USA IMO TST 2020 Solutions

United States of America — IMO Team Selection Tests

Ankan Bhattacharya and Evan Chen

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§1 Solutions to Day 1

§1.1 Solution to TST 1, by Carl Schildkraut and Milan Haiman

Choose positive integers $b_1, b_2, \dots$ satisfying

$$1 = \frac{b_1}{1^2} > \frac{b_2}{2^2} > \frac{b_3}{3^2} > \frac{b_4}{4^2} > \cdots$$

and let r denote the largest real number satisfying $\frac{b_n}{n^2} \ge r$ for all positive integers n. What are the possible values of r across all possible choices of the sequence (b_n)?

The answer is $0 \le r \le 1/2$. Obviously $r \ge 0$.

In one direction, we show that

Claim (Greedy bound) — For all integers n, we have

$$\frac{b_n}{n^2} \le \frac{1}{2} + \frac{1}{2n}.$$

Proof. This is by induction on n. For $n = 1$ it is given. For the inductive step we have

$$\begin{aligned} b_n &< n^2 \frac{b_{n-1}}{(n-1)^2} \le n^2 \left(\frac{1}{2} + \frac{1}{2(n-1)} \right) = \frac{n^3}{2(n-1)} \\ &= \frac{1}{2} \left[n^2 + n + 1 + \frac{1}{n-1} \right] \\ &= \frac{n(n+1)}{2} + \frac{1}{2} \left[1 + \frac{1}{n-1} \right] \\ &\le \frac{n(n+1)}{2} + 1 \end{aligned}$$

So $b_n < \frac{n(n+1)}{2} + 1$ and since b_n is an integer, $b_n \le \frac{n(n+1)}{2}$. This implies the result. $\square$

We now give a construction. For $r = 1/2$ we take $b_n = \frac{1}{2}n(n+1)$ for $r = 0$ we take $b_n = 1$.

Claim (Explicit construction, given by Nikolai Beluhov) — Fix $0 < r < 1/2$. Let N be large enough that $\lceil rn^2 + n \rceil < \frac{1}{2}n(n+1)$ for all $n \ge N$. Then the following sequence works:

$$b_n = \begin{cases} \lceil rn^2 + n \rceil & n \ge N \\ \frac{n^2+n}{2} & n < N. \end{cases}$$

Proof. We certainly have

$$\frac{b_n}{n^2} = \frac{rn^2 + n + O(1)}{n^2} \xrightarrow{n \to \infty} r.$$

Mainly, we contend $b_n n^{-2}$ is strictly decreasing. We need only check this for $n \ge N$; in fact

$$\frac{b_n}{n^2} \ge \frac{rn^2 + n}{n^2} > \frac{[r(n+1)^2 + (n+1)] + 1}{(n+1)^2} > \frac{b_{n+1}}{(n+1)^2}$$

where the middle inequality is true since it rearranges to $\frac{1}{n} > \frac{n+2}{(n+1)^2}$. $\square$

§1.2 Solution to TST 2, by Merlijn Staps

Two circles Γ_1 and Γ_2 have common external tangents ℓ_1 and ℓ_2 meeting at T. Suppose ℓ_1 touches Γ_1 at A and ℓ_2 touches Γ_2 at B. A circle Ω through A and B intersects Γ_1 again at C and Γ_2 again at D, such that quadrilateral $ABCD$ is convex.

Suppose lines AC and BD meet at point X, while lines AD and BC meet at point Y. Show that T, X, Y are collinear.

We present four solutions.

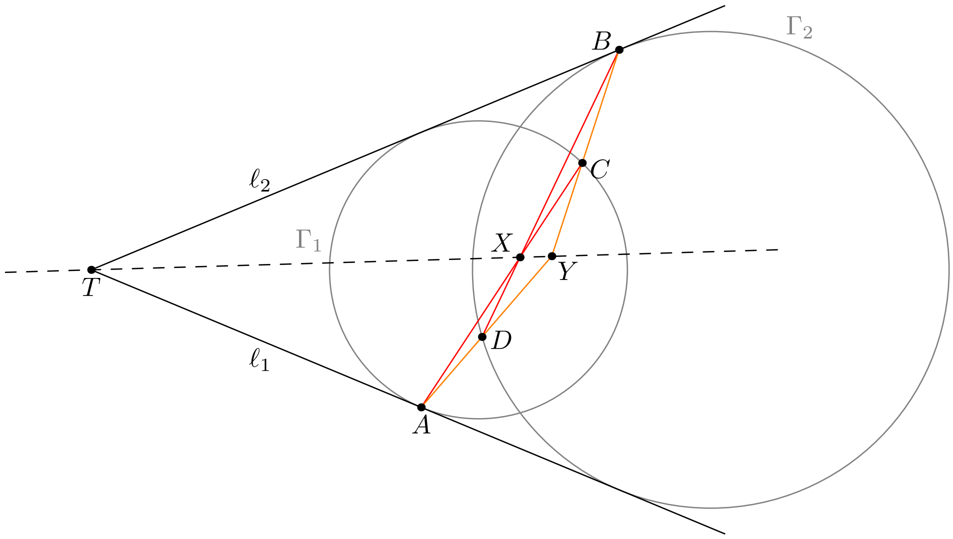
First solution, elementary (original) We have $\triangle YAC \sim \triangle YBD$, from which it follows

$$\frac{d(Y,AC)}{d(Y,BD)} = \frac{AC}{BD}.$$

Moreover, if we denote by r_1 and r_2 the radii of Γ_1 and Γ_2, then

$$\frac{d(T,AC)}{d(T,BD)} = \frac{TA\sin\angle(AC,\ell_1)}{TB\sin\angle(BD,\ell_2)} = \frac{2r_1\sin\angle(AC,\ell_1)}{2r_2\sin\angle(BD,\ell_2)} = \frac{AC}{BD}$$

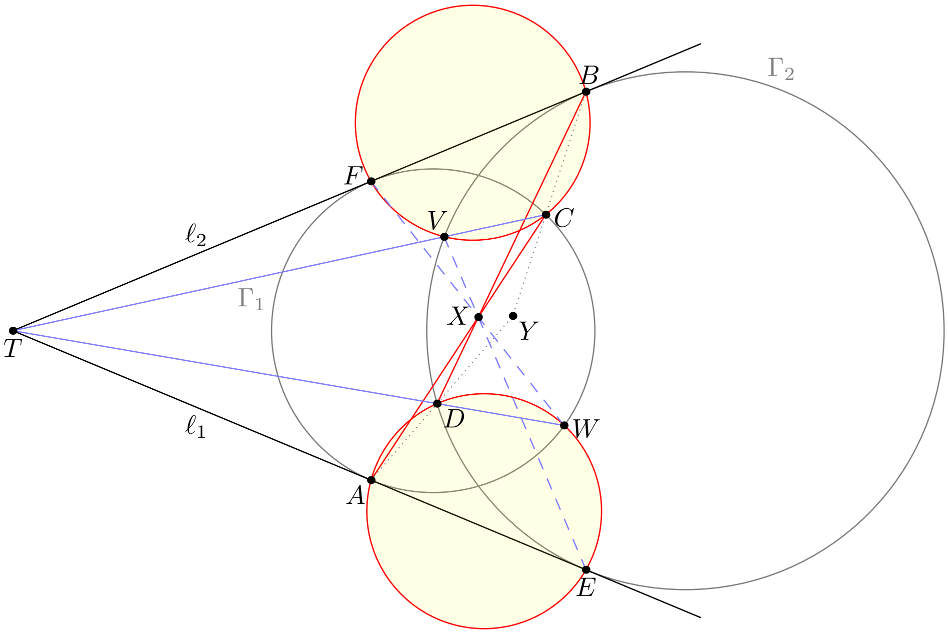
the last step by the law of sines.



This solves the problem up to configuration issues: we claim that Y and T both lie inside $\angle AXB \equiv \angle CXD$. WLOG $TA < TB$.

- The former is since Y lies outside segments BC and AD, since we assumed $ABCD$ was convex.
- For the latter, we note that X lies inside both Γ_1 and Γ_2 in fact on the radical axis of the two circles (since X was an interior point of both chords AC and BD). In particular, X is contained inside $\angle ATB$, and moreover $\angle ATB < 90^\circ$, and this is enough to imply the result.

Second solution, inversive This is based on the solution posted by `kapilpavase` on AoPS. Consider the inversion at T swapping Γ_1 and Γ_2; we let it send A to E, B to F, C to V, D to W, as shown. Draw circles $ADWE$ and $BCVF$.



Claim — Points T and Y lie on the radical axis of (ADE) and (BCF).

Proof. Because $TF \cdot TB = TA \cdot TE$ and $YA \cdot YD = YC \cdot YB$. □

Claim — Point X has equal power to (ADE) and (BCF).

Proof. Since $TV \cdot TC = TA \cdot TE$, quadrilateral $VCEA$ is cyclic too, so by radical axis with Γ_1 and Γ_2 we find X lies on VE. Similarly, X lies on FW. Thus, X is the center of negative inversion between (ADE) and (BCF).

But since $AE = BF$ and moreover

$$\begin{aligned} \measuredangle BCF + \measuredangle ADE &= (\measuredangle BCA + \measuredangle ACF) + (\measuredangle ADB + \measuredangle BDE) \\ &= (\measuredangle BCA + \measuredangle ADB) + (\measuredangle ACF + \measuredangle BDE) = 0 + 0 = 0 \end{aligned}$$

we conclude that (ADE) and (BCF) are *congruent*. As X was the center of negative inversion between them, we're done. □

Third solution, projective (Nikolai Beluhov) We start with some definitions. Let ℓ_1 touch Γ_2 at E, ℓ_2 touch Γ_1 at F, $K = \ell_1 \cap \overline{BD}$, $L = \ell_2 \cap \overline{AC}$, line FX meet Γ_1 again at M, line EX meet Γ_2 again at N, and lines AB, AD, and BC meet line TX at Z, Y_1, and Y_2. Thus the desired statement is equivalent to $Y_1 = Y_2$.

Claim — $(EB; ND)_{\Gamma_2} = (FA; MC)_{\Gamma_1}$.

Proof. Note that $AX \cdot XC = BX \cdot XD = EX \cdot XN$, so $AECN$ is cyclic. Likewise $BFDM$ is cyclic.

Consider the inversion with center T which swaps Γ_1 and Γ_2; it also swaps the pairs $\{A, E\}$ and $\{B, F\}$. Since $AECN$ is cyclic, C is on Γ_1, and N is on Γ_2, it also swaps $\{C, N\}$; similarly it swaps $\{D, M\}$.

Thus $(EB; ND)_{\Gamma_2} = (AF; CM)_{\Gamma_1} = (FA; MC)_{\Gamma_1}$ as desired. $\square$

With this claim, the remainder of the proof is chasing cross-ratios:

$$(TZ; XY_1) \stackrel{A}{=} (KB; XD) \stackrel{E}{=} (EB; ND)_{\Gamma_2} = (FA; MC)_{\Gamma_1} \stackrel{F}{=} (LA; XC) \stackrel{B}{=} (TZ; XY_2)$$

implies $Y_1 = Y_2$ as desired.

Fourth solution by untethered moving points Fix ℓ_1, ℓ_2, T, Γ_1 and Γ_2, and let Γ_1 and Γ_2 meet at U and V. By the radical axis theorem, X lies on UV.

Thus we instead treat X as a variable point on line UV and let $C = AX \cap \Gamma_1$, $D = BX \cap \Gamma_2$. By definition, X has degree 1 and T has degree 0.

We apply **Zack's lemma** to untethered point Y. Note that C and D move projectively on conics, and therefore have degree 2. Then, lines AD and BC each have degree at most $\deg(A) + \deg(D) = 0 + 2 = 2$, and so their intersection Y has degree at most $2 + 2 = 4$. But when $X \in AB$, the lines AD and BC are the same, so Zack's lemma implies that

$$\deg Y \le 4 - 1 = 3.$$

Thus the assertion that T, X, Y are collinear (which for example can be seen as a certain vanishing determinant) is a statement of degree at most $0 + 1 + 3 = 4$. Thus it suffices to find 5 values of X (other than $X \in \overline{AB}$, which we used already). This is remarkably easy:

1. When $X = U$ or $X = V$, then $X = C = D = Y$ and the statement is obvious

2. When $X \in \ell_1$, say, then $A = C$ and so Y lies on $AC = \ell_1$ as well. The case $X \in \ell_2$ is symmetric.

3. Finally, take X at infinity along UV. Then C and D are the other tangency points of the circles with ℓ_1, ℓ_2, and so $AC = \ell_1, BD = \ell_2$, so $Y = T$.

This finishes the problem.

§1.3 Solution to TST 3, by Nikolai Beluhov

Let $\alpha \geq 1$ be a real number. Hephaestus and Poseidon play a turn-based game on an infinite grid of unit squares. Before the game starts, Poseidon chooses a finite number of cells to be *flooded*. Hephaestus is building a *levee*, which is a subset of unit edges of the grid (called *walls*) forming a connected, non-self-intersecting path or loop.

The game then begins with Hephaestus moving first. On each of Hephaestus's turns, he adds one or more walls to the levee, as long as the total length of the levee is at most αn after his nth turn. On each of Poseidon's turns, every cell which is adjacent to an already flooded cell and with no wall between them becomes flooded as well.

Hephaestus wins if the levee forms a closed loop such that all flooded cells are contained in the interior of the loop — hence stopping the flood and saving the world. For which α can Hephaestus guarantee victory in a finite number of turns no matter how Poseidon chooses the initial cells to flood?

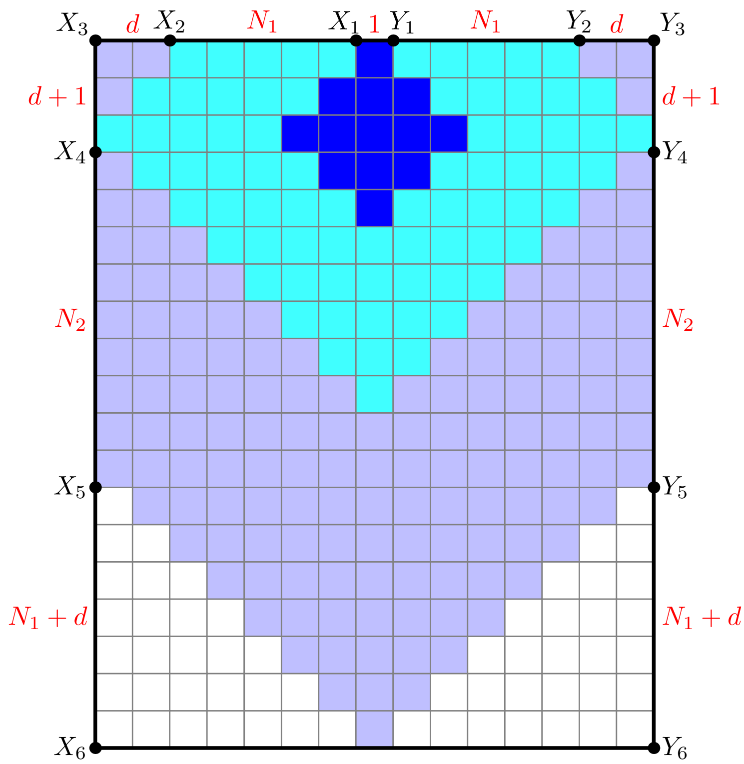
We show that if $\alpha > 2$ then Hephaestus wins, but when $\alpha = 2$ (and hence $\alpha \leq 2$) Hephaestus cannot contain even a single-cell flood initially.

Strategy for $\alpha > 2$: Impose $\mathbb{Z}^2$ coordinates on the cells. Adding more flooded cells does not make our task easier, so let us assume that initially the cells (x, y) with $|x| + |y| \leq d$ are flooded for some $d \geq 2$; thus on Hephaestus's kth turn, the water is contained in $|x| + |y| \leq d + k - 1$. Our goal is to contain the flood with a large rectangle.

We pick large integers N_1 and N_2 such that

$$\alpha N_1 > 2N_1 + (2d + 3)$$
$$\alpha(N_1 + N_2) > 2N_2 + (6N_1 + 8d + 4).$$

Mark the points X_i, Y_i as shown in the figure for $1 \leq i \leq 6$. The red figures indicate the distance between the marked points on the rectangle.



We follow the following plan.

- Turn 1: place wall X_1Y_1. This cuts off the flood to the north.
- Turns 2 through N_1+1: extend the levee to segment X_2Y_2. This prevents further flooding to the north.
- Turn N_1+2: add in broken lines $X_4X_3X_2$ and $Y_4Y_3Y_2$ all at once. This cuts off the flood west and east.
- Turns N_1+2 to N_1+N_2+1: extend the levee along segments X_4X_5 and Y_4Y_5. This prevents further flooding west and east.
- Turn N_1+N_2+2: add in the broken line $X_5X_6Y_6Y_5$ all at once and win.

Proof for $\alpha = 2$: Suppose Hephaestus contains the flood on his $(n+1)$st turn. We prove that $\alpha > 2$ by showing that in fact at least $2n+4$ walls have been constructed.

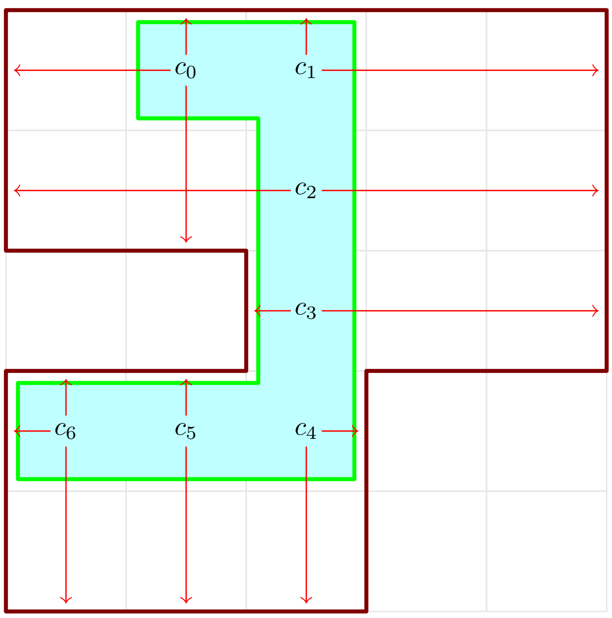
Let $c_0, c_1, \dots, c_n$ be a path of cells such that c_0 is the initial cell flooded, and in general c_i is flooded on Poseidon's ith turn from c_{i-1}. The levee now forms a closed loop enclosing all c_i.

Claim — If c_i and c_j are adjacent then $|i-j|=1$.

Proof. Assume c_i and c_j are adjacent but $|i-j|>1$. Then the two cells must be separated by a wall. But the levee forms a closed loop, and now c_i and c_j are on opposite sides. □

Thus the c_i actually form a path. We color *green* any edge of the unit grid (wall or not) which is an edge of exactly one c_i (i.e. the boundary of the polyomino). It is easy to see there are exactly $2n+4$ green edges.

Now, from the center of each cell c_i, shine a laser towards each green edge of c_i (hence a total of $2n+4$ lasers are emitted). An example below is shown for $n=6$, with the levee marked in brown.



Claim — No wall is hit by more than one laser.

Proof. Assume for contradiction that a wall w is hit by lasers from c_i and c_j. WLOG that laser is vertical, so c_i and c_j are in the same column (e.g. $(i,j)=(0,5)$ in figure).

We consider two cases on the position of w.

- If w is between c_i and c_j, then we have found a segment intersecting the levee exactly once. But the endpoints of the segment lie inside the levee. This contradicts the assumption that the levee is a closed loop.
- Suppose w lies above both c_i and c_j and assume WLOG $i<j$. Then we have found that there is no levee at all between c_i and c_j.

  Let $\rho \ge 1$ be the distance between the centers of c_i and c_j. Then c_j is flooded in a straight line from c_i within ρ turns, and this is the unique shortest possible path. So this situation can only occur if $j=i+\rho$ and c_i, ..., c_j form a column. But then no vertical lasers from c_i and c_j may point in the same direction, contradiction.

Since neither case is possible, the proof ends here. □

This implies the levee has at least $2n+4$ walls (the number of lasers) on Hephaestus's $(n+1)$st turn. So $\alpha \ge \frac{2n+4}{n+1} > 2$.

Remark (Author comments)**.** The author provides the following remarks.

- Even though the flood can be stopped when $\alpha = 2+\varepsilon$, it takes a very long time to do that. Starting from a single flooded cell, the strategy I have outlined requires $\Theta(1/\varepsilon^2)$ days. Starting from several flooded cells contained within an area of diameter D, it takes $\Theta(D/\varepsilon^2)$ days. I do not know any strategies that require fewer days than that.
- There is a gaping chasm between $\alpha \le 2$ and $\alpha > 2$. Since $\alpha \le 2$ does not suffice even when only one cell is flooded in the beginning, there are in fact no initial configurations at all for which it is sufficient. On the other hand, $\alpha > 2$ works for all initial configurations.
- The second half of the solution essentially estimates the perimeter of a polyomino in terms of its diameter (where diameter is measured entirely within the polyomino).

  It appears that this has not been done before, or at least I was unable to find any reference for it. I did find tons of references where the perimeter of a polyomino is estimated in terms of its area, but nothing concerning the diameter.

  My argument is a formalisation of the intuition that if P is any shortest path within some weirdly-shaped polyomino, then the boundary of that polyomino must hug P rather closely so that P cannot be shortened.

§2 Solutions to Day 2

§2.1 Solution to TST 4, by Mehtaab Sawhney and Zack Chroman

For a finite simple graph G, we define G' to be the graph on the same vertex set as G, where for any two vertices $u \neq v$, the pair $\{u, v\}$ is an edge of G' if and only if u and v have a common neighbor in G. Prove that if G is a finite simple graph which is isomorphic to $(G')'$, then G is also isomorphic to G'.

We say a vertex of a graph is *fatal* if it has degree at least 3, and some two of its neighbors are not adjacent.

Claim — The graph G' has at least as many triangles as G, and has strictly more if G has any fatal vertices.

Proof. Obviously any triangle in G persists in G'. Moreover, suppose v is a fatal vertex of G. Then the neighbors of G will form a clique in G' which was not there already, so there are more triangles. □

Thus we only need to consider graphs G with no fatal vertices. Looking at the connected components, the only possibilities are cliques (including single vertices), cycles, and paths. So in what follows we restrict our attention to graphs G only consisting of such components.

Remark (Warning)**.** Beware: assuming G is connected loses generality. For example, it could be that $G = G_1 \sqcup G_2$, where $G_1' \cong G_2$ and $G_2' \cong G_1$.

First, note that the following are stable under the operation:

- an isolated vertex,
- a cycle of odd length, or
- a clique with at least three vertices.

In particular, $G \cong G''$ holds for such graphs.

On the other hand, cycles of even length or paths of nonzero length will break into more connected components. For this reason, a graph G with any of these components will not satisfy $G \cong G''$ because G' will have strictly more connected components than G, and G'' will have at least as many as G'.

Therefore $G \cong G''$ if and only if G is a disjoint union of the three types of connected components named earlier. Since $G \cong G'$ holds for such graphs as well, the problem statement follows right away.

Remark. Note that the same proof works equally well for an arbitrary number of iterations $G^{'' \cdots ''} \cong G$, rather than just $G'' \cong G$.

Remark. The proposers included a variant of the problem where given any graph G, the operation stabilized after at most $O(\log n)$ operations, where n was the number of vertices of G.

§2.2 Solution to TST 5, by Carl Schildkraut

Find all integers $n \geq 2$ for which there exists an integer m and a polynomial $P(x)$ with integer coefficients satisfying the following three conditions:

- $m > 1$ and $\gcd(m, n) = 1$;
- the numbers $P(0)$, $P^2(0)$, $\dots$, $P^{m-1}(0)$ are not divisible by n; and
- $P^m(0)$ is divisible by n.

Here P^k means P applied k times, so $P^1(0) = P(0)$, $P^2(0) = P(P(0))$, etc.

The answer is that this is possible if and only if there exists primes $p' < p$ such that $p \mid n$ and $p' \nmid n$. (Equivalently, the radical $\operatorname{rad}(n)$ must not be the product of the first several primes.)

For a polynomial P, and an integer N, we introduce the notation

$$\mathbf{zord}(P \bmod N) \overset{\text{def}}{=} \min\left\{e > 0 \mid P^e(0) \equiv 0 \bmod N\right\}$$

where we set $\min \varnothing = 0$ by convention. Note that in general we have

$$\mathbf{zord}(P \bmod N) = \operatorname*{lcm}_{q \mid N}\left(\mathbf{zord}(P \bmod q)\right) \qquad (\dagger)$$

where the index runs over all prime powers $q \mid N$ (by Chinese remainder theorem). This will be used in both directions.

Construction: First, we begin by giving a construction. The idea is to first use the following prime power case.

Claim — Let p^e be a prime power, and $1 \leq k < p$. Then

$$f(X) = X + 1 - k \cdot \frac{X(X-1)(X-2)\dots(X-(k-2))}{(k-1)!}$$

viewed as a polynomial in $(\mathbb{Z}/p^e)[X]$ satisfies $\mathbf{zord}(f \bmod p^e) = k$.

Proof. Note $f(0) = 1$, $f(1) = 2$, $\dots$, $f(k-2) = k-1$, $f(k-1) = 0$ as needed. □

This gives us a way to do the construction now. For the prime power $p^e \mid n$, we choose $1 \leq p' < p$ and require $\mathbf{zord}(P \bmod p^e) = p'$ and $\mathbf{zord}(P \bmod q) = 1$ for every other prime power q dividing n. Then by $(\dagger)$ we are done.

> **Remark.** The claim can be viewed as a special case of Lagrange interpolation adapted to work over $\mathbb{Z}/p^e$ rather than $\mathbb{Z}/p$. Thus the construction of the polynomial f above is quite natural.

Necessity: by $(\dagger)$ again, it will be sufficient to prove the following claim.

Claim — For any prime power $q = p^e$, and any polynomial $f(x) \in \mathbb{Z}[x]$, if the quantity $\mathbf{zord}(f \bmod q)$ is nonzero then it has all prime factors at most p.

Proof. This is by induction on $e \geq 1$. For $e = 1$, the pigeonhole principle immediately implies that $\mathbf{zord}(P \bmod p) \leq p$.

Now assume $e \geq 2$. Let us define

$$k \stackrel{\text{def}}{=} \mathbf{zord}(P \bmod p^{e-1}), \qquad Q \stackrel{\text{def}}{=} P^k.$$

Since being periodic modulo p^e requires periodic modulo p^{e-1}, it follows that

$$\mathbf{zord}(P \bmod p^e) = k \cdot \mathbf{zord}(Q \bmod p^e).$$

However since $Q(0) \equiv 0 \bmod p^{e-1}$, it follows $\{Q(0), Q^2(0), \dots\}$ are actually all multiples of p^{e-1}. There are only p residues modulo p^e which are also multiples of p^{e-1}, so $\mathbf{zord}(Q \bmod p^e) \leq p$, as needed. $\square$

Remark. One reviewer submitted the following test-solving comments:

This is one of these problems where you can make many useful natural observations, and if you make enough of them eventually they cohere into a solution. For example, here are some things I noticed while solving:

- The polynomial $1 - x$ shows that $m = 2$ works for any odd n.
- In general, if ζ is a primitive mth root of unity modulo n, then $\zeta(x+1) - 1$ has the desired property (assuming $\gcd(m,n) = 1$). We can extend this using the Chinese remainder theorem to find that if $p \mid n$, $m \mid p-1$, and $\gcd(m,n) = 1$, then n works. So by this point I already have something about the prime factors of n being sort-of closed downwards.
- By iterating P we see it is enough to consider m prime.
- In the case where $n = 2^k$, it is not too difficult to show that no odd prime m works.

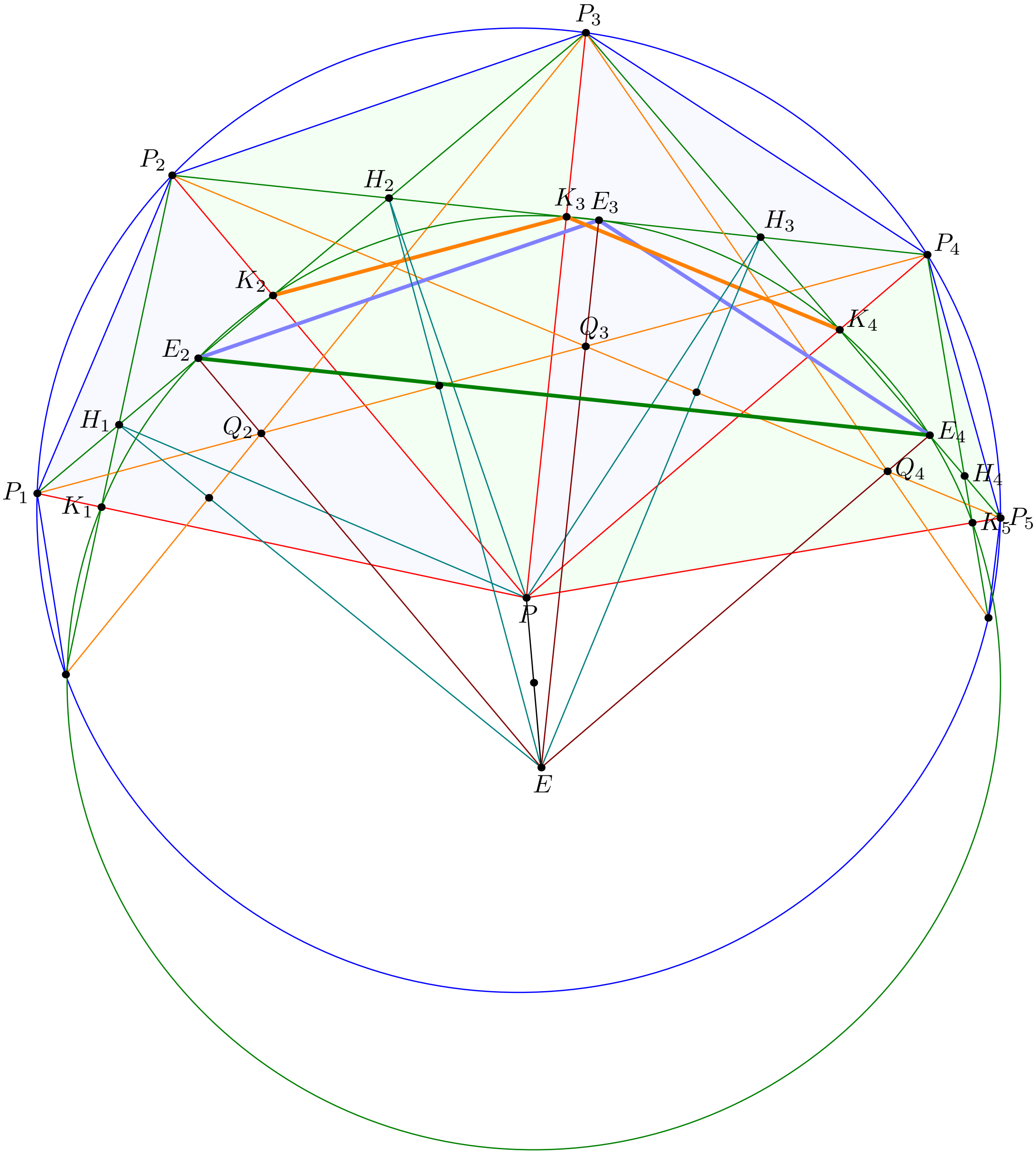
§2.3 Solution to TST 6, by Michael Ren

Let $P_1P_2\cdots P_{100}$ be a cyclic 100-gon, and let $P_i = P_{i+100}$ for all i. Define Q_i as the intersection of diagonals $\overline{P_{i-2}P_{i+1}}$ and $\overline{P_{i-1}P_{i+2}}$ for all integers i.

Suppose there exists a point P satisfying $\overline{PP_i} \perp \overline{P_{i-1}P_{i+1}}$ for all integers i. Prove that the points $Q_1, Q_2, \dots, Q_{100}$ are concyclic.

We show two solutions.

Solution to proposed problem We let $\overline{PP_2}$ and $\overline{P_1P_3}$ intersect (perpendicularly) at point K_2, and define $K_\bullet$ cyclically.



Claim — The points $K_\bullet$ are concyclic say with circumcircle γ.

Proof. Note that $PP_1 \times PK_1 = PP_2 \times PK_2 = \dots$ so the result follows by inversion at P. ∎

Let E_i be the second intersection of line $\overline{P_{i-1}K_iP_{i+1}}$ with γ; then it follows that the perpendiculars to $\overline{P_{i-1}P_{i+1}}$ at E_i all concur at a point E, which is the reflection of P across the center of γ.

We let $H_2 = \overline{P_1P_3} \cap \overline{P_2P_4}$ denote the orthocenter of $\triangle PP_2P_3$ and define $H_\bullet$ cyclically.

Claim — We have

$$\overline{EH_2} \perp \overline{P_1P_4} \parallel \overline{K_2K_3} \quad \text{and} \quad \overline{PH_2} \perp \overline{E_2E_3} \parallel \overline{P_2P_3}.$$

Proof. Both parallelisms follow by Reim's theorem through $\angle E_2H_2E_3 = \angle K_2H_2K_3$, So we need to show the perpendicularities.

Note that $\overline{H_2P}$ and $\overline{H_2E}$ are respectively circum-diameters of $\triangle H_2K_2K_3$ and $\triangle H_2E_2E_3$. As $\overline{K_2K_3}$ and $\overline{E_2E_3}$ are anti-parallel, it follows $\overline{H_2P}$ and $\overline{H_2E}$ are isogonal and we derive both perpendicularities. □

Claim — The points E, Q_3, E_3 are collinear.

Proof. We use the previous claim. The parallelisms imply that

$$\frac{E_3H_2}{E_3P_2} = \frac{E_2H_2}{E_2P_3} = \frac{E_4H_3}{E_4P_3} = \frac{E_3H_3}{E_3P_4}.$$

Now consider a homothety centered at E_3 sending H_2 to P_2 and H_3 to P_4. Then it should send the orthocenter of $\triangle EH_2H_3$ to Q_3, proving the result. □

From all this it follows that $\triangle EQ_2Q_3 \sim \triangle PK_2K_3$ as the opposite sides are all parallel. Repeating this we actually find a homothety of 100-gons

$$Q_1Q_2Q_3 \cdots \sim K_1K_2K_3 \cdots$$

and that concludes the proof.

Remark. The proposer remarks that in fact, if one lets s be an integer and instead defines $R_i = P_iP_{i+s} \cap P_{i+1}P_{i+s+1}$, then the $R_\bullet$ are concyclic. The present problem is the case $s = 3$. We comment on a few special cases:

- There is nothing to prove for $s = 1$.
- If $s = 0$, this amounts to proving that poles of $\overline{P_iP_{i+1}}$ are concyclic; by inversion this is equivalent to showing the midpoints of the sides are concyclic. This is an interesting problem but not as difficult.
- The problem for $s = 2$ is to show that our $H_\bullet$ are concyclic, which uses the $s = 0$ case as a lemma.

Solution to generalization (Nikolai Beluhov) We are going to need some well-known lemmas.

Lemma

Suppose that $ABCD$ is inscribed in a circle Γ. Let the tangents to Γ at A and B meet at E, let the tangents to Γ at C and D meet at F, and let diagonals AC and BD meet at P. Then points E, F, and P are collinear.

Proof. Let the circle of center E and radius $EA = EB$ meet lines AC and BD for the second time at points U and V. By a simple angle chase, triangles EUV and FCD are homothetic. □

Lemma

Suppose that points X and Y are isogonal conjugates in polygon $\mathcal{A} = A_1A_2 \dots A_n$. (This means that lines A_iX and A_iY are symmetric with respect to the interior angle bisector of $\angle A_{i-1}A_iA_{i+1}$ for all i, where $A_{n+j} \equiv A_j$ for all j.) Then the $2n$ projections of X and Y on the sides of $\mathcal{A}$ are concyclic.

Proof. By a simple angle chase, for all i we have that the four projections on sides $A_{i-1}A_i$ and A_iA_{i+1} are concyclic. Say that they lie on circle Γ_i. Consider the midpoint M of segment XY. For every side s of $\mathcal{A}$, we have that M is equidistant from the projections of X and Y on s. Therefore, M is the center of Γ_i for all i, and thus all of the Γ_i coincide. □

Lemma

Let Γ' and Γ'' be two circles and let r be some fixed real number. Then the locus of points X such that $\operatorname{Pow}(X, \Gamma') : \operatorname{Pow}(X, \Gamma'') = r$ is concyclic.

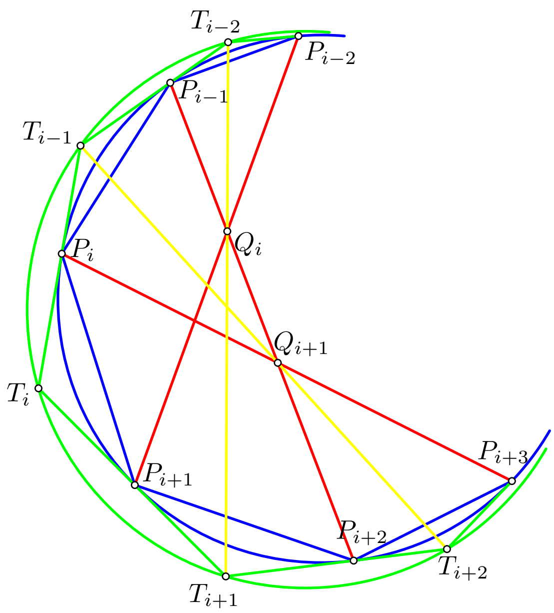
Proof. This is a classical result in circle geometry. A full proof is given, for example, in item 115 of Roger Johnson's *Advanced Euclidean Geometry*. □

We are ready to solve the problem. Let $\mathcal{P}$ be our polygon, let O be its the circumcenter, and let Γ be its circumcircle.

Fix any index i. In triangle $P_{i-1}P_iP_{i+1}$, we have that line P_iP contains the altitude through P_i and line P_iO contains the circumradius through P_i. Therefore, these two lines are symmetric with respect to the interior angle bisector of $\angle P_{i-1}P_iP_{i+1}$.

Thus points P and O are isogonal conjugates in $\mathcal{P}$. By Lemma 2, it follows that the projections of O onto the sides of $\mathcal{P}$ are concyclic. In other words, the midpoints of the sides of $\mathcal{P}$ lie on some circle ω.

Let M_i be the midpoint of segment P_iP_{i+1} and let the tangents to Γ at points P_i and P_{i+1} meet at T_i. Since inversion with respect to Γ swaps M_i and T_i for all i, and also since all of the M_i lie on the same circle ω, we obtain that all of the T_i lie on the same circle Ω.



Again, fix any index i. By Lemma 1 applied to cyclic quadrilateral $P_{i-2}P_{i-1}P_{i+1}P_{i+2}$, we have that point Q_i lies on line $T_{i-2}T_{i+1}$. Similarly, point Q_{i+1} lies on line $T_{i-1}T_{i+2}$.

Define
$$f_i = \frac{\operatorname{Pow}(Q_i, \Gamma)}{\operatorname{Pow}(Q_i, \Omega)}.$$

Claim — We have $f_i = f_{i+1}$ for all i.

Proof. Note that

$$\begin{aligned}
\operatorname{Pow}(Q_i, \Gamma) &= Q_iP_{i-1} \cdot Q_iP_{i+2} \\
\operatorname{Pow}(Q_{i+1}, \Gamma) &= Q_{i+1}P_{i-1} \cdot Q_{i+1}P_{i+2}. \\
\operatorname{Pow}(Q_i, \Omega) &= Q_iT_{i-2} \cdot Q_iT_{i+1} \\
\operatorname{Pow}(Q_{i+1}, \Omega) &= Q_{i+1}T_{i-1} \cdot Q_{i+1}T_{i+2}.
\end{aligned}$$

Consider cyclic quadrilateral $T_{i-2}T_{i-1}T_{i+1}T_{i+2}$. Since Γ touches its opposite sides $T_{i-2}T_{i-1}$ and $T_{i+1}T_{i+2}$ at points P_{i-1} and P_{i+2}, we have that line $P_{i-1}P_{i+2}$ makes equal angles with these opposite sides. From here, a simple angle chase shows that triangles $P_{i-1}Q_iT_{i-2}$ and $P_{i+2}Q_{i+1}T_{i+2}$ are similar. Thus

$$\frac{Q_iP_{i-1}}{Q_iT_{i-2}} = \frac{Q_{i+1}P_{i+2}}{Q_{i+1}T_{i+2}}.$$

Similarly,

$$\frac{Q_iP_{i+2}}{Q_iT_{i+1}} = \frac{Q_{i+1}P_{i-1}}{Q_{i+1}T_{i-1}}.$$

From these, the desired identity $f_i = f_{i+1}$ follows. □

Therefore, the power ratio f_i is the same for all i. By Lemma 3 for circles Γ and Ω, the solution is complete.

Remark. This solution applies to the full generalization (from 3 to s) mentioned in the end of the previous solution, essentially with no change.

Remark. Polygon $T_1T_2 \dots T_{100}$ is both circumscribed about a circle and inscribed inside a circle. Polygons like that are known as *Poncelet polygons*. See, for example, https://en.wikipedia.org/wiki/Poncelet's_closure_theorem. This solution borrows a lot from the discussion of Poncelet's closure theorem in *Advanced Euclidean Geometry*, referenced above for Lemma 3.