# USA IMO TST 2021 Solutions

## United States of America — IMO Team Selection Test

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## Contents

# §1 Solutions to February TST

## §1.1 Solution to TST 1, by Ankan Bhattacharya and Michael Ren

Determine all integers $s \geq 4$ for which there exist positive integers $a$, $b$, $c$, $d$ such that $s = a+b+c+d$ and $s$ divides $abc + abd + acd + bcd$.

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The answer is $s$ composite.

**Composite construction** Write $s = (w+x)(y+z)$, where $w$, $x$, $y$, $z$ are positive integers. Let $a = wy$, $b = wz$, $c = xy$, $d = xz$. Then

$$abc + abd + acd + bcd = wxyz(w + x)(y + z)$$

so this works.

**Prime proof** Choose suitable $a$, $b$, $c$, $d$. Then

$$(a + b)(a + c)(a + d) = (abc + abd + acd + bcd) + a^2(a + b + c + d) \equiv 0 \pmod{s}.$$

Hence $s$ divides a product of positive integers less than $s$, so $s$ is composite.

**Remark.** Here is another proof that $s$ is composite.

Suppose that $s$ is prime. Then the polynomial $(x - a)(x - b)(x - c)(x - d) \in \mathbb{F}_s[x]$ is even, so the roots come in two opposite pairs in $\mathbb{F}_s$. Thus the sum of each pair is at least $s$, so the sum of all four is at least $2s > s$, contradiction.

## §1.2 Solution to TST 2, by Andrew Gu and Frank Han

Points $A$, $V_1$, $V_2$, $B$, $U_2$, $U_1$ lie fixed on a circle $\Gamma$, in that order, and such that $BU_2 > AU_1 > BV_2 > AV_1$.

Let $X$ be a variable point on the arc $V_1V_2$ of $\Gamma$ not containing $A$ or $B$. Line $XA$ meets line $U_1V_1$ at $C$, while line $XB$ meets line $U_2V_2$ at $D$.

Prove there exists a fixed point $K$, independent of $X$, such that the power of $K$ to the circumcircle of $\triangle XCD$ is constant.

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For brevity, we let $\ell_i$ denote line $U_iV_i$ for $i = 1, 2$.

We first give an explicit description of the fixed point $K$. Let $E$ and $F$ be points on $\Gamma$ such that $\overline{AE} \parallel \ell_1$ and $\overline{BF} \parallel \ell_2$. The problem conditions imply that $E$ lies between $U_1$ and $A$ while $F$ lies between $U_2$ and $B$. Then we let

$$K = \overline{AF} \cap \overline{BE}.$$

This point exists because $AEFB$ are the vertices of a convex quadrilateral.

> **Remark** (How to identify the fixed point)**.** If we drop the condition that $X$ lies on the arc, then the choice above is motivated by choosing $X \in \{E, F\}$. Essentially, when one chooses $X \to E$, the point $C$ approaches an infinity point. So in this degenerate case, the only points whose power is finite to $(XCD)$ are bounded are those on line $BE$. The same logic shows that $K$ must lie on line $AF$. Therefore, if the problem is going to work, the fixed point must be exactly $\overline{AF} \cap \overline{BE}$.

We give two possible approaches for proving the power of $K$ with respect to $(XCD)$ is fixed.

**First approach by Vincent Huang** We need the following claim:

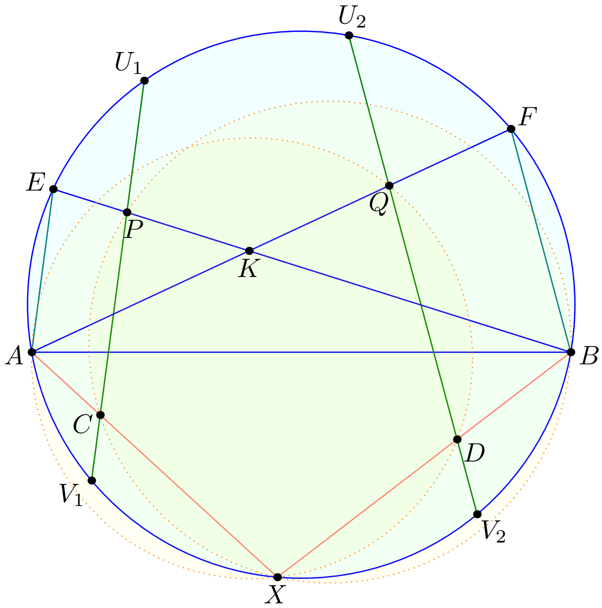
> **Claim —** Suppose distinct lines $AC$ and $BD$ meet at $X$. Then for any point $K$
>
> $$\operatorname{pow}(K, XAB) + \operatorname{pow}(K, XCD) = \operatorname{pow}(K, XAD) + \operatorname{pow}(K, XBC).$$

*Proof.* The difference between the left-hand side and right-hand side is a linear function in $K$, which vanishes at all of $A$, $B$, $C$, $D$. □

Construct the points $P = \ell_1 \cap \overline{BE}$ and $Q = \ell_2 \cap \overline{AF}$, which do not depend on $X$.

> **Claim —** Quadrilaterals $BPCX$ and $AQDX$ are cyclic.

*Proof.* By Reim's theorem: $\measuredangle CPB = \measuredangle AEB = \measuredangle AXB = \measuredangle CXB$, etc. □




Now, for the particular $K$ we choose, we have

$$\begin{aligned} \operatorname{pow}(K, XCD) &= \operatorname{pow}(K, XAD) + \operatorname{pow}(K, XBC) - \operatorname{pow}(K, XAB) \\ &= KA \cdot KQ + KB \cdot KP - \operatorname{pow}(K, \Gamma). \end{aligned}$$

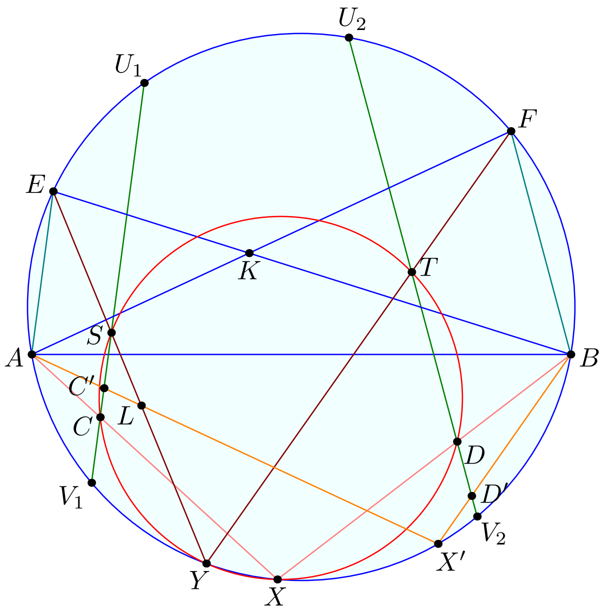
This is fixed, so the proof is completed.

**Second approach by authors** Let $Y$ be the second intersection of $(XCD)$ with $\Gamma$. Let $S = \overline{EY} \cap \ell_1$ and $T = \overline{FY} \cap \ell_2$.

**Claim —** Points $S$ and $T$ lies on $(XCD)$ as well.

*Proof.* By Reim's theorem: $\measuredangle CSY = \measuredangle AEY = \measuredangle AXY = \measuredangle CXY$, etc. $\square$

Now let $X'$ be any other choice of $X$, and define $C'$ and $D'$ in the obvious way. We are going to show that $K$ lies on the radical axis of $(XCD)$ and $(X'C'D')$.



The main idea is as follows:

**Claim** — The point $L = \overline{EY} \cap \overline{AX'}$ lies on the radical axis. By symmetry, so does the point $M = \overline{FY} \cap \overline{BX'}$ (not pictured).

*Proof.* Again by Reim's theorem, $SC'YX'$ is cyclic. Hence we have

$$\operatorname{pow}(L, X'C'D') = LC' \cdot LX' = LS \cdot LY = \operatorname{pow}(L, XCD). \qquad \square$$

To conclude, note that by Pascal theorem on

$$EYFAX'B$$

it follows $K$, $L$, $M$ are collinear, as needed.

**Remark.** All the conditions about $U_1$, $V_1$, $U_2$, $V_2$ at the beginning are there to eliminate configuration issues, making the problem less obnoxious to the contestant.

In particular, without the various assumptions, there exist configurations in which the point $K$ is at infinity. In these cases, the center of $XCD$ moves along a fixed line.

## §1.3 Solution to TST 3, by Gabriel Carroll

Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ that satisfy the inequality

$$f(y) - \left( \frac{z-y}{z-x} f(x) + \frac{y-x}{z-x} f(z) \right) \le f\left( \frac{x+z}{2} \right) - \frac{f(x)+f(z)}{2}$$

for all real numbers $x < y < z$.

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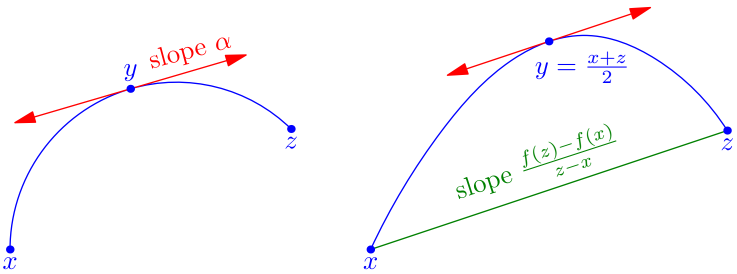
Answer: all functions of the form $f(y) = ay^2 + by + c$, where $a, b, c$ are constants with $a \le 0$.

If $I = (x, z)$ is an interval, we say that a real number $\alpha$ is a *supergradient* of $f$ at $y \in I$ if we always have

$$f(t) \le f(y) + \alpha(t - y)$$

for every $t \in I$. (This inequality may be familiar as the so-called "tangent line trick". A cartoon of this situation is drawn below for intuition.) We will also say $\alpha$ is a supergradient of $f$ at $y$, without reference to the interval, if $\alpha$ is a supergradient of *some* open interval containing $y$.




> **Claim** — The problem condition is equivalent to asserting that $\frac{f(z)-f(x)}{z-x}$ is a supergradient of $f$ at $\frac{x+z}{2}$ for the interval $(x, z)$, for every $x < z$.

*Proof.* This is just manipulation. □

At this point, we may readily verify that all claimed quadratic functions $f(x) = ax^2 + bx + c$ work — these functions are concave, so the graphs lie below the tangent line at any point. Given $x < z$, the tangent at $\frac{x+z}{2}$ has slope given by the derivative $f'(x) = 2ax + b$, that is

$$f'\left( \frac{x+z}{2} \right) = 2a \cdot \frac{x+z}{2} + b = \frac{f(z) - f(x)}{z - x}$$

as claimed. (Of course, it is also easy to verify the condition directly by elementary means, since it is just a polynomial inequality.)

Now suppose $f$ satisfies the required condition; we prove that it has the above form.

> **Claim** — The function $f$ is concave.

*Proof.* Choose any $\Delta > \max\{z - y, y - x\}$. Since $f$ has a supergradient $\alpha$ at $y$ over the interval $(y - \Delta, y + \Delta)$, and this interval includes $x$ and $z$, we have

$$\begin{aligned} \frac{z-y}{z-x} f(x) + \frac{y-x}{z-x} f(z) &\le \frac{z-y}{z-x}(f(y) + \alpha(x-y)) + \frac{y-x}{z-x}(f(y) + \alpha(z-y)) \\ &= f(y). \end{aligned}$$

That is, $f$ is a concave function. Continuity follows from the fact that any concave function on $\mathbb{R}$ is automatically continuous. $\square$

**Lemma** (see e.g. https://math.stackexchange.com/a/615161 for picture)
Any concave function $f$ on $\mathbb{R}$ is continuous.

*Proof.* Suppose we wish to prove continuity at $p \in \mathbb{R}$. Choose any real numbers $a$ and $b$ with $a < p < b$. For any $0 < \varepsilon < \max(b-p, p-a)$ we always have

$$f(p) + \frac{f(b)-f(p)}{b-p}\varepsilon \le f(p+\varepsilon) \le f(p) + \frac{f(p)-f(a)}{p-a}\varepsilon$$

which implies right continuity; the proof for left continuity is the same. $\square$

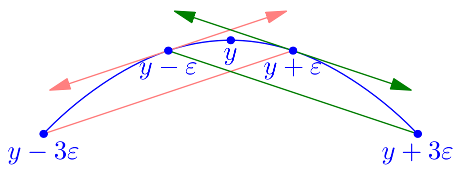
**Claim** — The function $f$ cannot have more than one supergradient at any given point.

*Proof.* Fix $y \in \mathbb{R}$. For $t > 0$, let's define the function

$$g(t) = \frac{f(y)-f(y-t)}{t} - \frac{f(y+t)-f(y)}{t}.$$

We contend that $g(\varepsilon) \le \frac{3}{5} g(3\varepsilon)$ for any $\varepsilon > 0$. Indeed by the problem condition,

$$\begin{aligned} f(y) &\le f(y-\varepsilon) + \frac{f(y+\varepsilon)-f(y-3\varepsilon)}{4} \\ f(y) &\le f(y+\varepsilon) - \frac{f(y+3\varepsilon)-f(y-\varepsilon)}{4}. \end{aligned}$$




Summing gives the desired conclusion.

Now suppose that $f$ has two supergradients $\alpha < \alpha'$ at point $y$. For small enough $\varepsilon$, we should have we have $f(y-\varepsilon) \le f(y) - \alpha'\varepsilon$ and $f(y+\varepsilon) \le f(y) + \alpha\varepsilon$, hence

$$g(\varepsilon) = \frac{f(y)-f(y-\varepsilon)}{\varepsilon} - \frac{f(y+\varepsilon)-f(y)}{\varepsilon} \ge \alpha' - \alpha.$$

This is impossible since $g(\varepsilon)$ may be arbitrarily small. $\square$

**Claim** — The function $f$ is quadratic on the rational numbers.

*Proof.* Consider any four-term arithmetic progression $x, x+d, x+2d, x+3d$. Because $(f(x+2d)-f(x+d))/d$ and $(f(x+3d)-f(x))/3d$ are both supergradients of $f$ at the point $x + 3d/2$, they must be equal, hence

$$f(x+3d) - 3f(x+2d) + 3f(x+d) - f(x) = 0. \tag{1}$$

If we fix $d = 1/n$, it follows inductively that $f$ agrees with a quadratic function $\widetilde{f}_n$ on the set $\frac{1}{n}\mathbb{Z}$. On the other hand, for any $m \ne n$, we apparently have $\widetilde{f}_n = \widetilde{f}_{mn} = \widetilde{f}_m$, so the quadratic functions on each "layer" are all equal. $\square$

Since $f$ is continuous, it follows $f$ is quadratic, as needed.

**Remark** (Alternate finish using differentiability due to Michael Ren)**.** In the proof of the main claim (about uniqueness of supergradients), we can actually notice the two terms $\frac{f(y)-f(y-t)}{t}$ and $\frac{f(y+t)-f(y)}{t}$ in the definition of $g(t)$ are both monotonic (by concavity). Since we supplied a proof that $\lim_{t\to 0} g(t) = 0$, we find $f$ is differentiable.

Now, if the derivative at some point exists, it must coincide with all the supergradients; (informally, this is why "tangent line trick" always has the slope as the derivative, and formally, we use the mean value theorem). In other words, we must have

$$f(x+y) - f(x-y) = 2f'(x) \cdot y$$

holds for all real numbers $x$ and $y$. By choosing $y = 1$ we obtain that $f'(x) = f(x+1) - f(x-1)$ which means $f'$ is also continuous.

Finally differentiating both sides with respect to $y$ gives

$$f'(x+y) - f'(x-y) = 2f'(x)$$

which means $f'$ obeys Jensen's functional equation. Since $f'$ is continuous, this means $f'$ is linear. Thus $f$ is quadratic, as needed.