USA IMO TST 2017 Solutions

United States of America — IMO Team Selection Tests

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Contents

Note that in this solutions file, we present slightly stronger versions of problems 4 and 6 on the January TST than actually appeared on the exams.

§1 Solutions to Day 1

§1.1 Solution to TST 1, by Po-Shen Loh

In a sports league, each team uses a set of at most t signature colors. A set S of teams is *color-identifiable* if one can assign each team in S one of their signature colors, such that no team in S is assigned *any* signature color of a different team in S. For all positive integers n and t, determine the maximum integer $g(n,t)$ such that: In any sports league with exactly n distinct colors present over all teams, one can always find a color-identifiable set of size at least $g(n,t)$.

Answer: $\lceil n/t \rceil$.

To see this is an upper bound, note that one can easily construct a sports league with that many teams anyways.

Here is a short "minimality" solution. Take the smallest (!) set S of teams such that every color is a signature color for some team. Then for each team $t \in S$, we can assign it a color (by minimality). But to be spanning, obviously $\#S \geq \lceil n/t \rceil$.

For an algorithmic approach, the idea is to greedy pick by color (rather than by team), taking at each step the least used color. Select the color C_1 with the *fewest* teams using it, and a team T_1 using it. Then delete all colors T_1 uses, and all teams which use C_1. Note that

- By problem condition, this deletes at most t teams total.
- Any remaining color C still has at least one user. Indeed, if not, then C had the same set of teams as C_1 did (by minimality of C), but then it should have deleted as a color of T_1.

Now repeat this algorithm with C_2 and T_2, and so on. This operations uses at most t colors each time, so we select at least $\lceil n/t \rceil$ colors.

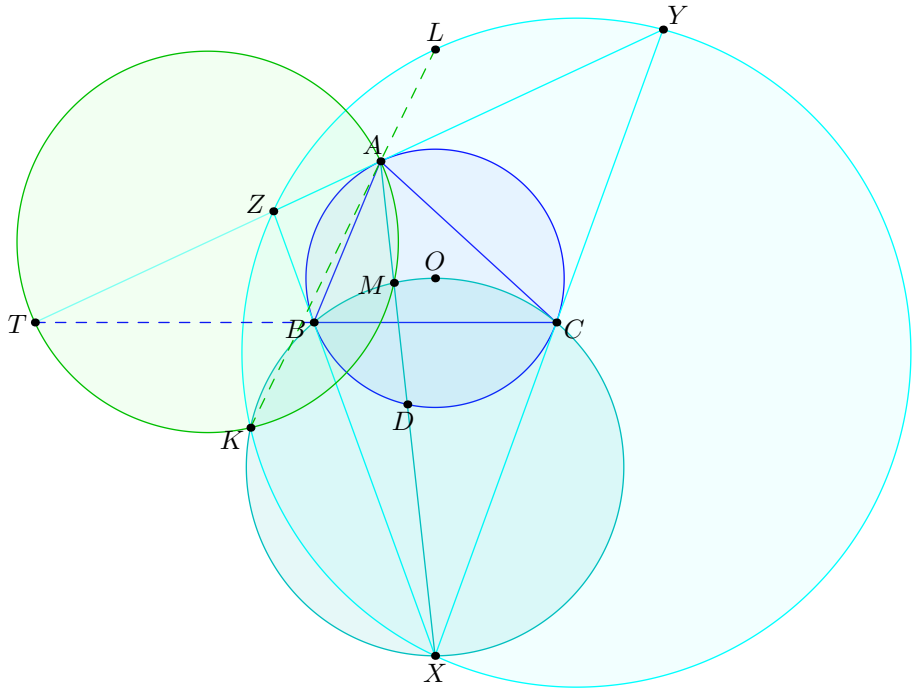
§1.2 Solution to TST 2, by Evan Chen

Let ABC be an acute scalene triangle with circumcenter O, and let T be on line BC such that $\angle TAO = 90^\circ$. The circle with diameter $\overline{AT}$ intersects the circumcircle of $\triangle BOC$ at two points A_1 and A_2, where $OA_1 < OA_2$. Points B_1, B_2, C_1, C_2 are defined analogously.

(a) Prove that $\overline{AA_1}$, $\overline{BB_1}$, $\overline{CC_1}$ are concurrent.

(b) Prove that $\overline{AA_2}$, $\overline{BB_2}$, $\overline{CC_2}$ are concurrent on the Euler line of triangle ABC.

Let triangle ABC have circumcircle Γ. Let $\triangle XYZ$ be the tangential triangle of $\triangle ABC$ (hence Γ is the incircle of $\triangle XYZ$), and denote by Ω its circumcircle. Suppose the symmedian $\overline{AX}$ meets Γ again at D, and let M be the midpoint of $\overline{AD}$. Finally, let K be the Miquel point of quadrilateral $ZBCY$, meaning it is the intersection of Ω and the circumcircle of $\triangle BOC$ (other than X).



We first claim that M and K are A_1 and A_2. In that case $OM < OA < OK$, so $M = A_1$, $K = A_2$.

To see that $M = A_1$, note that $\angle OMX = 90^\circ$, and moreover that $\overline{TA}$, $\overline{TD}$ are tangents to Γ, whence we also have $M = \overline{TO} \cap \overline{AD}$. Thus T lies on both (BOC) and (AT). This solves part (a) of the problem: the concurrency point is the symmedian point of $\triangle ABC$.

Now, note that since K is the Miquel point,

$$\frac{ZK}{YK} = \frac{ZB}{YC} = \frac{ZA}{YA}$$

and hence $\overline{KA}$ is an angle bisector of $\angle ZKY$. Thus from $(TA; YZ) = -1$ we obtain $\angle TKA = 90^\circ$.

It remains to show $\overline{AK}$ passes through a fixed point on the Euler line. We claim it is the exsimilicenter of Γ and Ω. Let L be the midpoint of the arc YZ of $\triangle XYZ$ not containing X. Then we know that K, A, L are collinear. Now the positive homothety sending Γ to Ω maps A to L; this proves the claim. Finally, it is well-known that the line through O and the circumcenter of $\triangle XYZ$ coincides with the Euler line of $\triangle ABC$; hence done.

A second approach to (b) presented by many contestants is to take an inversion around the circumcircle of ABC. In that situation, the part reduces to the following known lemma: if $\overline{AH_a}$, $\overline{BH_b}$, $\overline{CH_c}$ are the altitudes of a triangle, then the circumcircles of triangles OAH_a, BOH_b, COH_c are coaxial, and the radical axis coincides with the Euler line. Indeed one simply observes that the orthocenter has equal power to all three circles.

Authorship comments This problem was inspired by the fact that K, A, L are collinear in the figure, which was produced by one of my students (Ryan Kim) in a solution to a homework problem. I realized for example that this implied that line AK passed through the X_{56} point of $\triangle XYZ$ (which lies on the Euler line of $\triangle ABC$).

This problem was the result of playing around with the resulting very nice picture: all the power comes from the "magic" point K.

§1.3 Solution to TST 3, by Alison Miller

Let $P, Q \in \mathbb{R}[x]$ be relatively prime nonconstant polynomials. Show that there can be at most three real numbers λ such that $P + \lambda Q$ is the square of a polynomial.

This is true even with $\mathbb{R}$ replaced by $\mathbb{C}$, and it will be necessary to work in this generality. We will prove the claim in the following form:

> **Claim —** Assume $P, Q \in \mathbb{C}[x]$ are relatively prime. If $\alpha P + \beta Q$ is a square for four different choices of the ratio $[\alpha : \beta]$ then P and Q must be constant.

Call pairs (P, Q) as in the claim *bad*; so we wish to show the only bad pairs are pairs of constant polynomials. Assume not, and take a bad pair with $\deg P + \deg Q$ minimal.

By a suitable Möbius transformation, we may transform (P, Q) so that the four ratios are $[1 : 0]$, $[0 : 1]$, $[1 : -1]$ and $[1 : -k]$, so we find there are polynomials A and B such that

$$A^2 - B^2 = C^2$$
$$A^2 - kB^2 = D^2$$

where $A^2 = P + \lambda_1 Q$, $B^2 = P + \lambda_2 Q$, say. Of course $\gcd(A, B) = 1$.

Consequently, we have $C^2 = (A + B)(A - B)$ and $D^2 = (A + \mu B)(A - \mu B)$ where $\mu^2 = k$. Now $\gcd(A, B) = 1$, so $A + B$, $A - B$, $A + \mu B$ and $A - \mu B$ are squares; id est (A, B) is bad. This is a contradiction, since $\deg A + \deg B < \deg P + \deg Q$.

§2 Solutions to Day 2

§2.1 Solution to TST 4, by Linus Hamilton

You are cheating at a trivia contest. For each question, you can peek at each of the $n > 1$ other contestant's guesses before writing your own. For each question, after all guesses are submitted, the emcee announces the correct answer. A correct guess is worth 0 points. An incorrect guess is worth -2 points for other contestants, but only -1 point for you, because you hacked the scoring system. After announcing the correct answer, the emcee proceeds to read out the next question. Show that if you are leading by 2^{n-1} points at any time, then you can surely win first place.

We will prove the result with 2^{n-1} replaced even by $2^{n-2}+1$.

We first make the following reductions. First, change the weights to be $+1$, -1, 0 respectively (rather than 0, -2, -1); this clearly has no effect. Also, WLOG that all contestants except you initially have score zero (and that your score exceeds 2^{n-2}). WLOG ignore rounds in which all answers are the same. Finally, ignore rounds in which you get the correct answer, since that leaves you at least as well off as before — in other words, we'll assume your score is always fixed, but you can pick any group of people with the same answers and ensure they lose 1 point, while some other group gains 1 point.

The key observation is the following. Consider two rounds R_1 and R_2 such that:

- In round R_1, some set S of contestants gains a point.
- In round R_2, the set S of contestants all have the same answer.

Then, if we copy the answers of contestants in S during R_2, then the sum of the scorings in R_1 and R_2 cancel each other out. In other words we can then ignore R_1 and R_2 forever.

We thus consider the following strategy. We keep a list $\mathcal{L}$ of subsets of $\{1,\dots,n\}$, initially empty. Now do the following strategy:

- On a round, suppose there exists a set S of people with the same answer such that $S \in \mathcal{L}$. Then, copy the answer of S, causing them to lose a point. Delete S from $\mathcal{L}$. (Importantly, we do not add any new sets to $\mathcal{L}$.)
- Otherwise, copy any set T of contestants, selecting $|T| \ge n/2$ if possible. Let S be the set of contestants who answer correctly (if any), and add S to the list $\mathcal{L}$. Note that $|S| \le n/2$, since S is disjoint from T.

By construction, $\mathcal{L}$ has no duplicate sets. So the score of any contestant c is bounded above by the number of times that c appears among sets in $\mathcal{L}$. The number of such sets is clearly at most $\frac{1}{2}\cdot 2^{n-1}$. So, if you lead by $2^{n-2}+1$ then you ensure victory. This completes the proof!

> **Remark.** Several remarks are in order. First, we comment on the bound $2^{n-2}+1$ itself. The most natural solution using only the list idea gives an upper bound of $(2^n-2)+1$, which is the number of nonempty proper subsets of $\{1,\dots,n\}$. Then, there are two optimizations one can observe:
>
> - In fact we can improve to the number of times any particular contestant c appears in some set, rather than the total number of sets.
> - When adding new sets S to $\mathcal{L}$, one can ensure $|S| \le n/2$.
>
> Either observation alone improves the bound from 2^n-1 to 2^{n-1}, but both together give

the bound $2^{n-2}+1$. Additionally, when n is odd the calculation of subsets actually gives $2^{n-2} - \frac{1}{2}\binom{n-1}{\frac{n-1}{2}} + 1$. This gives the best possible value at both $n = 2$ and $n = 3$. It seems likely some further improvements are possible, and the true bound is suspected to be polynomial in n.

Secondly, the solution is highly motivated by considering a true/false contest in which only two distinct answers are given per question. However, a very natural mistake (which graders assessed as a two-point deduction) is to try and prove that in fact one can "WLOG" we are in the two-question case. The proof of this requires substantially more care than expected. For instance, set $n = 3$. If $\mathcal{L} = \{\{1\}, \{2\}, \{3\}\}$ then it becomes impossible to prevent a duplicate set from appearing in $\mathcal{L}$ if all contestants give distinct answers. One might attempt to fix this by instead adding to $\mathcal{L}$ the *complement* of the set T described above. The example $\mathcal{L} = \{\{1, 2\}, \{2, 3\}, \{3, 1\}\}$ (followed again by a round with all distinct answers) shows that this proposed fix does not work either. This issue affects all variations of the above approach.

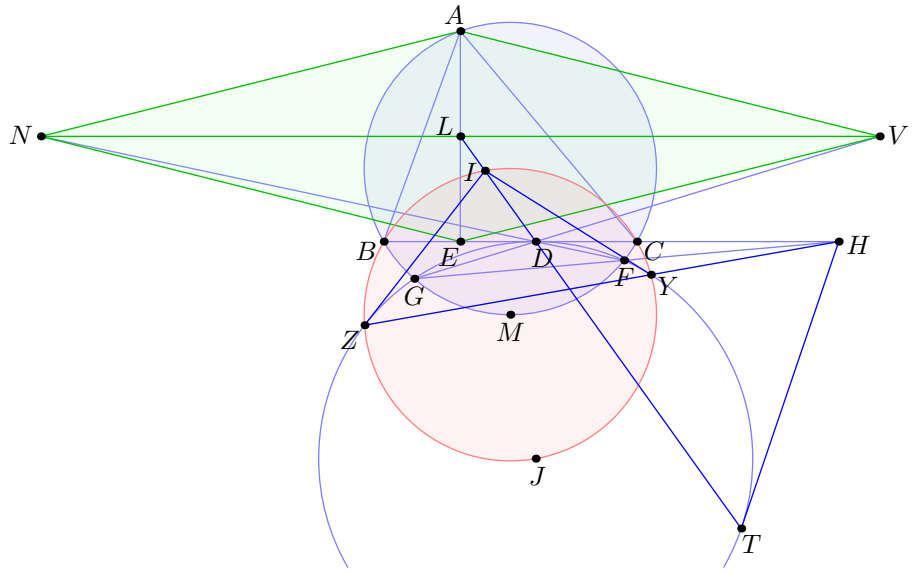
Remark. Here are some motivations for the solution:

1. The exponential bound 2^n suggests looking at subsets.
2. The $n = 2$ case suggests the idea of "repeated rounds".
3. The "two distinct answers" case suggests looking at rounds as partitions (even though the WLOG does not work, at least not without further thought).
4. There's something weird about this problem: it's a finite bound over unbounded time. This is a hint to *not* worry excessively about the actual scores, which turn out to be almost irrelevant.

§2.2 Solution to TST 5, by Danielle Wang

Let ABC be a triangle with altitude $\overline{AE}$. The A-excircle touches $\overline{BC}$ at D, and intersects the circumcircle at two points F and G. Prove that one can select points V and N on lines DG and DF such that quadrilateral $EVAN$ is a rhombus.

Let I denote the incenter, J the A-excenter, and L the midpoint of $\overline{AE}$. Denote by $\overline{IY}$, $\overline{IZ}$ the tangents from I to the A-excircle. Note that lines $\overline{BC}$, $\overline{GF}$, $\overline{YZ}$ then concur at H (unless $AB = AC$, but this case is obvious), as it's the radical center of cyclic hexagon $BICYJZ$, the circumcircle and the A-excircle.



Now let $\overline{HD}$ and $\overline{HT}$ be the tangents from H to the A-excircle. It follows that $\overline{DT}$ is the symmedian of $\triangle DZY$, hence passes through $I = \overline{YY} \cap \overline{ZZ}$. Moreover, it's well known that $\overline{DI}$ passes through L, the midpoint of the A-altitude (for example by homothety). Finally, $(DT; FG) = -1$, hence project through D onto the line through L parallel to $\overline{BC}$ to obtain $(\infty L; VN) = -1$ as desired.

Authorship comments This is a joint proposal with Danielle Wang (mostly by her). The formulation given was that the tangents to the A-excircle at F and G was on line $\overline{DI}$; I solved this formulation using the radical axis argument above. I then got the idea to involve the point L, already knowing it was on $\overline{DI}$. Observing the harmonic quadrilateral, I took perspectivity through M onto the line through L parallel to $\overline{BC}$ (before this I had tried to use the A-altitude with little luck). This yields the rhombus in the problem.

§2.3 Solution to TST 6, by Noam Elkies

Prove that there are infinitely many triples (a, b, p) of integers, with p prime and $0 < a \le b < p$, for which p^5 divides $(a+b)^p - a^p - b^p$.

The key claim is that if $p \equiv 1 \pmod 3$, then

$$p(x^2 + xy + y^2)^2 \text{ divides } (x+y)^p - x^p - y^p$$

as polynomials in x and y. Since it's known that one can select a and b such that $p^2 \mid a^2 + ab + b^2$, the conclusion follows. (The theory of quadratic forms tells us we can do it with $p^2 = a^2 + ab + b^2$; Thue's lemma lets us do it by solving $x^2 + x + 1 \equiv 0 \pmod{p^2}$.)

To prove this, it is the same to show that

$$(x^2 + x + 1)^2 \text{ divides } F(x) \stackrel{\text{def}}{=} (x+1)^p - x^p - 1.$$

since the binomial coefficients $\binom{p}{k}$ are clearly divisible by p. Let ζ be a third root of unity. Then $F(\zeta) = (1+\zeta)^p - \zeta^p - 1 = -\zeta^2 - \zeta - 1 = 0$. Moreover, $F'(x) = p(x+1)^{p-1} - px^{p-1}$, so $F'(\zeta) = p - p = 0$. Hence ζ is a double root of F as needed.

(Incidentally, $p = 2017$ works!)

Remark. One possible motivation for this solution is the case $p = 7$. It is nontrivial even to prove that p^2 can divide the expression if we exclude the situation $a + b = p$ (which provably never achieves p^3). As $p = 3, 5$ fails considering the $p = 7$ polynomial gives

$$(x+1)^7 - x^7 - 1 = 7x(x+1)\left(x^4 + 2x^3 + 3x^2 + 2x + 1\right).$$

The key is now to notice that the last factor is $(x^2 + x + 1)^2$, which suggests the entire solution.