USA IMO TST 2021 Solutions

United States of America — IMO Team Selection Test

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§0 Problems

1. Determine all integers $s \ge 4$ for which there exist positive integers a, b, c, d such that $s = a + b + c + d$ and s divides $abc + abd + acd + bcd$.

2. Points A, V_1, V_2, B, U_2, U_1 lie fixed on a circle Γ, in that order, and such that $BU_2 > AU_1 > BV_2 > AV_1$.

   Let X be a variable point on the arc V_1V_2 of Γ not containing A or B. Line XA meets line U_1V_1 at C, while line XB meets line U_2V_2 at D.

   Prove there exists a fixed point K, independent of X, such that the power of K to the circumcircle of $\triangle XCD$ is constant.

3. Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ that satisfy the inequality

$$f(y) - \left(\frac{z-y}{z-x} f(x) + \frac{y-x}{z-x} f(z)\right) \le f\left(\frac{x+z}{2}\right) - \frac{f(x)+f(z)}{2}$$

   for all real numbers $x < y < z$.

§1 Solutions to Day 1

§1.1 USA TST 2021/1, proposed by Ankan Bhattacharya, Michael Ren

Available online at https://aops.com/community/p20672573.

Problem statement

Determine all integers $s \geq 4$ for which there exist positive integers a, b, c, d such that $s = a + b + c + d$ and s divides $abc + abd + acd + bcd$.

The answer is s composite.

¶ Composite construction. Write $s = (w + x)(y + z)$, where w, x, y, z are positive integers. Let $a = wy$, $b = wz$, $c = xy$, $d = xz$. Then

$$abc + abd + acd + bcd = wxyz(w + x)(y + z)$$

so this works.

¶ Prime proof. Choose suitable a, b, c, d. Then

$$(a + b)(a + c)(a + d) = (abc + abd + acd + bcd) + a^2(a + b + c + d) \equiv 0 \pmod{s}.$$

Hence s divides a product of positive integers less than s, so s is composite.

> **Remark.** Here is another proof that s is composite.
> Suppose that s is prime. Then the polynomial $(x-a)(x-b)(x-c)(x-d) \in \mathbb{F}_s[x]$ is even, so the roots come in two opposite pairs in $\mathbb{F}_s$. Thus the sum of each pair is at least s, so the sum of all four is at least $2s > s$, contradiction.

§1.2 USA TST 2021/2, proposed by Andrew Gu, Frank Han

Available online at https://aops.com/community/p20672623.

Problem statement

Points A, V_1, V_2, B, U_2, U_1 lie fixed on a circle Γ, in that order, and such that $BU_2 > AU_1 > BV_2 > AV_1$.

Let X be a variable point on the arc V_1V_2 of Γ not containing A or B. Line XA meets line U_1V_1 at C, while line XB meets line U_2V_2 at D.

Prove there exists a fixed point K, independent of X, such that the power of K to the circumcircle of $\triangle XCD$ is constant.

For brevity, we let ℓ_i denote line U_iV_i for $i = 1, 2$.

We first give an explicit description of the fixed point K. Let E and F be points on Γ such that $\overline{AE} \parallel \ell_1$ and $\overline{BF} \parallel \ell_2$. The problem conditions imply that E lies between U_1 and A while F lies between U_2 and B. Then we let

$$K = \overline{AF} \cap \overline{BE}.$$

This point exists because $AEFB$ are the vertices of a convex quadrilateral.

> **Remark** (How to identify the fixed point)**.** If we drop the condition that X lies on the arc, then the choice above is motivated by choosing $X \in \{E, F\}$. Essentially, when one chooses $X \to E$, the point C approaches an infinity point. So in this degenerate case, the only points whose power is finite to (XCD) are bounded are those on line BE. The same logic shows that K must lie on line AF. Therefore, if the problem is going to work, the fixed point must be exactly $\overline{AF} \cap \overline{BE}$.

We give two possible approaches for proving the power of K with respect to (XCD) is fixed.

¶ First approach by Vincent Huang. We need the following claim:

Claim — Suppose distinct lines AC and BD meet at X. Then for any point K

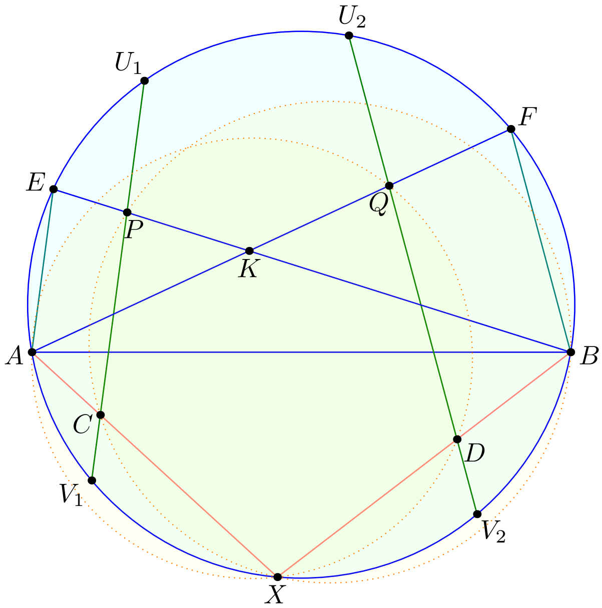
$$\operatorname{pow}(K, XAB) + \operatorname{pow}(K, XCD) = \operatorname{pow}(K, XAD) + \operatorname{pow}(K, XBC).$$

Proof. The difference between the left-hand side and right-hand side is a linear function in K, which vanishes at all of A, B, C, D. □

Construct the points $P = \ell_1 \cap \overline{BE}$ and $Q = \ell_2 \cap \overline{AF}$, which do not depend on X.

Claim — Quadrilaterals $BPCX$ and $AQDX$ are cyclic.

Proof. By Reim's theorem: $\measuredangle CPB = \measuredangle AEB = \measuredangle AXB = \measuredangle CXB$, etc. □



Now, for the particular K we choose, we have

$$\begin{aligned}\mathrm{pow}(K, XCD) &= \mathrm{pow}(K, XAD) + \mathrm{pow}(K, XBC) - \mathrm{pow}(K, XAB) \\ &= KA \cdot KQ + KB \cdot KP - \mathrm{pow}(K, \Gamma).\end{aligned}$$

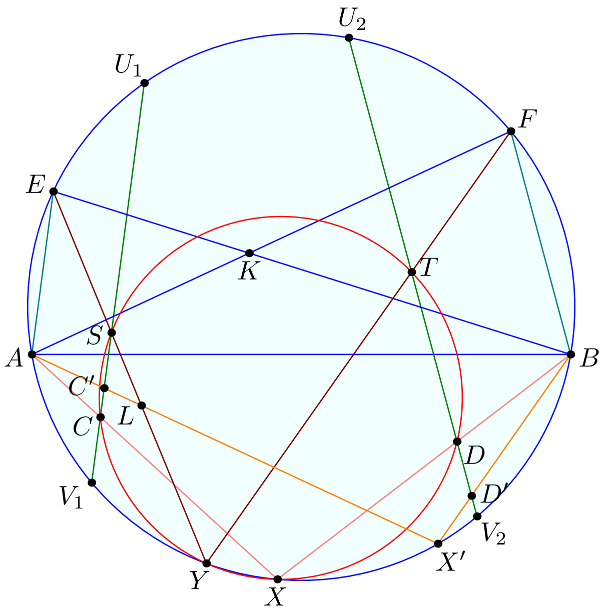
This is fixed, so the proof is completed.

¶ Second approach by authors. Let Y be the second intersection of (XCD) with Γ. Let $S = \overline{EY} \cap \ell_1$ and $T = \overline{FY} \cap \ell_2$.

Claim — Points S and T lies on (XCD) as well.

Proof. By Reim's theorem: $\measuredangle CSY = \measuredangle AEY = \measuredangle AXY = \measuredangle CXY$, etc. □

Now let X' be any other choice of X, and define C' and D' in the obvious way. We are going to show that K lies on the radical axis of (XCD) and $(X'C'D')$.



The main idea is as follows:

Claim — The point $L = \overline{EY} \cap \overline{AX'}$ lies on the radical axis. By symmetry, so does the point $M = \overline{FY} \cap \overline{BX'}$ (not pictured).

Proof. Again by Reim's theorem, $SC'YX'$ is cyclic. Hence we have

$$\operatorname{pow}(L, X'C'D') = LC' \cdot LX' = LS \cdot LY = \operatorname{pow}(L, XCD). \qquad \square$$

To conclude, note that by Pascal theorem on

$$EYFAX'B$$

it follows K, L, M are collinear, as needed.

Remark. All the conditions about U_1, V_1, U_2, V_2 at the beginning are there to eliminate configuration issues, making the problem less obnoxious to the contestant.

In particular, without the various assumptions, there exist configurations in which the point K is at infinity. In these cases, the center of XCD moves along a fixed line.

§1.3 USA TST 2021/3, proposed by Gabriel Carroll

Available online at https://aops.com/community/p20672681.

Problem statement

Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ that satisfy the inequality

$$f(y) - \left(\frac{z-y}{z-x}f(x) + \frac{y-x}{z-x}f(z)\right) \le f\left(\frac{x+z}{2}\right) - \frac{f(x)+f(z)}{2}$$

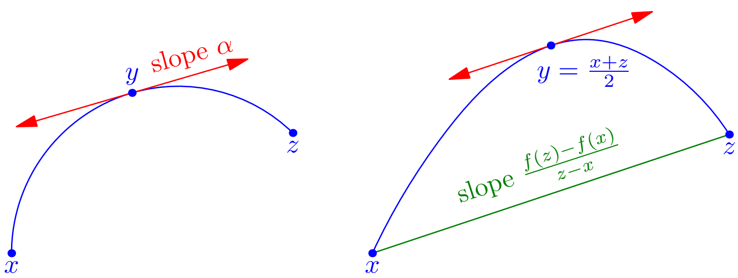
for all real numbers $x < y < z$.

Answer: all functions of the form $f(y) = ay^2 + by + c$, where a, b, c are constants with $a \le 0$.

If $I = (x, z)$ is an interval, we say that a real number α is a *supergradient* of f at $y \in I$ if we always have

$$f(t) \le f(y) + \alpha(t - y)$$

for every $t \in I$. (This inequality may be familiar as the so-called "tangent line trick". A cartoon of this situation is drawn below for intuition.) We will also say α is a supergradient of f at y, without reference to the interval, if α is a supergradient of *some* open interval containing y.



Claim — The problem condition is equivalent to asserting that $\frac{f(z)-f(x)}{z-x}$ is a supergradient of f at $\frac{x+z}{2}$ for the interval (x, z), for every $x < z$.

Proof. This is just manipulation. □

At this point, we may readily verify that all claimed quadratic functions $f(x) = ax^2 + bx + c$ work — these functions are concave, so the graphs lie below the tangent line at any point. Given $x < z$, the tangent at $\frac{x+z}{2}$ has slope given by the derivative $f'(x) = 2ax + b$, that is

$$f'\left(\frac{x+z}{2}\right) = 2a \cdot \frac{x+z}{2} + b = \frac{f(z) - f(x)}{z - x}$$

as claimed. (Of course, it is also easy to verify the condition directly by elementary means, since it is just a polynomial inequality.)

Now suppose f satisfies the required condition; we prove that it has the above form.

Claim — The function f is concave.

Proof. Choose any $\Delta > \max\{z-y, y-x\}$. Since f has a supergradient α at y over the interval $(y-\Delta, y+\Delta)$, and this interval includes x and z, we have

$$\begin{aligned} \frac{z-y}{z-x}f(x) + \frac{y-x}{z-x}f(z) &\le \frac{z-y}{z-x}(f(y)+\alpha(x-y)) + \frac{y-x}{z-x}(f(y)+\alpha(z-y)) \\ &= f(y). \end{aligned}$$

That is, f is a concave function. Continuity follows from the fact that any concave function on $\mathbb{R}$ is automatically continuous. □

Lemma (see e.g. https://math.stackexchange.com/a/615161 for picture)
Any concave function f on $\mathbb{R}$ is continuous.

Proof. Suppose we wish to prove continuity at $p \in \mathbb{R}$. Choose any real numbers a and b with $a < p < b$. For any $0 < \varepsilon < \max(b-p, p-a)$ we always have

$$f(p) + \frac{f(b)-f(p)}{b-p}\varepsilon \le f(p+\varepsilon) \le f(p) + \frac{f(p)-f(a)}{p-a}\varepsilon$$

which implies right continuity; the proof for left continuity is the same. □

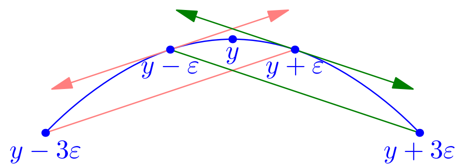
Claim — The function f cannot have more than one supergradient at any given point.

Proof. Fix $y \in \mathbb{R}$. For $t > 0$, let's define the function

$$g(t) = \frac{f(y)-f(y-t)}{t} - \frac{f(y+t)-f(y)}{t}.$$

We contend that $g(\varepsilon) \le \frac{3}{5}g(3\varepsilon)$ for any $\varepsilon > 0$. Indeed by the problem condition,

$$\begin{aligned} f(y) &\le f(y-\varepsilon) + \frac{f(y+\varepsilon)-f(y-3\varepsilon)}{4} \\ f(y) &\le f(y+\varepsilon) - \frac{f(y+3\varepsilon)-f(y-\varepsilon)}{4}. \end{aligned}$$



Summing gives the desired conclusion.

Now suppose that f has two supergradients $\alpha < \alpha'$ at point y. For small enough ε, we should have we have $f(y-\varepsilon) \le f(y) - \alpha'\varepsilon$ and $f(y+\varepsilon) \le f(y) + \alpha\varepsilon$, hence

$$g(\varepsilon) = \frac{f(y)-f(y-\varepsilon)}{\varepsilon} - \frac{f(y+\varepsilon)-f(y)}{\varepsilon} \ge \alpha' - \alpha.$$

This is impossible since $g(\varepsilon)$ may be arbitrarily small. □

Claim — The function f is quadratic on the rational numbers.

Proof. Consider any four-term arithmetic progression $x, x+d, x+2d, x+3d$. Because $(f(x+2d)-f(x+d))/d$ and $(f(x+3d)-f(x))/3d$ are both supergradients of f at the point $x+3d/2$, they must be equal, hence

$$f(x+3d) - 3f(x+2d) + 3f(x+d) - f(x) = 0. \tag{1}$$

If we fix $d = 1/n$, it follows inductively that f agrees with a quadratic function $\widetilde{f_n}$ on the set $\frac{1}{n}\mathbb{Z}$. On the other hand, for any $m \neq n$, we apparently have $\widetilde{f_n} = \widetilde{f_{mn}} = \widetilde{f_m}$, so the quadratic functions on each "layer" are all equal. $\square$

Since f is continuous, it follows f is quadratic, as needed.

> **Remark** (Alternate finish using differentiability due to Michael Ren)**.** In the proof of the main claim (about uniqueness of supergradients), we can actually notice the two terms $\frac{f(y)-f(y-t)}{t}$ and $\frac{f(y+t)-f(y)}{t}$ in the definition of $g(t)$ are both monotonic (by concavity). Since we supplied a proof that $\lim_{t\to 0} g(t) = 0$, we find f is differentiable.
>
> Now, if the derivative at some point exists, it must coincide with all the supergradients; (informally, this is why "tangent line trick" always has the slope as the derivative, and formally, we use the mean value theorem). In other words, we must have
>
> $$f(x+y) - f(x-y) = 2f'(x) \cdot y$$
>
> holds for all real numbers x and y. By choosing $y = 1$ we obtain that $f'(x) = f(x+1) - f(x-1)$ which means f' is also continuous.
>
> Finally differentiating both sides with respect to y gives
>
> $$f'(x+y) - f'(x-y) = 2f'(x)$$
>
> which means f' obeys Jensen's functional equation. Since f' is continuous, this means f' is linear. Thus f is quadratic, as needed.