USA IMO TST 2021 Solutions

United States of America — IMO Team Selection Test

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Contents

§0 Problems

1. Determine all integers $s \geq 4$ for which there exist positive integers a, b, c, d such that $s = a + b + c + d$ and s divides $abc + abd + acd + bcd$.

2. Points A, V_1, V_2, B, U_2, U_1 lie fixed on a circle Γ, in that order, and such that $BU_2 > AU_1 > BV_2 > AV_1$.

   Let X be a variable point on the arc V_1V_2 of Γ not containing A or B. Line XA meets line U_1V_1 at C, while line XB meets line U_2V_2 at D.

   Prove there exists a fixed point K, independent of X, such that the power of K to the circumcircle of $\triangle XCD$ is constant.

3. Find all functions $f \colon \mathbb{R} \to \mathbb{R}$ that satisfy the inequality

$$f(y) - \left(\frac{z - y}{z - x} f(x) + \frac{y - x}{z - x} f(z) \right) \leq f\left(\frac{x + z}{2} \right) - \frac{f(x) + f(z)}{2}$$

   for all real numbers $x < y < z$.

§1 Solutions to Day 1

§1.1 USA TST 2021/1, proposed by Ankan Bhattacharya, Michael Ren

Available online at https://aops.com/community/p20672573.

Problem statement

Determine all integers $s \geq 4$ for which there exist positive integers a, b, c, d such that $s = a + b + c + d$ and s divides $abc + abd + acd + bcd$.

The answer is s composite.

¶ **Composite construction.**　Write $s = (w + x)(y + z)$, where w, x, y, z are positive integers. Let $a = wy$, $b = wz$, $c = xy$, $d = xz$. Then

$$abc + abd + acd + bcd = wxyz(w + x)(y + z)$$

so this works.

¶ **Prime proof.**　Choose suitable a, b, c, d. Then

$$(a + b)(a + c)(a + d) = (abc + abd + acd + bcd) + a^2(a + b + c + d) \equiv 0 \pmod{s}.$$

Hence s divides a product of positive integers less than s, so s is composite.

> **Remark.** Here is another proof that s is composite.
> Suppose that s is prime. Then the polynomial $(x - a)(x - b)(x - c)(x - d) \in \mathbb{F}_s[x]$ is even, so the roots come in two opposite pairs in $\mathbb{F}_s$. Thus the sum of each pair is at least s, so the sum of all four is at least $2s > s$, contradiction.

§1.2 USA TST 2021/2, proposed by Andrew Gu, Frank Han

Available online at https://aops.com/community/p20672623.

> **Problem statement**
>
> Points A, V_1, V_2, B, U_2, U_1 lie fixed on a circle Γ, in that order, and such that $BU_2 > AU_1 > BV_2 > AV_1$.
>
> Let X be a variable point on the arc V_1V_2 of Γ not containing A or B. Line XA meets line U_1V_1 at C, while line XB meets line U_2V_2 at D.
>
> Prove there exists a fixed point K, independent of X, such that the power of K to the circumcircle of $\triangle XCD$ is constant.

For brevity, we let ℓ_i denote line U_iV_i for $i = 1, 2$.

We first give an explicit description of the fixed point K. Let E and F be points on Γ such that $\overline{AE} \parallel \ell_1$ and $\overline{BF} \parallel \ell_2$. The problem conditions imply that E lies between U_1 and A while F lies between U_2 and B. Then we let

$$K = \overline{AF} \cap \overline{BE}.$$

This point exists because $AEFB$ are the vertices of a convex quadrilateral.

> **Remark** (How to identify the fixed point)**.** If we drop the condition that X lies on the arc, then the choice above is motivated by choosing $X \in \{E, F\}$. Essentially, when one chooses $X \to E$, the point C approaches an infinity point. So in this degenerate case, the only points whose power is finite to (XCD) are bounded are those on line BE. The same logic shows that K must lie on line AF. Therefore, if the problem is going to work, the fixed point must be exactly $\overline{AF} \cap \overline{BE}$.

We give two possible approaches for proving the power of K with respect to (XCD) is fixed.

¶ **First approach by Vincent Huang.**   We need the following claim:

> **Claim —** Suppose distinct lines AC and BD meet at X. Then for any point K
>
> $$\mathrm{pow}(K, XAB) + \mathrm{pow}(K, XCD) = \mathrm{pow}(K, XAD) + \mathrm{pow}(K, XBC).$$

Proof. The difference between the left-hand side and right-hand side is a linear function in K, which vanishes at all of A, B, C, D. $\square$

Construct the points $P = \ell_1 \cap \overline{BE}$ and $Q = \ell_2 \cap \overline{AF}$, which do not depend on X.

> **Claim —** Quadrilaterals $BPCX$ and $AQDX$ are cyclic.

Proof. By Reim's theorem: $\measuredangle CPB = \measuredangle AEB = \measuredangle AXB = \measuredangle CXB$, etc. $\square$
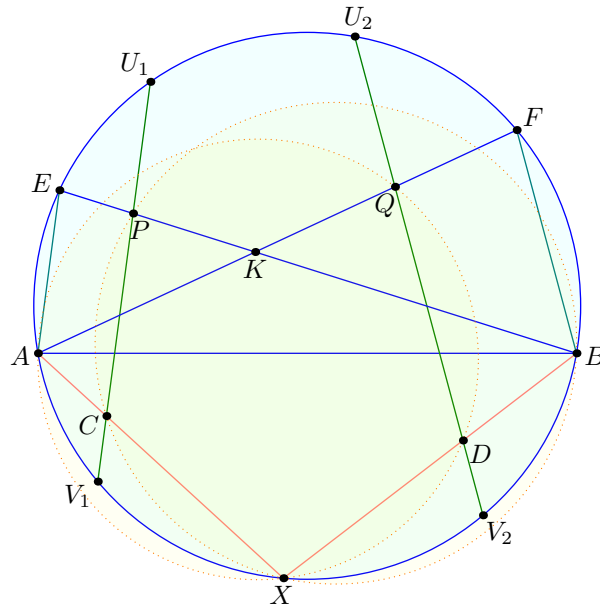
Now, for the particular K we choose, we have

$$\text{pow}(K, XCD) = \text{pow}(K, XAD) + \text{pow}(K, XBC) - \text{pow}(K, XAB)$$
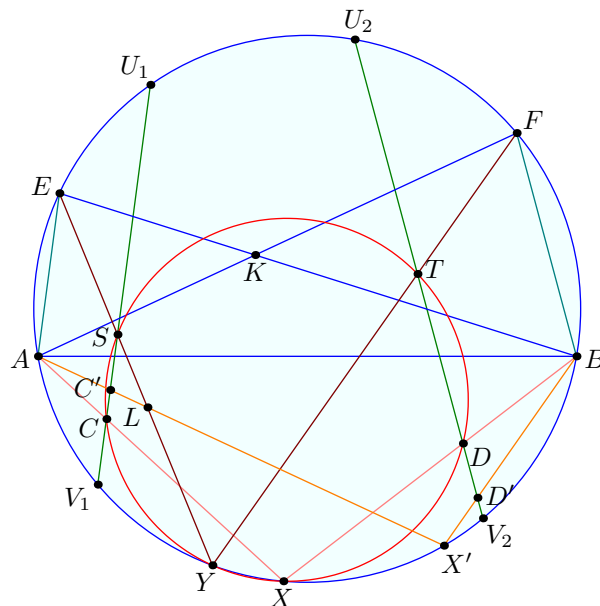$$= KA \cdot KQ + KB \cdot KP - \text{pow}(K, \Gamma).$$

This is fixed, so the proof is completed.

¶ **Second approach by authors.** Let Y be the second intersection of (XCD) with Γ. Let $S = \overline{EY} \cap \ell_1$ and $T = \overline{FY} \cap \ell_2$.

> **Claim —** Points S and T lies on (XCD) as well.

Proof. By Reim's theorem: $\measuredangle CSY = \measuredangle AEY = \measuredangle AXY = \measuredangle CXY$, etc. $\square$

Now let X' be any other choice of X, and define C' and D' in the obvious way. We are going to show that K lies on the radical axis of (XCD) and $(X'C'D')$.

The main idea is as follows:

> **Claim —** The point $L = \overline{EY} \cap \overline{AX'}$ lies on the radical axis. By symmetry, so does the point $M = \overline{FY} \cap \overline{BX'}$ (not pictured).

Proof. Again by Reim's theorem, $SC'YX'$ is cyclic. Hence we have

$$\text{pow}(L, X'C'D') = LC' \cdot LX' = LS \cdot LY = \text{pow}(L, XCD). \qquad \square$$

To conclude, note that by Pascal theorem on

$$EYFAX'B$$

it follows K, L, M are collinear, as needed.

> **Remark.** All the conditions about U_1, V_1, U_2, V_2 at the beginning are there to eliminate configuration issues, making the problem less obnoxious to the contestant.
>
> In particular, without the various assumptions, there exist configurations in which the point K is at infinity. In these cases, the center of XCD moves along a fixed line.

§1.3 USA TST 2021/3, proposed by Gabriel Carroll

Available online at https://aops.com/community/p20672681.

Problem statement

Find all functions $f \colon \mathbb{R} \to \mathbb{R}$ that satisfy the inequality

$$f(y) - \left(\frac{z-y}{z-x} f(x) + \frac{y-x}{z-x} f(z) \right) \leq f\left(\frac{x+z}{2} \right) - \frac{f(x) + f(z)}{2}$$

for all real numbers $x < y < z$.

Answer: all functions of the form $f(y) = ay^2 + by + c$, where a, b, c are constants with $a \leq 0$.

If $I = (x, z)$ is an interval, we say that a real number α is a *supergradient* of f at $y \in I$ if we always have

$$f(t) \leq f(y) + \alpha(t - y)$$

for every $t \in I$. (This inequality may be familiar as the so-called "tangent line trick". A cartoon of this situation is drawn below for intuition.) We will also say α is a supergradient of f at y, without reference to the interval, if α is a supergradient of *some* open interval containing y.
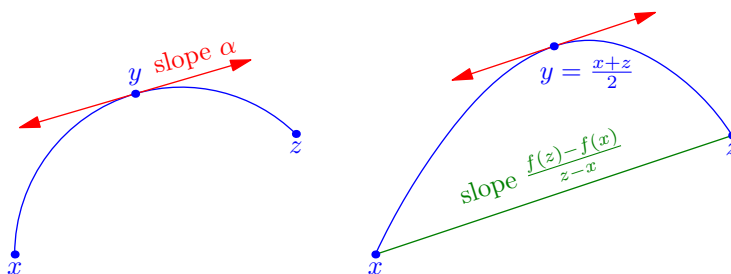
> **Claim —** The problem condition is equivalent to asserting that $\frac{f(z)-f(x)}{z-x}$ is a supergradient of f at $\frac{x+z}{2}$ for the interval (x, z), for every $x < z$.

Proof. This is just manipulation. $\square$

At this point, we may readily verify that all claimed quadratic functions $f(x) = ax^2 + bx + c$ work — these functions are concave, so the graphs lie below the tangent line at any point. Given $x < z$, the tangent at $\frac{x+z}{2}$ has slope given by the derivative $f'(x) = 2ax + b$, that is

$$f'\left(\frac{x+z}{2} \right) = 2a \cdot \frac{x+z}{2} + b = \frac{f(z) - f(x)}{z - x}$$

as claimed. (Of course, it is also easy to verify the condition directly by elementary means, since it is just a polynomial inequality.)

Now suppose f satisfies the required condition; we prove that it has the above form.

> **Claim —** The function f is concave.

Proof. Choose any $\Delta > \max\{z - y, y - x\}$. Since f has a supergradient α at y over the interval $(y - \Delta, y + \Delta)$, and this interval includes x and z, we have

$$\frac{z - y}{z - x}f(x) + \frac{y - x}{z - x}f(z) \le \frac{z - y}{z - x}(f(y) + \alpha(x - y)) + \frac{y - x}{z - x}(f(y) + \alpha(z - y))$$
$$= f(y).$$

That is, f is a concave function. Continuity follows from the fact that any concave function on $\mathbb{R}$ is automatically continuous. $\qquad\square$

> **Lemma** (see e.g. https://math.stackexchange.com/a/615161 for picture)
>
> Any concave function f on $\mathbb{R}$ is continuous.

Proof. Suppose we wish to prove continuity at $p \in \mathbb{R}$. Choose any real numbers a and b with $a < p < b$. For any $0 < \varepsilon < \max(b - p, p - a)$ we always have

$$f(p) + \frac{f(b) - f(p)}{b - p}\varepsilon \le f(p + \varepsilon) \le f(p) + \frac{f(p) - f(a)}{p - a}\varepsilon$$

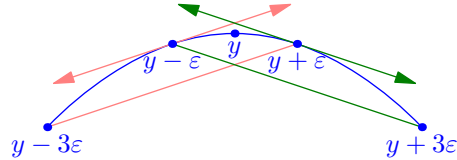which implies right continuity; the proof for left continuity is the same. $\qquad\square$

> **Claim** — The function f cannot have more than one supergradient at any given point.

Proof. Fix $y \in \mathbb{R}$. For $t > 0$, let's define the function

$$g(t) = \frac{f(y) - f(y - t)}{t} - \frac{f(y + t) - f(y)}{t}.$$

We contend that $g(\varepsilon) \le \frac{3}{5}g(3\varepsilon)$ for any $\varepsilon > 0$. Indeed by the problem condition,

$$f(y) \le f(y - \varepsilon) + \frac{f(y + \varepsilon) - f(y - 3\varepsilon)}{4}$$
$$f(y) \le f(y + \varepsilon) - \frac{f(y + 3\varepsilon) - f(y - \varepsilon)}{4}.$$

Summing gives the desired conclusion.

Now suppose that f has two supergradients $\alpha < \alpha'$ at point y. For small enough ε, we should have we have $f(y - \varepsilon) \le f(y) - \alpha'\varepsilon$ and $f(y + \varepsilon) \le f(y) + \alpha\varepsilon$, hence

$$g(\varepsilon) = \frac{f(y) - f(y - \varepsilon)}{\varepsilon} - \frac{f(y + \varepsilon) - f(y)}{\varepsilon} \ge \alpha' - \alpha.$$

This is impossible since $g(\varepsilon)$ may be arbitrarily small. $\qquad\square$

> **Claim** — The function f is quadratic on the rational numbers.

Proof. Consider any four-term arithmetic progression $x, x + d, x + 2d, x + 3d$. Because $(f(x + 2d) - f(x + d))/d$ and $(f(x + 3d) - f(x))/3d$ are both supergradients of f at the point $x + 3d/2$, they must be equal, hence

$$f(x + 3d) - 3f(x + 2d) + 3f(x + d) - f(x) = 0. \tag{1}$$

If we fix $d = 1/n$, it follows inductively that f agrees with a quadratic function $\widetilde{f}_n$ on the set $\frac{1}{n}\mathbb{Z}$. On the other hand, for any $m \neq n$, we apparently have $\widetilde{f}_n = \widetilde{f}_{mn} = \widetilde{f}_m$, so the quadratic functions on each "layer" are all equal. $\qquad\square$

Since f is continuous, it follows f is quadratic, as needed.

> **Remark** (Alternate finish using differentiability due to Michael Ren). In the proof of the main claim (about uniqueness of supergradients), we can actually notice the two terms $\frac{f(y)-f(y-t)}{t}$ and $\frac{f(y+t)-f(y)}{t}$ in the definition of $g(t)$ are both monotonic (by concavity). Since we supplied a proof that $\lim_{t\to 0} g(t) = 0$, we find f is differentiable.
>
> Now, if the derivative at some point exists, it must coincide with all the supergradients; (informally, this is why "tangent line trick" always has the slope as the derivative, and formally, we use the mean value theorem). In other words, we must have
>
> $$f(x+y) - f(x-y) = 2f'(x) \cdot y$$
>
> holds for all real numbers x and y. By choosing $y = 1$ we obtain that $f'(x) = f(x+1) - f(x-1)$ which means f' is also continuous.
>
> Finally differentiating both sides with respect to y gives
>
> $$f'(x+y) - f'(x-y) = 2f'(x)$$
>
> which means f' obeys Jensen's functional equation. Since f' is continuous, this means f' is linear. Thus f is quadratic, as needed.