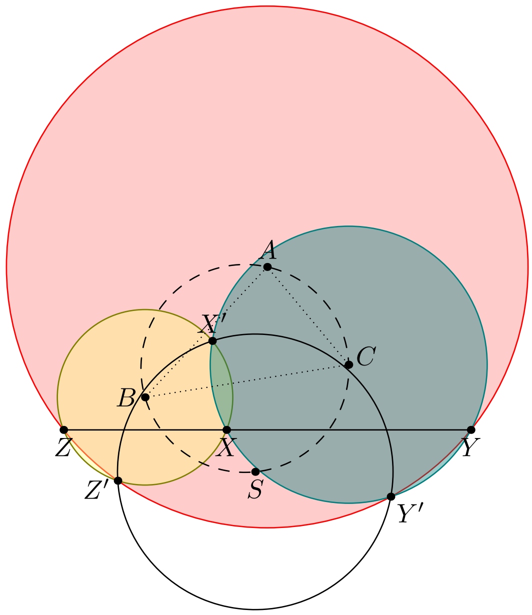
This means that clockwise rotation by θ_Q is just counterclockwise rotation by θ_P. The combination of these two implies that PQO_QO_P is an isosceles trapezoid with O_PO_Q parallel to PQ, which proves part (b).

§2 Solutions to Day 2

§2.1 Solution to TST 4, by Michael Ren

Let ABC be a triangle, and let X, Y, and Z be collinear points such that $AY = AZ$, $BZ = BX$, and $CX = CY$. Points X', Y', and Z' are the reflections of X, Y, and Z over BC, CA, and AB, respectively. Prove that if $X'Y'Z'$ is a nondegenerate triangle, then its circumcenter lies on the circumcircle of ABC.

¶ Solution 1 (Pitchayut Saengrungkongka)



Let S denote the circumcenter of $\triangle X'Y'Z'$. Observe that $AY = AZ = AY' = AZ'$, so $YZY'Z'$ is cyclic and $AS \perp Y'Z'$. Similarly, $BS \perp Z'X'$ and $CS \perp X'Y'$.

The rest is angle chasing. Let $\measuredangle \ell$ denote the angle between line ℓ and a fixed line. Then, we have

$$\begin{aligned} \measuredangle AS &= 90^\circ + \measuredangle Y'Z' = 90^\circ + \measuredangle YY' + \measuredangle ZZ' - \measuredangle YZ \\ &= 90^\circ + \measuredangle CA + \measuredangle AB - \measuredangle YZ. \end{aligned}$$

Analogously, we get

$$\measuredangle BS = 90^\circ + \measuredangle AB + \measuredangle BC - \measuredangle XZ,$$

so subtracting these gives $\measuredangle ASB = \measuredangle ACB$, as desired.

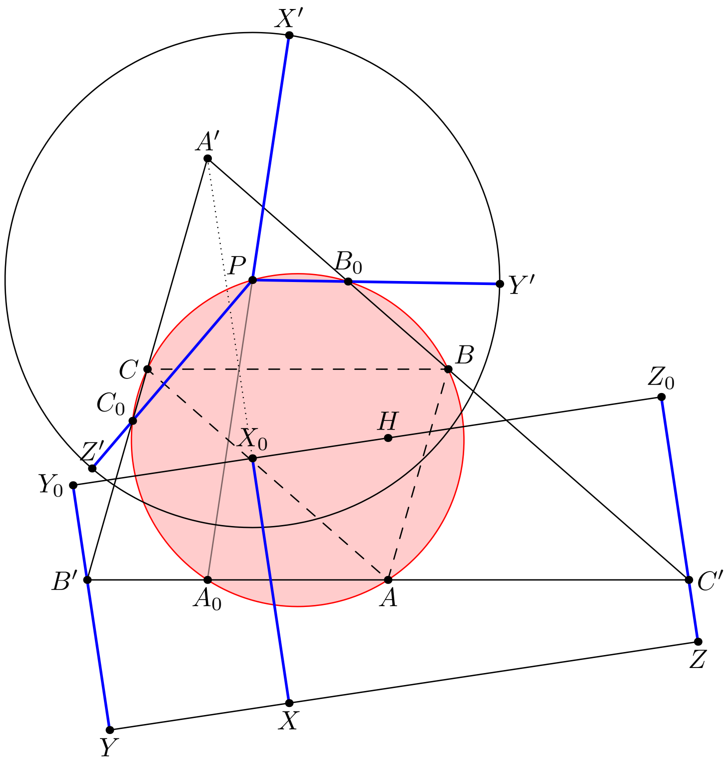
> **Remark.** There are some other angle chasing solutions that use the fact that XX', YY', and ZZ' meet at a point on $(X'Y'Z')$. This one is featured as it does not require any additional points.

¶ Solution 2 (author) Let $A'B'C'$ be the anticomplimentary triangle of ABC. Note that X, Y, and Z are the projections of A', B', and C' onto line XYZ. Let A_0, B_0, and C_0 be the reflections of A', B', and C' over BC, CA, and AB, respectively. If we take $A'B'C'$ as our reference triangle, we see that $A_0B_0C_0$ is the orthic triangle so $(ABCA_0B_0C_0)$ is the nine-point circle of $A'B'C'$.

Let A_0X' meet (ABC) again at P. Then

$$\begin{aligned} \measuredangle BB_0P &= \measuredangle AA_0P + \measuredangle BCA \\ &= \measuredangle AA_0X + \measuredangle BCA \\ &= \measuredangle(A'X, BC) + \measuredangle BCA \\ &= \measuredangle(B'Y, CA) \\ &= \measuredangle BB_0Y'. \end{aligned}$$

Therefore P lies on B_0Y', and likewise C_0Z' by symmetry.



Let X_0, Y_0, and Z_0 be the reflections of P across BC, CA, and AB respectively. By reflection, X_0 lies on $A'X$ and $PX' = X_0X$. We also know that if H is the orthocenter of ABC, then $X_0Y_0Z_0H$ is the Steiner line of P. Let D be the reflection of H across BC, which is also the antipode of A_0 on (ABC). Then $\angle A'X_0H = \angle A_0PD = 90°$, so $X_0Y_0Z_0H$ is perpendicular to $A'X_0X$ and parallel to XYZ.

This means that $XX_0 = YY_0 = ZZ_0$. Combined with $PX' = XX_0$ and analogous statements, we have $PX' = PY' = PZ'$ so P is the circumcenter of $X'Y'Z'$. This solves the problem.

> **Remark.** If the condition that X, Y, and Z are collinear is removed, then in general the circumcenters of XYZ and $X'Y'Z'$ are isogonal conjugates with respect to ABC.

§2.2 Solution to TST 5, by Linus Tang

A pond has 2025 lily pads arranged in a circle. Two frogs, Alice and Bob, begin on different lily pads. A frog jump is a jump which travels 2, 3, or 5 positions clockwise. Alice and Bob each make a series of frog jumps, and each frog ends on the same lily pad that it started from. Given that each lily pad is the destination of exactly one jump, prove that each frog completes exactly two laps around the pond (i.e. travels 4050 positions in total).

Let $\pi : \mathbb{Z}/2025\mathbb{Z} \to \mathbb{Z}/2025\mathbb{Z}$ be the permutation where the jump with source i has destination $\pi(i)$. We know that π has exactly two cycles, corresponding to the paths of each frog.

Suppose that the frogs complete a total of ℓ laps around the circle, so the sum of the lengths of all the jumps made is 2025ℓ. Thus the average jump moves ℓ positions, so $\ell \in \{2, 3, 4, 5\}$. We now split into cases.

¶ Case 1: $\ell \in \{2, 5\}$ If $\ell = 2$, then every jump travels 2 positions, so $\pi(i) = i + 2$ for all i. This permutation has exactly one cycle, contradiction. If $\ell = 5$, then every jump travels 5 positions, so $\pi(i) = i + 5$ for all i. This permutation has exactly five cycles, contradiction.

¶ Case 2: $\ell = 3$ The key idea is to consider the cycle decomposition of the auxillary permutation $\pi' : \mathbb{Z}/2025\mathbb{Z} \to \mathbb{Z}/2025\mathbb{Z}$ by $\pi'(i) = \pi(i) - 3$. The average jump in π' (i.e. the distance from i to $\pi'(i)$ traveling clockwise) travels 0 positions. Thus, either each cycle of π' either travels net zero positions, or there is some cycle that travels net negative positions. Since $\pi'(i) \in \{i - 1, i, i + 2\}$, the latter case can only happen if $\pi'(i) = i - 1$ for al i, which has average jump -1, not 0, contradiction. Thus, each cycle travels net 0 positions. It is easy to see that the only such cycles are $i \mapsto i$ and $i \mapsto i + 2 \mapsto i + 1 \mapsto i$.

Say that i is a *unicycle start* if $i \mapsto i$ is a cycle in π', a *tricycle start* if $i \mapsto i + 2 \mapsto i + 1 \mapsto i$ is a cycle in π', and a *cycle start* if it is either a unicycle start or a tricycle start. It is easy to see that if i is a unicycle start, then $i + 1$ must be a cycle start, and if i is a tricycle start, then $i + 3$ must be a cycle start.

Given this structural classification of π', we will now generate a contradiction by showing that π can't have exactly two cycles. Place auxillary frogs C, D, and E on lily pads i, $i + 1$, and $i + 2$ respectively, where i is a cycle start. If i is a unicycle start, then have C jump to $\pi(i) = i + 3$. If i is a tricycle start, then have C, D, and E jump to $\pi(i) = i + 5$, $\pi(i + 1) = i + 3$, and $\pi(i + 2) = i + 4$, respectively. Note that C, D, and E are still on consecutive lily pads in some order, with the first lily pad number being a cycle start. We can repeat this process, having either the first frog or all three frogs jump according to permutation π, depending on whether the first lily pad number is a unicycle start or a tricycle start. When all of the cycles of π' have been exhausted in this manner, C, D, and E are back to positions i, $i + 1$, and $i + 2$, in some order. Note that each step of the above process permutes C, D, and E with an even permutation, so the final permutation of C, D, and E must be CDE, DEC, or ECD. The first case implies that π has three cycles, while the other two cases imply that π has only one cycle, a contradiction.

¶ Case 3: $\ell = 4$ Now consider the auxillary permutation $\pi'' : \mathbb{Z}/2025\mathbb{Z} \to \mathbb{Z}/2025\mathbb{Z}$ given by $\pi''(i) = \pi(i) - 4$. Note that $\pi''(i) \in \{i-2, i-1, i+1\}$, so a similar analysis as the $\ell = 3$ case shows that the only two possible types of cycles in π'' are $i \to i+1 \to i$ and $i \to i+1 \to i+2 \to i$. Again, define i to be a *bicycle start* if $i \mapsto i+1 \mapsto i$ is a cycle in π'' a *tricycle start* if $i \mapsto i+1 \mapsto i+2 \mapsto i$ is a cycle in π'', and a *cycle start* if it is either a bicycle or tricycle start. It is easy to see that if i is a bicycle start, then $i+2$ must be a cycle start, and if i is a tricycle start, then $i+3$ must be a cycle start.

Now, place auxillary frogs C, D, E, and F on lily pads i, $i+1$, $i+2$, and $i+3$ respectively, where i is a cycle start. If i is a bicycle start, have C jump to $\pi(i) = i+5$ and D jump to $\pi(i+1) = i+4$, and if i is a tricycle start, have C jump to $\pi(i) = i+5$, D jump to $\pi(i+1) = i+6$, and E jump to $\pi(i+2) = i+4$. Note that C, D, E, and F are still on consecutive lily pads in some order, with the first lily pad number being a cycle start. If i is a bicycle start, then it causes the frogs to permute according to $CDEF \mapsto EFDC$, and if i is a tricycle start, then it causes the frogs to permute according to $CDEF \mapsto FECD$.

Thus, when all of the cycles of π'' have been exhausted in this manner, C, D, E, and F are back to positions i, $i+1$, $i+2$, and $i+3$, in one of the permutations $CDEF$, $EFDC$, $DCFE$, or $FECD$. The first case implies that π has four cycles, while the second and fourth cases imply that π has only one cycle, a contradiction, so we must be in the third case, i.e. the frogs end up on lily pads i, $i+1$, $i+2$, and $i+3$ in the order $DCFE$. Therefore, one of Alice or Bob travels the combined path of C and D, while the other travels the combined path of E and F, each of which completes two laps around the circle, as desired.

Remark. 2025 can be replaced with any odd number (sufficiently large as to avoid confusion about what a "lap" is), and the problem statement will still be true. If 2025 is replaced with an even number, then the problem statement is false only due to the $\ell = 2$ case.

Remark. Here are some variations of the problem (not checked thoroughly):

If 2025 is replaced by any odd number and the allowed distances are changed from 2, 3, 5 to 1, 2, 4, then you can conclude that one frog completes one lap and the other completes two laps.

If 2025 is replaced by any number and the allowed distances are changed to 1, 3, 4, then you can conclude that each frog completes only one lap.

§2.3 Solution to TST 6, by Pitchayut Saengrungkongka

Prove that there exists a real number $\varepsilon > 0$ such that there are infinitely many sequences of integers $0 < a_1 < a_2 < \ldots < a_{2025}$ satisfying

$$\gcd(a_1^2 + 1, a_2^2 + 1, \ldots, a_{2025}^2 + 1) > a_{2025}^{1+\varepsilon}.$$

¶ Solution 1 (Alexander Wang) By the Chinese Remainder Theorem for $\mathbb{Q}[x]$, for any choice of 2^{12} signs there exists a unique polynomial f_i of degree at most 23 such that

$$f_i \equiv \pm cx \pmod{c^2x^2 + 1}$$

for $c = 1, 2, \ldots, 12$. Furthermore, these polynomials are multiples of x and come in pairs which sum to zero, so we can pick 2048 of these polynomials which have positive leading coefficients and label them $f_1, f_2, \ldots, f_{2048}$.

Let N be a positive integer such that any coefficient of the polynomials is $\frac{1}{N}$ times an integer. For a suffiiently large positive integer x, take $a_i = f_i(Nx)$ for $i = 1, 2, \ldots, 2025$, which will be positive integers. Then,

$$\gcd(a_1^2 + 1, a_2^2 + 1, \ldots, a_{2025}^2 + 1) \geq \prod_{c=1}^{12}(c^2x^2 + 1)$$

because $a_i^2 + 1 \equiv 0 \pmod{c^2x^2 + 1}$ for any i by construction. The right hand side is asymptotically x^{24} while a_{2025} is x^{23} up to constant factors, so any $\varepsilon < \frac{1}{23}$ works.

> **Remark.** In terms of $n = 2025$, this solution achieves $\varepsilon = \Omega\left(\frac{1}{\log n}\right)$ which is the best that we know of. Solution 2 achieves $\varepsilon = \Omega\left(\frac{1}{n^{\log_2(3)}}\right)$ while solutions 3 and 4 achieve $\varepsilon = \Omega\left(\frac{1}{n}\right)$.

¶ Solution 2 (Luke Robitaille) Define the sequence of polynomials P_i by

$$P_1(x) = x^2 + 1,$$
$$P_n(x) = \left(\prod_{i=1}^{n-1} P_i(x)\right)^2 + 1 \text{ for } n > 1.$$

Due to the recurrence, we have $\gcd(P_i(x), P_j(x)) = 1$ for $i \neq j$. Let

$$P(x) = P_1(x)P_2(x)\ldots P_m(x),$$

which is a polynomial of degree $2 \cdot 3^m$. For $i \geq 2$ we have

$$\left(\frac{P(x)}{P_i(x)}\right)^2 = \left(\prod_{i=1}^{n-1} P_i(x)\right)^2 \times (P_{i+1}(x))^2 \times \cdots \times (P_m(x))^2$$
$$\equiv -1 \times 1 \times \cdots \times 1 \equiv -1 \pmod{P_i(x)}.$$

and additionally

$$\left(\frac{xP(x)}{P_1(x)}\right)^2 = x^2 \times \left(\prod_{i=2}^{m} P_i(x)\right)^2$$
$$\equiv -1 \pmod{P_1(x)}.$$

Now consider all 2^{m-1} polynomials of the form

$$Q(x) = \frac{xP(x)}{P_1(x)} \pm \frac{P(x)}{P_2(x)} \pm \frac{P(x)}{P_3(x)} \pm \dots \pm \frac{P(x)}{P_m(x)}.$$

Each such polynomial has leading coefficient 1 and degree $2 \cdot 3^m - 1$, and they are all distinct (the terms are in decreasing order of degree from left to right). Furthermore, each $Q(x)$ satisfies $Q(x)^2 \equiv -1 \pmod{P(x)}$ by the Chinese Remainder Theorem.

Now we can construct the solution. Let $m = 12$ (so $2^{m-1} > 2025$), $\varepsilon < \frac{1}{2 \cdot 3^{m-1} - 1}$, x be a (large) positive integer and take $a_1, \dots, a_{2025}$ to be distinct values of $Q(x)$ as described above. Then $\gcd(a_1^2+1, a_2^2+1, \dots, a_{2025}^2+1) \geq P(x)$ and

$$a_{2025}^{1+\varepsilon} < a_{2025}^{\frac{2 \cdot 3^m}{2 \cdot 3^m - 1}} < x^{2 \cdot 3^m} < P(x)$$

for sufficiently large x. Since there are infinitely many such x, we are done.

Remark. This solution takes some inspiration from the other solutions that follow, but stands out for not having to deal with any polynomials outside of $\mathbb{Z}[x]$.

¶ Solution 3 (author) We first explain the general idea of the construction. The idea is to work in polynomial ring $\mathbb{Q}[x]$. Suppose that the target gcd is $g(x)$, which will have a lot of factors. Then, the equation $f(x)^2 \equiv -1 \pmod{g(x)}$ will have multiple solutions $f_1, f_2, \dots, f_n$, and we will have $\deg f_i < \deg g$. Thus, we may take $\varepsilon = \frac{1}{\deg g}$. However, one also needs to find x for which $f_1(x), \dots, f_n(x)$ are integers. The way we get around this is to find *some* explicit choice of f_i and g as described above, and then pick x manually to ensure integrality.

Let $p_1, p_2, \dots, p_n \equiv 1 \pmod 4$ be distinct primes that we will constrain later. Let $N = p_1 p_2 \dots p_n$. Our g will be

$$g(x) = x^{2N} + 1.$$

Then, we claim that

Claim — the polynomial

$$f_i(x) = \frac{2p_i}{N} \cdot \frac{x^{p_i}(x^{2N}+1)}{x^{2p_i}+1} - x^N$$

satisfies $f_i^2 \equiv -1 \pmod g$.

Proof. A direct computation shows that

$$f_i \equiv x^{p_i} \pmod{x^{2p_i}+1}$$
$$f_i \equiv -x^N \pmod{\tfrac{x^{2N}+1}{x^{2p_i}+1}}$$

□

In the next claim, we will show that one may choose $p_1, \ldots, p_n$ so that $f_1(2), \ldots, f_n(2)$ are all integers. This will imply that if $a \equiv 2 \pmod{N}$, then $f_1(a), \ldots, f_n(a)$ are integers all divisible by $\frac{g(a)}{N}$. By taking a large, this gives a construction for $\varepsilon < \frac{1}{2N}$.

To finish the problem, we prove the claim.

Claim — There exists n distinct primes $p_1, p_2, \ldots, p_n \equiv 1 \pmod 4$ such that for any indices $i \neq j$,

$$p_j \mid \frac{4^{p_1 p_2 \cdots p_n} + 1}{4^{p_i} + 1}.$$

Proof. First, we will establish a prime sequence which $p_1 p_2 \ldots p_n \mid 4^{p_1 p_2 \cdots p_n} + 1$. This is essentially IMO 2000 P5: begin with $p_1 = 5$ and for each i, assign (by Zsigmondy) p_i a primitive prime divisor of $4^{p_1 p_2 \cdots p_{i-1}} + 1$; it's easy to see that this works as advertised.

Now, for each i, j, the divisiblity relation is verified unless $p_j \mid 4^{p_i} + 1$, in which case we can apply Lifting the Exponent lemma to complete the proof. Finally, note that $p_i \equiv 1 \pmod 4$ because each divide $4^{p_1 p_2 \cdots p_n} + 1$, a perfect square plus one. □

¶ Solution 4 (Carl Schildkraut, Mihir Singhal, Brandon Wang) We show the result for each $n = 2^{m-1}$. For positive integers i, define polynomials $h_i(x) = x^{2^{i-1}}$ and $f_i(x) = h_i(x)^2 + 1 = x^{2^i} + 1$. The main claim is the following:

Claim — For every positive integer m, there exist polynomials $g_1, \ldots, g_m \in \mathbb{Z}[x]$ satisfying

$$\sum_{i=1}^{m} \frac{g_i(x)}{f_i(x)} = \frac{2^{m-1}}{f_1(x) \cdots f_m(x)}.$$

Proof. We prove the result by induction on m. In the base case of $m = 1$, we may simply take $g_1(x) = 1$.

For the inductive step, suppose $m \geq 2$, and the result holds for $m - 1$. Let $p_1, \ldots, p_{m-1}$ be polynomials for which

$$\sum_{i=1}^{m-1} \frac{p_i(x)}{f_i(x)} = \frac{2^{m-2}}{f_1(x) \cdots f_{m-1}(x)}.$$

Write $p_0(x) = p_m(x)$ for notational simplicity. Since $f_{i+1}(x) = f_i(x^2)$, we also have

$$\sum_{i=1}^{m-1} \frac{p_i(x^2)}{f_{i+1}(x)} = \frac{2^{m-2}}{f_2(x) \cdots f_m(x)}.$$

Defining $g_i(x) = p_i(x) - p_{i-1}(x^2) \cdot \frac{x^{2^m} - 1}{x^2 + 1}$, we have

$$\begin{aligned} \sum_{i=1}^{m} \frac{g_i(x)}{f_i(x)} &= \sum_{i=1}^{m} \frac{p_i(x)}{f_i(x)} - \frac{x^{2^m} - 1}{x^2 + 1} \sum_{i=1}^{m} \frac{p_{i-1}(x^2)}{f_i(x)} \\ &= \frac{2^{m-2}}{f_1 \cdots f_{m-1}} - \frac{f_m - 2}{f_1} \cdot \frac{2^{m-2}}{f_2 \cdots f_m} = \frac{2^{m-2}(f_m - (f_m - 2))}{f_1 \cdots f_m} = \frac{2^{m-1}}{f_1 \cdots f_m}, \end{aligned}$$

as desired. □

Fix m, and let $g_1, \dots, g_m$ be as guaranteed by the lemma. Now, for each $\theta \in \{-1,1\}^m$, define a polynomial $q_\theta(x)$ by

$$q_\theta(x) = \sum_{i=1}^{m} \theta_i g_i(x) h_i(x) \prod_{\substack{j=1 \\ j \neq i}}^{m} f_i(x).$$

For each i, we have

$$q_\theta(x) \equiv \theta_i h_i(x) \left(g_i(x) \prod_{\substack{j=1 \\ j \neq i}}^{m} f_i(x) \right) \equiv 2^{m-1} \theta_i h_i(x) \pmod{f_i(x)}.$$

As a result, since $h_i^2 \equiv -1 \pmod{f_i}$, we have

$$q_\theta(x)^2 + 2^{2m-2} \equiv 2^{2m-2} \left(\theta_i^2 h_i(x)^2 + 1 \right) = 2^{2m-2} f_i(x) \equiv 0 \pmod{f_i}$$

for each i. The polynomials f_i are relatively prime, so we conclude in fact that

$$f_1 \cdots f_m \mid q_\theta^2 + 2^{2m-2}. \tag{$\star$}$$

Let p_θ be the remainder when q_θ is divided by the product $f_1 \cdots f_m$, so that $\deg p_\theta < \deg(f_1 \cdots f_m)$; since $f_1 \cdots f_m$ is monic, p_θ has integer coefficients. Moreover, there exists a polynomial s_θ so that

$$p_\theta = s_\theta \cdot f_1 \cdots f_m + \sum_{i=1}^{m} \theta_i g_i h_i \prod_{\substack{j=1 \\ j \neq i}}^{m} f_i.$$

In particular, since $f_i(X)$ is even for any i and any odd integer X, $p_\theta(X)$ is a multiple of 2^{m-1}. Let X be an odd integer, and define $a_{\theta,X} = p_\theta(X)/2^{m-1} \in \mathbb{Z}$. Since ($\star$) holds for p_θ as well as q_θ, there is a monic polynomial t_θ for which

$$p_\theta^2 + 2^{2m-2} = t_\theta f_1 \cdots f_m,$$

whence

$$a_{\theta,X}^2 + 1 = \frac{1}{2^{2m-2}} \left(p_\theta(X)^2 + 1 \right) = t_\theta(x) \prod_{i=1}^{m} \frac{f_i(X)}{2}.$$

Therefore

$$\prod_{i=1}^{m} \frac{f_i(X)}{2} \text{ divides } \gcd\left(\left\{ a_{\theta,X} : \theta \in \{-1,1\}^m \right\} \right).$$

As X grows, the gcd above is at least $2^{-m} X^{\deg f_1 \cdots f_m}$. But each p_θ has degree at most $(\deg f_1 \cdots f_m) - 1$, so each $a_{\theta,X}$ is at most some constant C times $X^{\deg(f_1 \cdots f_m)-1}$. Therefore, for any $0 < \varepsilon < \frac{1}{\deg(f_1 + \cdots f_m) - 1}$, we have for large enough X that

$$\gcd\left(\left\{ a_{\theta,X} : \theta \in \{-1,1\}^m \right\} \right) \geq \max\left(\left\{ |a_{\theta,X}| : \theta \in \{-1,1\}^m \right\} \right)^{1+\varepsilon}.$$

As $a_{\theta,X} = -a_{-\theta,X}$, at least half of the $a_{\theta,X}$ are nonnegative. We conclude the result for $n = 2^{m-1}$ and any $\varepsilon < \frac{1}{2^{m+1}-3}$.