USA TST 2024 Solutions

United States of America — Team Selection Test

Andrew Gu, Evan Chen, Gopal Goel, Luke Robitaille

65th IMO 2024 United Kingdom and 13th EGMO 2024 Georgia

Contents

§0 Problems

1. Find the smallest constant $C > 1$ such that the following statement holds: for every integer $n \ge 2$ and sequence of non-integer positive real numbers a_1, a_2, ..., a_n satisfying
$$\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n} = 1,$$
it's possible to choose positive integers b_i such that

(i) for each $i = 1, 2, \dots, n$, either $b_i = \lfloor a_i \rfloor$ or $b_i = \lfloor a_i \rfloor + 1$; and

(ii) we have
$$1 < \frac{1}{b_1} + \frac{1}{b_2} + \dots + \frac{1}{b_n} \le C.$$

2. Let ABC be a triangle with incenter I. Let segment AI intersect the incircle of triangle ABC at point D. Suppose that line BD is perpendicular to line AC. Let P be a point such that $\angle BPA = \angle PAI = 90^\circ$. Point Q lies on segment BD such that the circumcircle of triangle ABQ is tangent to line BI. Point X lies on line PQ such that $\angle IAX = \angle XAC$. Prove that $\angle AXP = 45^\circ$.

3. Let $n > k \ge 1$ be integers and let p be a prime dividing $\binom{n}{k}$. Prove that the k-element subsets of $\{1, \dots, n\}$ can be split into p classes of equal size, such that any two subsets with the same sum of elements belong to the same class.

4. Find all integers $n \ge 2$ for which there exists a sequence of $2n$ pairwise distinct points $(P_1, \dots, P_n, Q_1, \dots, Q_n)$ in the plane satisfying the following four conditions:

(i) no three of the $2n$ points are collinear;

(ii) $P_iP_{i+1} \ge 1$ for all $i = 1, 2, \dots, n$, where $P_{n+1} = P_1$;

(iii) $Q_iQ_{i+1} \ge 1$ for all $i = 1, 2, \dots, n$, where $Q_{n+1} = Q_1$; and

(iv) $P_iQ_j \le 1$ for all $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, n$.

5. Suppose $a_1 < a_2 < \dots < a_{2024}$ is an arithmetic sequence of positive integers, and $b_1 < b_2 < \dots < b_{2024}$ is a geometric sequence of positive integers. Find the maximum possible number of integers that could appear in both sequences, over all possible choices of the two sequences.

6. Solve over $\mathbb{R}$ the functional equation
$$f(xf(y)) + f(y) = f(x + y) + f(xy).$$

§1 Solutions to Day 1

§1.1 USA TST 2024/1, proposed by Merlijn Staps

Available online at https://aops.com/community/p29409075.

Problem statement

Find the smallest constant $C > 1$ such that the following statement holds: for every integer $n \geq 2$ and sequence of non-integer positive real numbers $a_1, a_2, \dots, a_n$ satisfying

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} = 1,$$

it's possible to choose positive integers b_i such that

(i) for each $i = 1, 2, \dots, n$, either $b_i = \lfloor a_i \rfloor$ or $b_i = \lfloor a_i \rfloor + 1$; and

(ii) we have

$$1 < \frac{1}{b_1} + \frac{1}{b_2} + \cdots + \frac{1}{b_n} \leq C.$$

¶ **Answer.** The answer is $C = \frac{3}{2}$.

¶ **Lower bound.** Note that if $a_1 = \frac{4n-3}{2n-1}$ and $a_i = \frac{4n-3}{2}$ for $i > 1$, then we must have $b_1 \in \{1, 2\}$ and $b_i \in \{2n-2, 2n-1\}$ for $i > 1$. If we take $b_1 = 2$ then we obtain

$$\frac{1}{b_1} + \frac{1}{b_2} + \cdots + \frac{1}{b_n} \leq \frac{1}{2} + (n-1) \cdot \frac{1}{2n-2} = 1,$$

whereas if we take $b_1 = 1$ then we obtain

$$\frac{1}{b_1} + \frac{1}{b_2} + \cdots + \frac{1}{b_n} \geq 1 + (n-1) \cdot \frac{1}{2n-1} = \frac{3n-2}{2n-1}.$$

This shows that $C \geq \frac{3n-2}{2n-1}$, and as $n \to \infty$ this shows that $C \geq \frac{3}{2}$.

¶ **Upper bound.** For $0 \leq k \leq n$, define

$$c_i = \sum_{i=1}^{k} \frac{1}{\lfloor a_i \rfloor} + \sum_{i=k+1}^{n} \frac{1}{\lfloor a_i \rfloor + 1}.$$

Note that $c_0 < c_1 < \cdots < c_n$ and

$$c_0 < \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} = 1 < c_n.$$

This means there exists a unique value of k for which $c_{k-1} < 1 < c_k$. For this k we have

$$1 < c_k = c_{k-1} + \frac{1}{(\lfloor a_k \rfloor)(\lfloor a_k \rfloor + 1)} < 1 + \frac{1}{1 \cdot 2} = \frac{3}{2}.$$

Therefore we may choose $b_i = \lfloor a_i \rfloor$ for $i \leq k$ and $b_i = \lfloor a_i \rfloor + 1$ for $i > k$.

Remark. The solution can be phrased in the following "motion-based" way. Imagine starting with all floors (corresponding to c_0), then changing each floor to a ceiling one by one until after n steps every floor is a ceiling (arriving at c_n). As we saw, $c_0 < 1 < c_n$, but $c_0 < \dots < c_n$. Moreover, each discrete step increases the sum by at most

$$\frac{1}{\lfloor a_i \rfloor} - \frac{1}{\lfloor a_i \rfloor + 1} \le \frac{1}{2}$$

and so the changing sum must be in the interval $[1, 3/2]$ at some point.

¶ Upper bound (alternate). First suppose $a_i < 2$ for some i. Assume without loss of generality $i = 1$ here. Let $b_1 = 1$ and $b_i = \lfloor a_i \rfloor + 1$ for all other i. Then

$$\begin{aligned} 1 < \frac{1}{b_1} + \dots + \frac{1}{b_n} &= 1 + \frac{1}{\lfloor a_2 \rfloor + 1} + \dots + \frac{1}{\lfloor a_n \rfloor + 1} \\ &< \left(\frac{1}{2} + \frac{1}{a_1}\right) + \frac{1}{a_2} + \dots + \frac{1}{a_n} = \frac{3}{2}. \end{aligned}$$

Now suppose $a_i > 2$ always. Then $\frac{a_i}{\lfloor a_i \rfloor} < \frac{3}{2}$, so

$$1 = \frac{1}{a_1} + \dots + \frac{1}{a_n} < \frac{1}{\lfloor a_1 \rfloor} + \dots + \frac{1}{\lfloor a_n \rfloor} < \frac{3}{2}\left(\frac{1}{a_1} + \dots + \frac{1}{a_n}\right) = \frac{3}{2}.$$

Therefore we may let $b_i = \lfloor a_i \rfloor$ for all i.

Remark. The original proposal asked to find the optimal C for a fixed n. The answer is $\frac{3n-2}{2n-1}$, i.e. the lower bound construction in the solution is optimal.

§1.2 USA TST 2024/2, proposed by Luke Robitaille

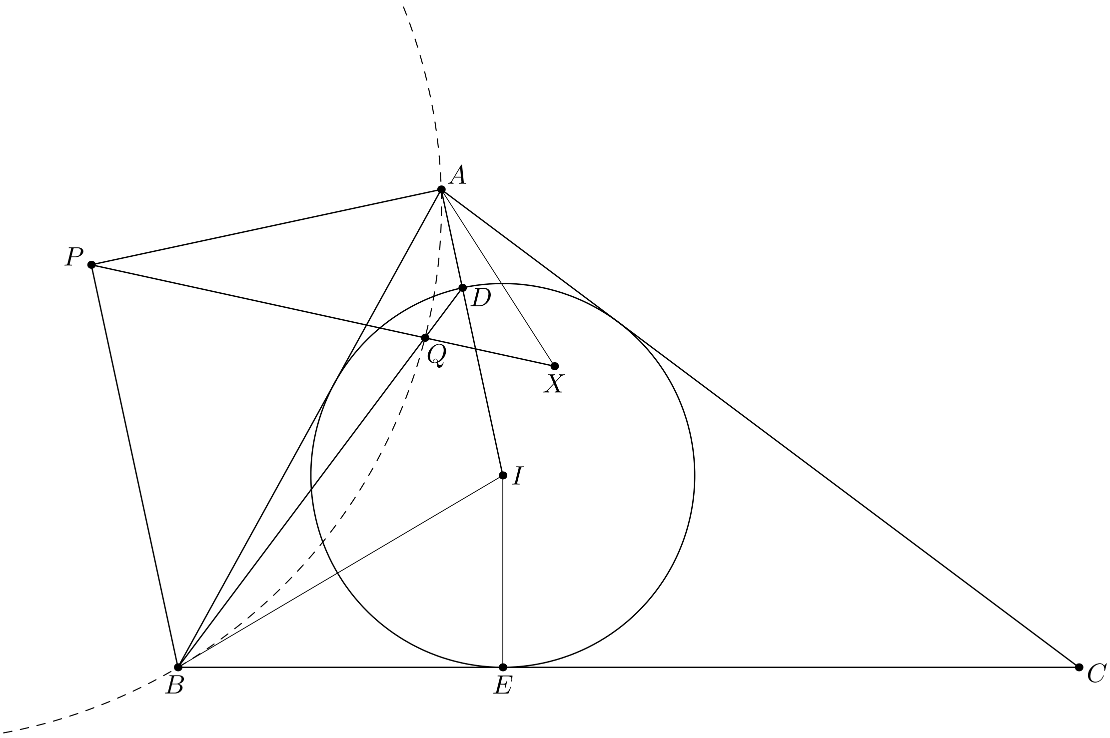
Available online at https://aops.com/community/p29409083.

Problem statement

Let ABC be a triangle with incenter I. Let segment AI intersect the incircle of triangle ABC at point D. Suppose that line BD is perpendicular to line AC. Let P be a point such that $\angle BPA = \angle PAI = 90^\circ$. Point Q lies on segment BD such that the circumcircle of triangle ABQ is tangent to line BI. Point X lies on line PQ such that $\angle IAX = \angle XAC$. Prove that $\angle AXP = 45^\circ$.

We show several approaches.

¶ First solution, by author.



Claim — We have $BP = BQ$.

Proof. For readability, we split the proof into three unconditional parts.

- We translate the condition $\overline{BD} \perp \overline{AC}$. It gives $\angle DBA = 90^\circ - A$, so that

$$\angle DBI = \left| \frac{B}{2} - (90^\circ - A) \right| = \frac{|A - C|}{2}$$

$$\angle BDI = \angle DBA + \angle BAD = (90^\circ - A) + \frac{A}{2} = 90^\circ - \frac{A}{2}.$$

Hence, letting r denote the inradius, we can translate $\overline{BD} \perp \overline{AC}$ into the following trig condition:

$$\sin \frac{B}{2} = \frac{r}{BI} = \frac{DI}{BI} = \frac{\sin \angle DBI}{\sin \angle BDI} = \frac{\sin \frac{|A-C|}{2}}{\sin\left(90^\circ - \frac{A}{2}\right)}.$$

- The length of BP is given from right triangle APB as

$$BP = BA \cdot \sin \angle PAB = BA \cdot \sin\left(90^\circ - \frac{A}{2}\right).$$

- The length of BQ is given from the law of sines on triangle ABQ. The tangency gives $\angle BAQ = \angle DBI$ and $\angle BQA = 180^\circ - \angle ABI = 180^\circ - \angle IBE$ and thus

$$BQ = BA \cdot \frac{\sin \angle BAQ}{\sin \angle AQB} = BA \cdot \frac{\sin \angle DBI}{\sin \angle ABI} = BA \cdot \frac{\sin \frac{|A-C|}{2}}{\sin \frac{B}{2}}.$$

The first bullet implies the expressions in the second and third bullet for BP and BQ are equal, as needed. □

Remark. In the above proof, one dos not actually need to compute $\angle DBI = \frac{|A-C|}{2}$. The proof works equally leaving that expression intact as $\sin \angle DBI$ in place of $\sin \frac{|A-C|}{2}$.

Now we can finish by angle chasing. We have

$$\angle PBQ = \angle PBA + \angle ABD = \frac{A}{2} + 90^\circ - A = 90^\circ - \frac{A}{2}.$$

Then

$$\angle BPQ = \frac{180^\circ - \angle PBQ}{2} = 45^\circ + \frac{A}{4},$$

so $\angle APQ = 90^\circ - \angle BPQ = 45^\circ - \frac{A}{4}$. Also, if we let J be the incenter of IAC, then $\angle PAJ = 90^\circ + \frac{A}{4}$, and clearly X lies on line AJ. Then $\angle APQ + \angle PAJ = 135^\circ < 180^\circ$, so X lies on the same side of AP as Q and J (by the parallel postulate). Therefore $\angle AXP = 180^\circ - 135^\circ = 45^\circ$, as desired.

Remark. The problem was basically written backwards by starting from the $BD \perp AC$ condition, turning that into trig, and then contriving P and Q so that the $BD \perp AC$ condition implied $BP = BQ$.

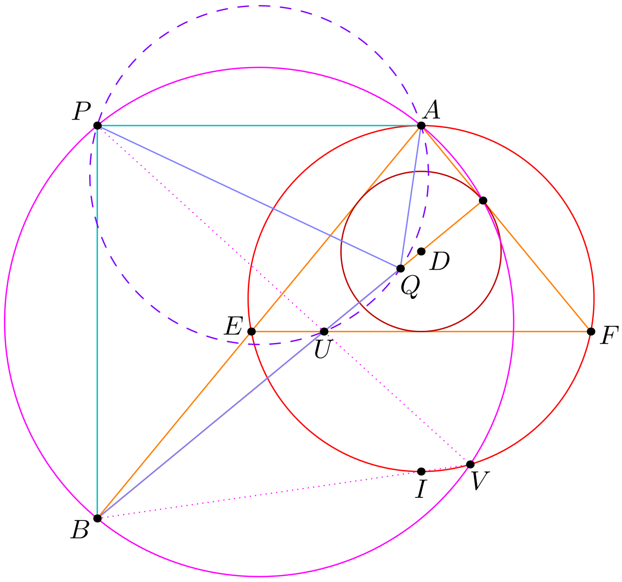
¶ Second solution, by Jeffrey Kwan. We prove the following restatement:

> Consider isosceles triangle AEF with $AE = AF$ and incenter D. Let B be the point on ray AE such that $BD \perp AF$, and let P be the projection of B onto the line through A parallel to EF. Let I be the point diametrically opposite A in the circumcircle of AEF, and let Q be the point on line BD such that BI is tangent to the circumcircle of AQB. Then $\angle APQ = 45^\circ - \angle A/4$.

First note that $\angle DFE = 45^\circ - \angle A/4$, so it suffices to show that $\overline{PQ} \parallel \overline{DF}$. Let $U = \overline{BD} \cap \overline{EF}$, and let $V = BI \cap (AEF)$. Observe that:

- P and V both lie on the circle with diameter AB, so $\angle BVP = \angle PAB = 90^\circ - \angle A/2$.
- We have $\angle EVB = \angle EVI = \angle A/2 = \angle DUF = \angle BUE$. Hence $BEUV$ is cyclic.

Now $\angle BVU = \angle AEU = 90^\circ - \angle A/2 = \angle BVP$, so $\overline{PUV}$ are collinear.



From the tangency condition, we have that $\angle AQB = 180^\circ - \angle ABI$, which implies that

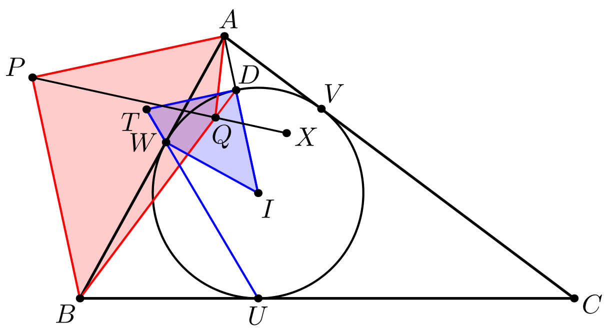
$$\angle AQU + \angle APU = \angle AQB + \angle APV = (180^\circ - \angle ABI) + \angle ABI = 180^\circ,$$

and so $APUQ$ is cyclic. Finally, note that D is the orthocenter of $\triangle AUF$, which implies that

$$\angle APQ = \angle AUQ = \angle AUD = \angle AFD = \angle DFE.$$

This forces $\overline{PQ} \parallel \overline{DF}$, as desired.

¶ Third solution by Pitchayut Saengrungkongka and Maxim Li. We provide yet another proof that $BP = BQ$.



Let the incircle be ω and touch BC and AB at point U and W. Let the tangent to ω at D meet UW at T. Notice that T is the pole of BD with respect to ω, so $IT \perp BD$. Now, we make the following critical claim, which in particular implies $BP = BQ$.

Claim — Quadrilaterals $DIWT$ and $PBQA$ are inversely similar.

Proof. This follows from four angle relations.

- $\measuredangle IDT = \measuredangle BPA = 90^\circ$.
- $\measuredangle TIW = \measuredangle ABQ$.
- $\measuredangle DIT = \measuredangle IAC = \measuredangle BAI = \measuredangle ABP$.
- $\measuredangle ITW = \measuredangle QBI = \measuredangle QAB$. □

With $BP = BQ$ obtained, one finishes with the same angle chasing as in the first solution.

§1.3 USA TST 2024/3, proposed by Ankan Bhattacharya

Available online at https://aops.com/community/p29409068.

Problem statement

Let $n > k \ge 1$ be integers and let p be a prime dividing $\binom{n}{k}$. Prove that the k-element subsets of $\{1, \dots, n\}$ can be split into p classes of equal size, such that any two subsets with the same sum of elements belong to the same class.

Let $\sigma(S)$ denote the sum of the elements of S, so that

$$P(x) := \sum_{\substack{S \subseteq \{1,\dots,n\} \\ |S|=k}} x^{\sigma(S)}$$

is the generating function for the sums of k-element subsets of $\{1, \dots, n\}$.

By Legendre's formula,

$$\nu_p\left(\binom{n}{k}\right) = \sum_{r=1}^{\infty} \left(\left\lfloor \frac{n}{p^r} \right\rfloor - \left\lfloor \frac{k}{p^r} \right\rfloor - \left\lfloor \frac{n-k}{p^r} \right\rfloor \right)$$

so there exists a positive integer r with

$$\left\lfloor \frac{n}{p^r} \right\rfloor - \left\lfloor \frac{k}{p^r} \right\rfloor - \left\lfloor \frac{n-k}{p^r} \right\rfloor > 0.$$

The main claim is the following:

Claim — $P(x)$ is divisible by

$$\Phi_{p^r}(x) = x^{(p-1)p^{r-1}} + \cdots + x^{p^{r-1}} + 1.$$

Before proving this claim, we will show how it solves the problem. It implies that there exists a polynomial Q with integer coefficients satisfying

$$\begin{aligned} P(x) &= \Phi_{p^r}(x)Q(x) \\ &= (x^{(p-1)p^{r-1}} + \cdots + x^{p^{r-1}} + 1)Q(x). \end{aligned}$$

Let $c_0, c_1, \dots$ denote the coefficients of P, and define

$$s_i = \sum_{j \equiv i \pmod{p^r}} c_j.$$

Then it's easy to see that

$$\begin{aligned} s_0 &= s_{p^{r-1}} &= \cdots = s_{(p-1)p^{r-1}} \\ s_1 &= s_{p^{r-1}+1} &= \cdots = s_{(p-1)p^{r-1}+1} \\ &\vdots \\ s_{p^{r-1}-1} &= s_{2p^{r-1}-1} &= \cdots = s_{p^r-1}. \end{aligned}$$

This means we can construct the p classes by placing a set with sum z in class $\left\lfloor \frac{z \bmod p^r}{p^{r-1}} \right\rfloor$.

Now we present two ways to prove the claim.

¶ First proof of claim. There's a natural bijection between k-element subsets of $\{1,\dots,n\}$ and binary strings consisting of k zeroes and ℓ ones: the set $\{a_1,\dots,a_k\}$ corresponds to the string which has zeroes at positions a_1, ..., a_k. Moreover, the inversion count of this string is simply $(a_1+\dots+a_k)-\frac{1}{2}k(k+1)$, so we only deal with these inversion counts (equivalently, we are factoring $x^{\frac{k(k+1)}{2}}$ out of P).

Recall that the generating function for these inversion counts is given by the q-binomial coefficient

$$P(x) = \frac{(x-1)\cdots(x^{k+\ell}-1)}{\left[(x-1)\cdots(x^k-1)\right]\times\left[(x-1)\cdots(x^\ell-1)\right]}.$$

By choice of r, the numerator of $P(x)$ has more factors of $\Phi_{p^r}(x)$ than the denominator, so $\Phi_{p^r}(x)$ divides $P(x)$.

> **Remark.** Here is a proof that $P(x)$ is divisible by $\Phi_{p^r}(x)$ for some r using the q-binomial formula, without explicitly identifying r. We know that $P(x)$ is the product of several cyclotomic polynomials, and that $P(1)$ is a multiple of p. Thus there is a factor $\Phi_q(x)$ for which $\Phi_q(1)$ is a multiple of p, which is equivalent to q being a power of p.

¶ Second proof of claim. Note that $P(x)$ is the coefficient of y^k in the polynomial

$$Q(x,y) \coloneqq (1+xy)(1+x^2y)\cdots(1+x^ny).$$

Let a be the remainder when n is divided by p^r, and let b be the remainder when k is divided by p^r; then we have $a<b$ by the choice of r. Let $q=\lfloor n/p^r\rfloor$ so $n=p^rq+a$. Consider taking x to be a primitive p^rth root of unity, say ω. Then

$$Q(\omega,y) = \left[(1+\omega y)(1+\omega^2y)\cdots(1+\omega^{p^r}y)\right]^q(1+\omega y)(1+\omega^2y)\dots(1+\omega^ay).$$

Now ω, ω^2, ..., ω^{p^r} are all the p^rth roots of unity, each exactly once; then we can see that

$$\begin{aligned}&(1+\omega y)(1+\omega^2y)\cdots(1+\omega^{p^r}y)\\&=(1-\omega(-y))(1-\omega^2(-y))\cdots(1-\omega^{p^r}(-y))\\&=1-(-y)^{p^r},\end{aligned}$$

so

$$Q(\omega,y) = (1-(-y)^{p^r})^q(1+\omega y)(1+\omega^2y)\dots(1+\omega^ay).$$

In particular, for any m, if the coefficient of y^m in $Q(w,x)$ is nonzero, then m must be congruent to one of $0,1,\dots,a \pmod{p^r}$. Therefore the coefficient of y^k in $Q(\omega,y)$ is zero. This means that $P(\omega)=0$ whenever ω is a primitive p^rth root of unity, which proves the claim.

§2 Solutions to Day 2

§2.1 USA TST 2024/4, proposed by Ray Li

Available online at https://aops.com/community/p29643081.

Problem statement

Find all integers $n \geq 2$ for which there exists a sequence of $2n$ pairwise distinct points $(P_1, \dots, P_n, Q_1, \dots, Q_n)$ in the plane satisfying the following four conditions:

(i) no three of the $2n$ points are collinear;

(ii) $P_iP_{i+1} \geq 1$ for all $i = 1, 2, \dots, n$, where $P_{n+1} = P_1$;

(iii) $Q_iQ_{i+1} \geq 1$ for all $i = 1, 2, \dots, n$, where $Q_{n+1} = Q_1$; and

(iv) $P_iQ_j \leq 1$ for all $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, n$.

¶ **Answer.** Even integers only.

¶ **Proof that even n work.** If we ignore the conditions that the points are pairwise distinct and form no collinear triples, we may take

$$P_{2i+1} = (0.51, 0), \quad P_{2i} = (-0.51, 0), \quad Q_{2i+1} = (0, 0.51), \quad Q_{2i} = (0, -0.51).$$

The distances P_iP_{i+1} and Q_iQ_{i+1} are $1.02 > 1$, while the distances P_iQ_j are $0.51\sqrt{2} < 1$. We may then perturb each point by a small amount to ensure that the distance inequalities still hold and have the points in general position.

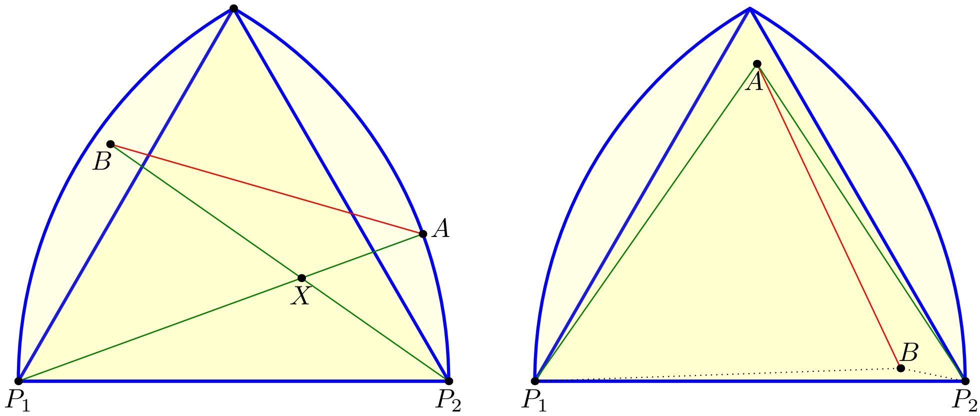
¶ **Proof that odd n do not work.** The main claim is the following.

Claim — For $1 \leq i \leq n$, points Q_i and Q_{i+1} must lie on opposite sides of line P_1P_2.

To isolate the geometry component of the problem, we rewrite the claim in the following contrapositive form, without referencing the points Q_i and Q_{i+1}:

Lemma

Suppose A and B are two points such that $\max(P_1A, P_1B, P_2A, P_2B) \leq 1$, and moreover A and B lie on the same side of line P_1P_2. Further assume no three of $\{P_1, P_2, A, B\}$ are collinear. Then $AB < 1$.



Proof of lemma. Suppose for the sake of contradiction that A and B lie on the same side of P_1P_2. The convex hull of these four points is either a quadrilateral or a triangle.

- If the convex hull is a quadrilateral, assume WLOG that the vertices are P_1P_2AB in order. Let X denote the intersection of segments P_1A and P_2B. Then

$$1 + AB = P_1P_2 + AB < P_1X + XP_2 + AX + XB = P_1A + P_2B \le 2.$$

- Otherwise, assume WLOG that B is in the interior of triangle P_1P_2A. Since $\angle P_1BA + \angle P_2BA = 360^\circ - \angle P_1BP_2 > 180^\circ$, at least one of $\angle P_1BA$ and $\angle P_2BA$ is obtuse. Assume WLOG the former angle is obtuse; then $AB < P_1A \le 1$. $\square$

Remark. Another proof of the lemma can be found by replacing segment AB with the intersection of this line on the boundary of the blue region above, which does not decrease the distance. In other words, one can assume WLOG that A and B lie on either segment AB or one of the two circular arcs. One then proves that $AB \le 1$, and that for equality to occur, one of A and B must lie on segment P_1P_2 However, this approach seems to involve a fair bit more calculation.

Yet another clever approach uses the trivia-fact that a Reuleaux triangle happens to have constant width.

In any case, it's important to realize that this claim is *not trivial*; while it looks like it is easy to prove, it is not, owing to the two near-equality cases.

It follows from the claim that Q_i is on the same side of line P_1P_2 as Q_1 if i is odd, and on the opposite side if i is even. Since $Q_1 = Q_{n+1}$, this means the construction is not possible when n is odd.

Remark. The fact that n cannot be odd follows from Theorem 3 of EPTAS for Max Clique on Disks and Unit Balls. In the language of that paper, if G is a unit ball graph, then the *induced odd cycle parking number* of $\bar{G}$ is at most 1.

In earlier versions of the proposed problem, the points were not necessarily distinct to make the even n case nicer, but this resulted in annoying boundary conditions for the odd n case.

§2.2 USA TST 2024/5, proposed by Ray Li

Available online at https://aops.com/community/p29642892.

Problem statement

Suppose $a_1 < a_2 < \cdots < a_{2024}$ is an arithmetic sequence of positive integers, and $b_1 < b_2 < \cdots < b_{2024}$ is a geometric sequence of positive integers. Find the maximum possible number of integers that could appear in both sequences, over all possible choices of the two sequences.

¶ **Answer.** 11 terms.

¶ **Construction.** Let $a_i = i$ and $b_i = 2^{i-1}$.

¶ **Bound.** We show a ν_p-based approach communicated by Derek Liu, which seems to be the shortest one. At first, we completely ignore the geometric sequence b_i and focus only on the arithmetic sequence.

Claim — Let p be any prime, and consider the sequence

$$\nu_p(a_1),\ \nu_p(a_2),\ \dots,\ \nu_p(a_{2024}).$$

Set $C := \left\lfloor \log_p(2023) \right\rfloor$. Then there are at most $C+2$ different values in this sequence.

Proof. By scaling, assume the a_i do not have a common factor, so that $a_i = a + di$ where $\gcd(a,d) = 1$.

- If $p \mid d$, then $p \nmid a$ and $\nu_p(a_i)$ is constant.
- Otherwise, assume $p \nmid d$. We will in fact prove that every term in the sequence is contained in $\{0, 1, \dots, C\}$ with at most one exception.

  Define $M := \max_i \nu_p(a_i)$. If $M \le C$, there is nothing to prove. Otherwise, fix some index m such that $\nu_p(a_m) = M$. We know $\nu_p(i-m) \le C$ since $|i-m| \le 2023$. But now for any other index $i \ne m$;

$$\begin{aligned} &\nu_p(d(i-m)) = \nu_p(i-m) \le C < M = \nu_p(a_m) \\ \implies\ &\nu_p(a_i) = \nu_p(a_m + d(i-m)) = \nu_p(i-m) \le C. \end{aligned}$$

  so $\nu_p(a_m)$ is the unique exceptional term of the sequence exceeding C. □

Remark. The bound in the claim is best possible by taking $a_i = p^M + (i-1)$ for any $M > C$. Then indeed, the sequence $\nu_p(a_i)$ takes on values in $\{0, 1, \dots, C\}$ for $i > 1$ while $\nu_p(a_1) = M$.

Back to the original problem with (b_i). Consider the common ratio $r \in \mathbb{Q}$ of the geometric sequence (b_i). If p is any prime with $\nu_p(r) \neq 0$, then every term of (b_i) has a different ν_p. So there are two cases to think about.

- Suppose any $p \geq 3$ has $\nu_p(r) \neq 0$. Then there are at most

  $$2 + \log_p(2023) = 2 + \log_3(2023) \approx 8.929 < 11$$

  overlapping terms, as needed.

- Otherwise, suppose r is a power of 2 (in particular, $r \geq 2$ is an integer). We already have an upper bound of 12; we need to improve it to 11.

  As in the proof before, we may assume WLOG by scaling down by a power of 2 that the common difference of a_i is odd. (This may cause some b_i to become rational non-integers, but that's OK. However, unless $\nu_2(a_i)$ is constant, the a_i will still be integers.)

  Then in order for the bound of 12 to be achieved, the sequence $\nu_2(a_i)$ must be contained in $\{0, 1, \dots, 10, M\}$ for some $M \geq 11$. In particular, we only need to work with $r = 2$.

  Denote by b the unique odd-integer term in the geometric sequence, which must appear among (a_i). Then $2b$ appears too, so the common difference of a_i is at most b.

  But if (a_i) is an arithmetic progression of integers that contains b and has common difference at most b, then no term of the sequence can ever exceed $b + 2023 \cdot b = 2024b$. Hence $2^M b$ cannot appear for any $M \geq 11$. This completes the proof.

Remark. There are several other approaches to the problem, but most take some time to execute. The primary issue is that the common difference of the a_i's could share prime factors with the common ratio in the b_i's, which means that merely trying to write out a lot of modular arithmetic equations leads to a lot of potential technical traps that are not pleasant to defuse.

One unusual thing is that many solutions end up proving a bound of 12 (in the case the common ratio is 2) and then having to adjust it to 11 later.

§2.3 USA TST 2024/6, proposed by Milan Haiman

Available online at https://aops.com/community/p29642897.

Problem statement

Solve over $\mathbb{R}$ the functional equation

$$f(xf(y)) + f(y) = f(x+y) + f(xy).$$

In addition to all constant functions, $f(x) \equiv x+1$ clearly works too. We prove these are the only solutions. The solution that follows is by the original proposer.

Let $P(x,y)$ denote the given assertion.

Claim 2.1 — If f is periodic, then f is constant.

Proof. Let f have period $d \neq 0$. From $P(x, y+d)$, we have

$$f(x(y+d)) = f(x+y+d) - f(y+d) - f(xf(y+d)) = f(x+y) - f(y) - f(xf(y)).$$

Applying $P(x,y)$ gives

$$f(x(y+d)) = f(xy).$$

In particular, taking $y=0$ yields that $f(dx) = f(0)$. Thus f is constant, as $d \neq 0$. □

Claim 2.2 — For all real numbers x and y, we have $f(f(x)+y) = f(f(y)+x)$.

Proof. Applying $P(f(x), y)$ and then $P(y,x)$ gives us

$$\begin{aligned} f(f(x)f(y)) &= f(f(x)+y) + f(f(x)y) - f(y) \\ &= f(f(x)+y) + f(x+y) + f(xy) - f(x) - f(y). \end{aligned}$$

Now swapping x and y gives us $f(f(x)+y) = f(f(y)+x)$. □

Claim 2.3 — If f is nonconstant, then $f(f(x)+y) = f(x) + f(y)$ for all reals x, y.

Proof. Let $x, y \in \mathbb{R}$ and let $d := f(f(x)+y) - f(x) - f(y)$. Let z be an arbitrary real number. By repeatedly applying Claim 2.2, we have

$$\begin{aligned} f(z + f(f(x)+y)) &= f(f(z) + f(x) + y) \\ &= f(f(x) + f(z) + y) \\ &= f(x + f(f(z)+y)) \\ &= f(x + f(f(y)+z)) \\ &= f(f(x) + f(y) + z) \\ &= f(z + f(x) + f(y)). \end{aligned}$$

If $d \neq 0$, then f is periodic with period d, contradicting Claim 2.1. □

Claim 2.4 — If f is nonconstant, then $f(0) = 1$ and $f(x+1) = f(x) + 1$.

Proof. From $P(z, 0)$ we have $f(zf(0)) = f(z)$ for all real z. Then $P(xf(0), y)$ and $P(x, y)$ give us

$$\begin{aligned} f(xf(0) + y) &= f(y) + f(xf(0)f(y)) - f(xf(0)y) \\ &= f(y) + f(xf(y)) - f(xy) = f(x + y). \end{aligned}$$

If $f(0) \neq 1$, then $xf(0) - x$ is a period of f for all x, violating Claim 2.1. So we must have $f(0) = 1$.

Now putting $x = 0$ in Claim 2.3 gives $f(x + 1) = f(x) + 1$. □

Claim 2.5 — If f is nonconstant, then $f(x) + f(y) = f(x + y) + 1$.

Proof. From $P(x + 1, y)$ and Claim 2.4, we have

$$f((x+1)f(y)) = f(x+y+1) + f(xy+y) - f(y) = f(x+y) + f(xy+y) - f(y) + 1.$$

Also, from Claim 2.3 and $P(x, y)$, we have

$$f((x + 1)f(y)) = f(xf(y)) + f(y) = f(x + y) + f(xy).$$

Thus $f(xy) = f(xy + y) - f(y) + 1$. Replacing x with $\frac{x}{y}$ gives the claim for all $y \neq 0$ (whereas $y = 0$ follows from Claim 2.4). □

Claim 2.6 — If f is nonconstant, then $f(x) \equiv x + 1$.

Proof. We apply Claim 2.3, Claim 2.5, and Claim 2.4:

$$f(f(x) + y) = f(x) + f(y) = f(x + y) + 1 = f(x + y + 1).$$

If $f(x) \neq x + 1$ for some x, then Claim 2.1 again gives a contradiction. □