USA IMO TST 2016 Solutions

United States of America — IMO Team Selection Tests

Evan Chen 《陳誼廷》

60^{th} IMO 2016 Hong Kong

Contents

§1 Solutions to Day 1

§1.1 Solution to TST 1, by Maria Monks

Let $S = \{1, \dots, n\}$. Given a bijection $f : S \to S$ an *orbit* of f is a set of the form $\{x, f(x), f(f(x)), \dots\}$ for some $x \in S$. We denote by $c(f)$ the number of distinct orbits of f. For example, if $n = 3$ and $f(1) = 2$, $f(2) = 1$, $f(3) = 3$, the two orbits are $\{1, 2\}$ and $\{3\}$, hence $c(f) = 2$.

Given k bijections f_1, ..., f_k from S to itself, prove that

$$c(f_1) + \dots + c(f_k) \le n(k - 1) + c(f)$$

where $f : S \to S$ is the composed function $f_1 \circ \dots \circ f_k$.

Most motivated solution is to consider $n - c(f)$ and show this is the transposition distance. Dumb graph theory works as well.

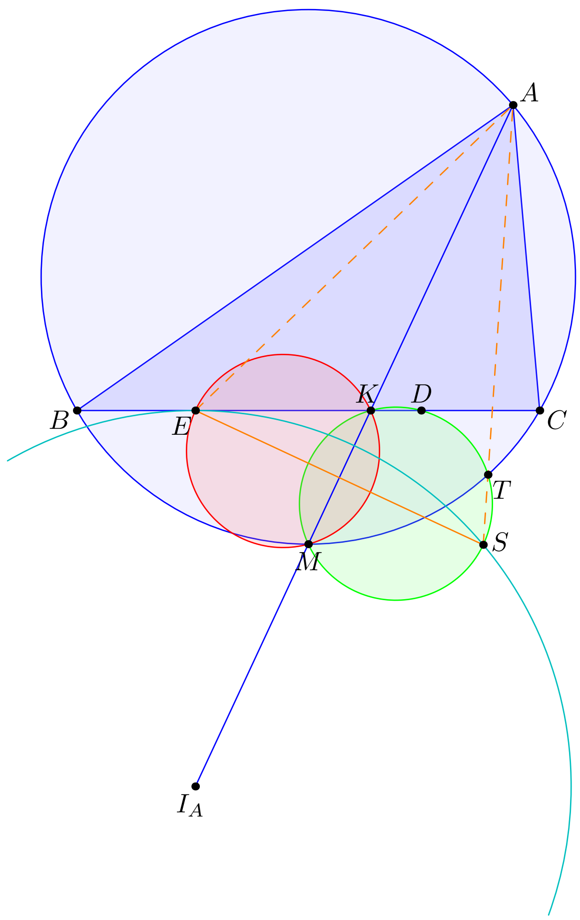
§1.2 Solution to TST 2, by Evan Chen

Let ABC be a scalene triangle with circumcircle Ω, and suppose the incircle of ABC touches BC at D. The angle bisector of $\angle A$ meets BC and Ω at K and M. The circumcircle of $\triangle DKM$ intersects the A-excircle at S_1, S_2, and Ω at $T \neq M$. Prove that line AT passes through either S_1 or S_2.

We present an angle-chasing solution, and then a more advanced alternative finish.

First solution (angle chasing) Assume for simplicity $AB < AC$. Let E be the contact point of the A-excircle on BC; also let ray TD meet Ω again at L. From the fact that $\angle MTL = \angle MTD = 180^\circ - \angle MKD$, we can deduce that $\angle MTL = \angle ACM$, meaning that L is the reflection of A across the perpendicular bisector ℓ of BC. If we reflect T, D, L over ℓ, we deduce A, E and the reflection of T across ℓ are collinear, which implies that $\angle BAT = \angle CAE$.

Now, consider the reflection point E across line AI, say S. Since ray AI passes through the A-excenter, S lies on the A-excircle. Since $\angle BAT = \angle CAE$, S also lies on ray AT. But the circumcircles of triangles DKM and KME are congruent (from $DM = EM$), so S lies on the circumcircle of $\triangle DKM$ too. Hence S is the desired intersection point.



Second solution (advanced) It's known that T is the touch-point of the A-mixtilinear incircle. Let E be contact point of A-excircle on BC. Now the circumcircles of $\triangle DKM$ and $\triangle KME$ are congruent, since $DM = ME$ and the angles at K are supplementary. Let S be the reflection of E across line KM, which by the above the above comment lies on the circumcircle of $\triangle DKM$. Since KM passes through the A-excenter, S also lies on the A-excircle. But S also lies on line AT, since lines AT and AE are isogonal (the mixtilinear cevian is isogonal to the Nagel line). Thus S is the desired intersection point.

Authorship comments This problem comes from an observation of mine: let ABC be a triangle, let the $\angle A$ bisector meet $\overline{BC}$ and (ABC) at E and M. Let W be the tangency point of the A-mixtilinear excircle with the circumcircle of ABC. Then A-Nagel line passed through a common intersection of the circumcircle of $\triangle MEW$ and the A-mixtilinear incircle.

This problem is the inverted version of this observation.

§1.3 Solution to TST 3, by Mark Sellke

Let p be a prime number. Let $\mathbb{F}_p$ denote the integers modulo p, and let $\mathbb{F}_p[x]$ be the set of polynomials with coefficients in $\mathbb{F}_p$. Define $\Psi\colon \mathbb{F}_p[x] \to \mathbb{F}_p[x]$ by

$$\Psi\left(\sum_{i=0}^{n} a_i x^i\right) = \sum_{i=0}^{n} a_i x^{p^i}.$$

Prove that for nonzero polynomials $F, G \in \mathbb{F}_p[x]$,

$$\Psi(\gcd(F,G)) = \gcd(\Psi(F), \Psi(G)).$$

Observe that Ψ is also a linear map of $\mathbb{F}_p$ vector spaces, and that $\Psi(xP) = \Psi(P)^p$ for any $P \in \mathbb{F}_p[x]$. (In particular, $\Psi(1) = x$, not 1, take caution!)

First solution (Ankan Bhattacharya) We start with:

Claim — If $P \mid Q$ then $\Psi(P) \mid \Psi(Q)$.

Proof. Set $Q = PR$, where $R = \sum_{i=0}^{k} r_i x^i$. Then

$$\Psi(Q) = \Psi\left(P \sum_{i=0}^{k} r_i x^i\right) = \sum_{i=0}^{k} \Psi\left(P \cdot r_i x^i\right) = \sum_{i=0}^{k} r_i \Psi(P)^{p^i}$$

which is divisible by $\Psi(P)$. □

This already implies

$$\Psi(\gcd(F,G)) \mid \gcd(\Psi(F), \Psi(G)).$$

For the converse, by Bezout there exists $A, B \in \mathbb{F}_p[x]$ such that $AF + BG = \gcd(F,G)$, so taking Ψ of both sides gives

$$\Psi(AF) + \Psi(BG) = \Psi\left(\gcd(F,G)\right).$$

The left-hand side is divisible by $\gcd(\Psi(F), \Psi(G))$ since the first term is divisible by $\Psi(F)$ and the second term is divisible by $\Psi(G)$. So $\gcd(\Psi(F), \Psi(G)) \mid \Psi(\gcd(F,G))$ and noting both sides are monic we are done.

Second solution Here is an alternative (longer but more conceptual) way to finish without Bezout lemma. Let $\daleth \subseteq \mathbb{F}_p[x]$ denote the set of polynomials in the image of Ψ, thus $\Psi\colon \mathbb{F}_p[x] \to \daleth$ is a bijection on the level of sets.

Claim — If $A, B \in \daleth$ then $\gcd(A,B) \in \daleth$.

Proof. It suffices to show that if A and B are monic, and $\deg A > \deg B$, then the remainder when A is divided by B is in $\daleth$. Suppose $\deg A = p^k$ and $B = x^{p^{k-1}} - c_2 x^{p^{k-2}} - \cdots - c_k$. Then

$$\begin{aligned} x^{p^k} &\equiv \left(c_2 x^{p^{k-2}} + c_3 x^{p^{k-3}} + \cdots + c_k\right)^p \pmod B \\ &\equiv c_2 x^{p^{k-1}} + c_3 x^{p^{k-2}} \cdots + c_k \pmod B \end{aligned}$$

since exponentiation by p commutes with addition in $\mathbb{F}_p$. This is enough to imply the conclusion. The proof if $\deg B$ is smaller less than p^{k-1} is similar. □

Thus, if we view $\mathbb{F}_p[x]$ and $\beth$ as partially ordered sets under polynomial division, then gcd is the "greatest lower bound" or "meet" in both partially ordered sets. We will now prove that Ψ is an *isomorphism* of the posets. We have already seen that $P \mid Q \implies \Psi(P) \mid \Psi(Q)$ from the first solution. For the converse:

Claim — If $\Psi(P) \mid \Psi(Q)$ then $P \mid Q$.

Proof. Suppose $\Psi(P) \mid \Psi(Q)$, but $Q = PA + B$ where $\deg B < \deg P$. Thus $\Psi(P) \mid \Psi(PA) + \Psi(B)$, hence $\Psi(P) \mid \Psi(B)$, but $\deg \Psi(P) > \deg \Psi(B)$ hence $\Psi(B) = 0 \implies B = 0$. □

This completes the proof.

Remark. In fact $\Psi \colon \mathbb{F}_p[x] \to \beth$ is a ring isomorphism if we equip $\beth$ with function composition as the ring multiplication. Indeed in the proof of the first claim (that $P \mid Q \implies \Psi(P) \mid \Psi(Q)$) we saw that

$$\Psi(RP) = \sum_{i=0}^{k} r_i \Psi(P)^{p^i} = \Psi(R) \circ \Psi(P).$$

§2 Solutions to Day 2

§2.1 Solution to TST 4, by Iurie Boreico

Let $\sqrt{3} = 1.b_1b_2b_3\dots_{(2)}$ be the binary representation of $\sqrt{3}$. Prove that for any positive integer n, at least one of the digits $b_n, b_{n+1}, \dots, b_{2n}$ equals 1.

Assume the contrary, so that for some integer k we have

$$k < 2^{n-1}\sqrt{3} < k + \frac{1}{2^{n+1}}.$$

Squaring gives

$$\begin{aligned} k^2 < 3 \cdot 2^{2n-2} &< k^2 + \frac{k}{2^n} + \frac{1}{2^{2n+2}} \\ &\le k^2 + \frac{2^{n-1}\sqrt{3}}{2^n} + \frac{1}{2^{2n+2}} \\ &= k^2 + \frac{\sqrt{3}}{2} + \frac{1}{2^{2n+2}} \\ &\le k^2 + \frac{\sqrt{3}}{2} + \frac{1}{16} \\ &< k^2 + 1 \end{aligned}$$

and this is a contradiction.

§2.2 Solution to TST 5, by Zilin Jiang

Let $n \geq 4$ be an integer. Find all functions $W \colon \{1, \dots, n\}^2 \to \mathbb{R}$ such that for every partition $[n] = A \cup B \cup C$ into disjoint sets,

$$\sum_{a \in A} \sum_{b \in B} \sum_{c \in C} W(a,b)W(b,c) = |A||B||C|.$$

Of course, $W(k,k)$ is arbitrary for $k \in [n]$. We claim that $W(a,b) = \pm 1$ for any $a \neq b$, with the sign fixed. (These evidently work.)

First, let $X_{abc} = W(a,b)W(b,c)$ for all distinct a, b, c, so the given condition is

$$\sum_{a,b,c \in A \times B \times C} X_{abc} = |A||B||C|.$$

Consider the given equation with the particular choices

- $A = \{1\}$, $B = \{2\}$, $C = \{3, 4, \dots, n\}$.
- $A = \{1\}$, $B = \{3\}$, $C = \{2, 4, \dots, n\}$.
- $A = \{1\}$, $B = \{2, 3\}$, $C = \{4, \dots, n\}$.

This gives

$$\begin{aligned} X_{123} + X_{124} + \dots + X_{12n} &= n - 2 \\ X_{132} + X_{134} + \dots + X_{13n} &= n - 2 \\ (X_{124} + \dots + X_{12n}) + (X_{134} + \dots + X_{13n}) &= 2(n-3). \end{aligned}$$

Adding the first two and subtracting the last one gives $X_{123} + X_{132} = 2$. Similarly, $X_{123} + X_{321} = 2$, and in this way we have $X_{321} = X_{132}$. Thus $W(3,2)W(2,1) = W(1,3)W(3,2)$, and since $W(3,2) \neq 0$ (clearly) we get $W(2,1) = W(3,2)$.

Analogously, for any distinct a, b, c we have $W(a,b) = W(b,c)$. For $n \geq 4$ this is enough to imply $W(a,b) = \pm 1$ for $a \neq b$ where the choice of sign is the same for all a and b.

Remark. Surprisingly, the $n = 3$ case has "extra" solutions for $W(1,2) = W(2,3) = W(3,1) = \pm 1$, $W(2,1) = W(3,2) = W(1,3) = \mp 1$.

Remark (Intuition)**.** It should still be possible to solve the problem with X_{abc} in place of $W(a,b)W(b,c)$, because we have about far more equations than variables $X_{a,b,c}$ so linear algebra assures us we almost certainly have a unique solution.

§2.3 Solution to TST 6, by Ivan Borsenco

Let ABC be an acute scalene triangle and let P be a point in its interior. Let A_1, B_1, C_1 be projections of P onto triangle sides BC, CA, AB, respectively. Find the locus of points P such that AA_1, BB_1, CC_1 are concurrent and $\angle PAB + \angle PBC + \angle PCA = 90^\circ$.

In complex numbers with ABC the unit circle, it is equivalent to solving the following two cubic equations in p and $q = \overline{p}$:

$$\begin{aligned}(p-a)(p-b)(p-c) &= (abc)^2(q-1/a)(q-1/b)(q-1/c) \\ 0 &= \prod_{\text{cyc}}(p+c-b-bcq) + \prod_{\text{cyc}}(p+b-c-bcq).\end{aligned}$$

Viewing this as two cubic curves in $(p,q) \in \mathbb{C}^2$, by *Bézout's Theorem* it follows there are at most nine solutions (unless both curves are not irreducible, but it's easy to check the first one cannot be factored). Moreover it is easy to name nine solutions (for ABC scalene): the three vertices, the three excenters, and I, O, H. Hence the answer is just those three triangle centers I, O and H.

Remark. On the other hand it is not easy to solve the cubics by hand; I tried for an hour without success. So I think this solution is only feasible with knowledge of algebraic geometry.

Remark. These two cubics have names:

- The locus of $\angle PAB + \angle PBC + \angle PCA = 90^\circ$ is the **McCay cubic**, which is the locus of points P for which P, P^*, O are collinear.
- The locus of the pedal condition is the **Darboux cubic**, which is the locus of points P for which P, P^*, L are collinear, L denoting the de Longchamps point.

Assuming $P \neq P^*$, this implies P and P^* both lie on the Euler line of $\triangle ABC$, which is possible only if $P = O$ or $P = H$.

Some other synthetic solutions are posted at `https://aops.com/community/c6h1243902p6368189`.