USA IMO TST 2015 Solutions

United States of America — IMO Team Selection Tests

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Note that in this solutions file, we present slightly stronger versions of problems 4 and 6 on the January TST than actually appeared on the exams.

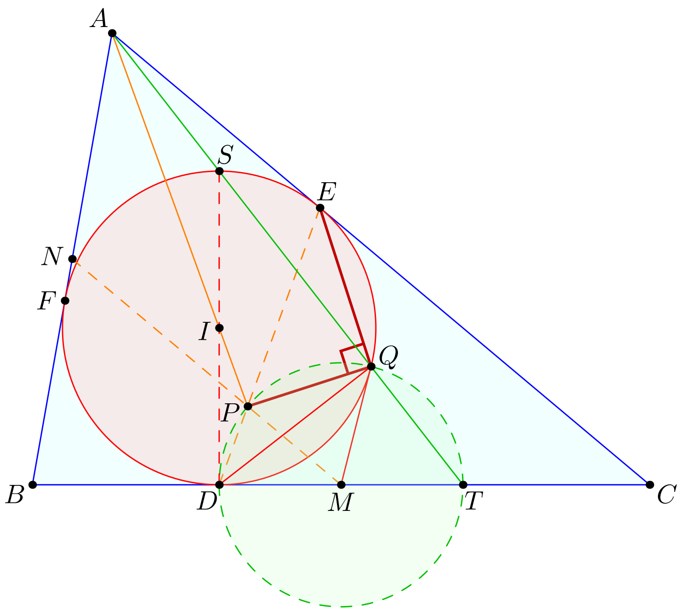
§1 Solutions to Day 1

§1.1 Solution to TST 1, by Evan Chen

Let ABC be a scalene triangle with incenter I whose incircle is tangent to $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ at D, E, F, respectively. Denote by M the midpoint of $\overline{BC}$ and let P be a point in the interior of $\triangle ABC$ so that $MD = MP$ and $\angle PAB = \angle PAC$. Let Q be a point on the incircle such that $\angle AQD = 90°$. Prove that either $\angle PQE = 90°$ or $\angle PQF = 90°$.

We present two solutions.

Official solution Assume without loss of generality that $AB < AC$; we show $\angle PQE = 90°$.



First, we claim that D, P, E are collinear. Let N be the midpoint of $\overline{AB}$. It is well-known that the three lines MN, DE, AI are concurrent at a point (see for example problem 6 of USAJMO 2014). Let P' be this intersection point, noting that P' actually lies on segment DE. Then P' lies inside $\triangle ABC$ and moreover

$$\triangle DP'M \sim \triangle DEC$$

so $MP' = MD$. Hence $P' = P$, proving the claim.

Let S be the point diametrically opposite D on the incircle, which is also the second intersection of $\overline{AQ}$ with the incircle. Let $T = \overline{AQ} \cap \overline{BC}$. Then T is the contact point of the A-excircle; consequently,

$$MD = MP = MT$$

and we obtain a circle with diameter $\overline{DT}$. Since $\angle DQT = \angle DQS = 90°$ we have Q on this circle as well.

As $\overline{SD}$ is tangent to the circle with diameter $\overline{DT}$, we obtain

$$\angle PQD = \angle SDP = \angle SDE = \angle SQE.$$

Since $\angle DQS = 90°$, $\angle PQE = 90°$ too.

Solution using spiral similarity We will ignore for now the point P. As before define S, T and note $\overline{AQST}$ collinear, as well as $DPQT$ cyclic on circle ω with diameter $\overline{DT}$.

Let τ be the spiral similarity at Q sending ω to the incircle. We have $\tau(T) = D$, $\tau(D) = S$, $\tau(Q) = Q$. Now

$$I = \overline{DD} \cap \overline{QQ} \implies \tau(I) = \overline{SS} \cap \overline{QQ}$$

and hence we conclude $\tau(I)$ is the pole of $\overline{ASQT}$ with respect to the incircle, which lies on line EF.

Then since $\overline{AI} \perp \overline{EF}$ too, we deduce τ sends line AI to line EF, hence $\tau(P)$ must be either E or F as desired.

Authorship comments Written April 2014. I found this problem while playing with GeoGebra. Specifically, I started by drawing in the points A, B, C, I, D, M, T, common points. I decided to add in the circle with diameter DT, because of the synergy it had with the rest of the picture. After a while of playing around, I intersected ray AI with the circle to get P, and was surprised to find that D, P, E were collinear, which I thought was impossible since the setup should have been symmetric. On further reflection, I realized it was because AI intersected the circle twice, and set about trying to prove this. I noticed the relation $\angle PQE = 90^\circ$ in my attempts to prove the result, even though this ended up being a corollary rather than a useful lemma.

§1.2 Solution to TST 2, by Iurie Boreico

Prove that for every positive integer n, there exists a set S of n positive integers such that for any two distinct $a, b \in S$, $a - b$ divides a and b but none of the other elements of S.

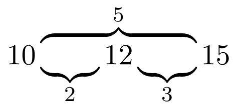
The idea is to look for a sequence $d_1, \dots, d_{n-1}$ of "differences" such that the following two conditions hold. Let $s_i = d_1 + \dots + d_{i-1}$, and $t_{i,j} = d_i + \dots + d_{j-1}$ for $i \le j$.

(i) No two of the $t_{i,j}$ divide each other.

(ii) There exists an integer a satisfying the CRT equivalences

$$a \equiv -s_i \pmod{t_{i,j}} \quad \forall i \le j$$

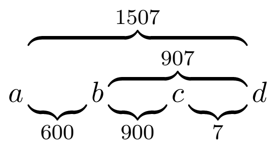
Then the sequence $a + s_1$, $a + s_2$, ..., $a + s_n$ will work. For example, when $n = 3$ we can take $(d_1, d_2) = (2, 3)$ giving



because the only conditions we need satisfy are

$$\begin{aligned} a &\equiv 0 \pmod 2 \\ a &\equiv 0 \pmod 5 \\ a &\equiv -2 \pmod 3. \end{aligned}$$

But with this setup we can just construct the d_i inductively. To go from n to $n+1$, take a $d_1, \dots, d_{n-1}$ and let p be a prime not dividing any of the d_i. Moreover, let $M = \prod_{i=1}^{n-1} d_i$. Then we claim that $d_1M, d_2M, \dots, d_{n-1}M, p$ is such a difference sequence. For example, the previous example extends as follows.



The new numbers p, $p + Md_{n-1}$, $p + Md_{n-2}$, ...are all relatively prime to everything else. Hence (i) still holds. To see that (ii) still holds, just note that we can still get a family of solutions for the first n terms, and then the last $(n + 1)$st term can be made to work by Chinese Remainder Theorem since all the new $p + Md_k$ are coprime to everything.

§1.3 Solution to TST 3, by Linus Hamilton

A physicist encounters 2015 atoms called usamons. Each usamon either has one electron or zero electrons, and the physicist can't tell the difference. The physicist's only tool is a diode. The physicist may connect the diode from any usamon A to any other usamon B. (This connection is directed.) When she does so, if usamon A has an electron and usamon B does not, then the electron jumps from A to B. In any other case, nothing happens. In addition, the physicist cannot tell whether an electron jumps during any given step. The physicist's goal is to isolate two usamons that she is 100% sure are currently in the same state. Is there any series of diode usage that makes this possible?

The answer is no. Call the usamons $U_1, \dots, U_m$ (here $m = 2015$). Consider models M_k of the following form: $U_1, \dots, U_k$ are all charged for some $0 \le k \le m$ and the other usamons are not charged. Note that for any pair there's a model where they are different states, by construction.

We can consider the physicist as acting on these $m+1$ models simultaneously, and trying to reach a state where there's a pair in all models which are all the same charge. (This is a necessary condition for a winning strategy to exist.)

But we claim that any diode operation $U_i \to U_j$ results in the $m+1$ models being an isomorphic copy of the previous set. If $i < j$ then the diode operation can be interpreted as just swapping U_i with U_j, which doesn't change anything. Moreover if $i > j$ the operation never does anything. The conclusion follows from this.

Remark. This problem is not a "standard" olympiad problem, so I can't say it's trivial. But the idea is pretty natural I think.

You can motivate it as follows: there's a sequence of diode operations you can do which forces the situation to be one of the M_k above: first, use the diode into U_1 for all other U_i's, so that either no electrons exist at all or U_1 has an electron. Repeat with the other U_i. This leaves us at the situation described at the start of the problem. Then you could guess the answer was "no" just based on the fact that it's impossible for $n = 2, 3$ and that there doesn't seem to be a reasonable strategy.

In this way it's possible to give a pretty good description of what it's possible to do. One possible phrasing: "the physicist can arrange the usamons in a line such that all the charged usamons are to the left of the un-charged usamons, but can't determine the *number* of charged usamons".

§2 Solutions to Day 2

§2.1 Solution to TST 4, by Victor Wang

Let $f \colon \mathbb{Q} \to \mathbb{Q}$ be a function such that for any $x, y \in \mathbb{Q}$, the number $f(x+y) - f(x) - f(y)$ is an integer. Decide whether there must exist a constant c such that $f(x) - cx$ is an integer for every rational number x.

No, such a constant need not exist.

One possible solution is as follows: define a sequence by $x_0 = 1$ and

$$\begin{aligned} 2x_1 &= x_0 \\ 2x_2 &= x_1 + 1 \\ 2x_3 &= x_2 \\ 2x_4 &= x_3 + 1 \\ 2x_5 &= x_4 \\ 2x_6 &= x_5 + 1 \\ &\vdots \end{aligned}$$

Set $f(2^{-k}) = x_k$ and $f(2^k) = 2^k$ for $k = 0, 1, \dots$. Then, let

$$f\left(a \cdot 2^k + \frac{b}{c}\right) = af\left(2^k\right) + \frac{b}{c}$$

for odd integers a, b, c. One can verify this works.

A second shorter solution (given by the proposer) is to set, whenever $\gcd(p, q) = 1$ and $q > 0$,

$$f\left(\frac{p}{q}\right) = \frac{p}{q}\left(1! + 2! + \cdots + q!\right).$$

> **Remark.** Silly note: despite appearances, $f(x) = \lfloor x \rfloor$ is not a counterexample since one can take $c = 0$.

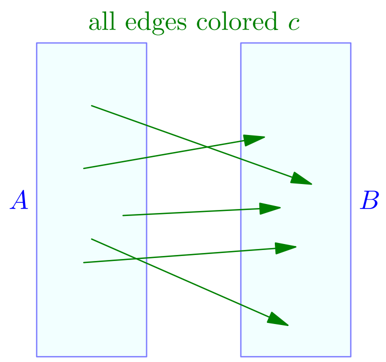
§2.2 Solution to TST 5, by Po-Shen Loh

Fix a positive integer n. A tournament on n vertices has all its edges colored by χ colors, so that any two directed edges $u \to v$ and $v \to w$ have different colors. Over all possible tournaments on n vertices, determine the minimum possible value of χ.

The answer is

$$\chi = \lceil \log_2 n \rceil .$$

First, we prove by induction on n that $\chi \ge \log_2 n$ for any coloring and any tournament. The base case $n = 1$ is obvious. Now given any tournament, consider any used color c. Then it should be possible to divide the tournament into two subsets A and B such that all c-colored edges point from A to B (for example by letting A be all vertices which are the starting point of a c-edge).



One of A and B has size at least $n/2$, say A. Since A has no c edges, and uses at least $\log_2 |A|$ colors other than c, we get

$$\chi \ge 1 + \log_2(n/2) = \log_2 n$$

completing the induction.

One can read the construction off from the argument above, but here is a concrete description. For each integer n, consider the tournament whose vertices are the binary representations of $S = \{0, \dots, n-1\}$. Instantiate colors c_1, c_2, ... Then for $v, w \in S$, we look at the smallest order bit for which they differ; say the kth one. If v has a zero in the kth bit, and w has a one in the kth bit, we draw $v \to w$. Moreover we color the edge with color c_k. This works and uses at most $\lceil \log_2 n \rceil$ colors.

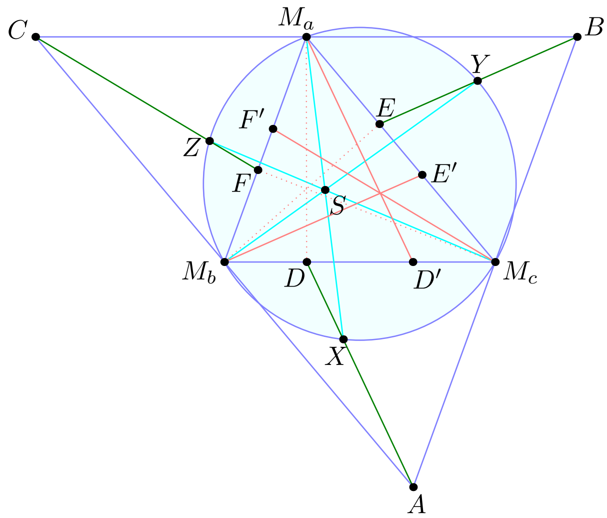
> **Remark** (Motivation)**.** The philosophy "combinatorial optimization" applies here. The idea is given any color c, we can find sets A and B such that all c-edges point A to B. Once you realize this, the next insight is to realize that you may as well color *all* the edges from A to B by c; after all, this doesn't hurt the condition and makes your life easier. Hence, if f is the answer, we have already a proof that $f(n) = 1 + \max(f(|A|), f(|B|))$ and we choose $|A| \approx |B|$. This optimization also gives the inductive construction.

§2.3 Solution to TST 6

Let ABC be a non-equilateral triangle and let M_a, M_b, M_c be the midpoints of the sides BC, CA, AB, respectively. Let S be a point lying on the Euler line. Denote by X, Y, Z the second intersections of M_aS, M_bS, M_cS with the nine-point circle. Prove that AX, BY, CZ are concurrent.

We assume now and forever that ABC is scalene since the problem follows by symmetry in the isosceles case. We present four solutions.

First solution by barycentric coordinates (Evan Chen) Let AX meet M_bM_c at D, and let X reflected over M_bM_c's midpoint be X'. Let Y', Z', E, F be similarly defined.



By Cevian Nest Theorem it suffices to prove that M_aD, M_bE, M_cF are concurrent. Taking the isotomic conjugate and recalling that $M_aM_bAM_c$ is a parallelogram, we see that it suffices to prove M_aX', M_bY', M_cZ' are concurrent.

We now use barycentric coordinates on $\triangle M_aM_bM_c$. Let

$$S = \left(a^2S_A + t : b^2S_B + t : c^2S_C + t\right)$$

(possibly $t = \infty$ if S is the centroid). Let $v = b^2S_B + t$, $w = c^2S_C + t$. Hence

$$X = \left(-a^2vw : (b^2w + c^2v)v : (b^2w + c^2v)w\right).$$

Consequently,

$$X' = \left(a^2vw : -a^2vw + (b^2w + c^2v)w : -a^2vw + (b^2w + c^2v)v\right)$$

We can compute

$$b^2w + c^2v = (bc)^2(S_B + S_C) + (b^2 + c^2)t = (abc)^2 + (b^2 + c^2)t.$$

Thus

$$-a^2v + b^2w + c^2v = (b^2 + c^2)t + (abc)^2 - (ab)^2S_B - a^2t = S_A((ab)^2 + t).$$

Finally

$$X' = \left(a^2vw : S_A(c^2S_C + t)\left((ab)^2 + 2t\right) : S_A(b^2S_B + t)\left((ac)^2 + 2t\right)\right)$$

and from this it's evident that AX', BY', CZ' are concurrent.

Second solution by moving points (Anant Mudgal) Let H_a, H_b, H_c be feet of altitudes, and let γ denote the nine-point circle. The main claim is that:

Claim — Lines XH_a, YH_b, ZH_c are concurrent,

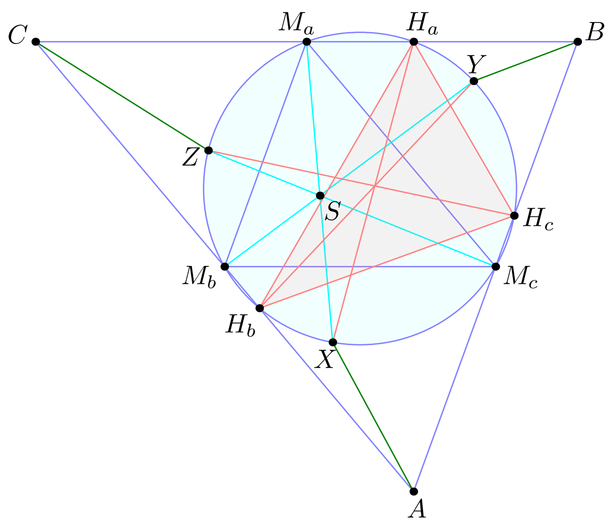
Proof. In fact, we claim that the concurrence point lies on the Euler line ℓ. This gives us a way to apply the moving points method: fix triangle ABC and animate $S \in \ell$; then the map

$$\begin{aligned} \ell &\to \gamma \to \ell \\ S &\mapsto X \mapsto S_a := \ell \cap \overline{H_a X} \end{aligned}$$

is projective, because it consists of two perspectivities. So we want the analogous maps $S \mapsto S_b$, $S \mapsto S_c$ to coincide. For this it suffices to check three positions of S; since you're such a good customer here are four.

- If S is the orthocenter of $\triangle M_aM_bM_c$ (equivalently the circumcenter of $\triangle ABC$) then S_a coincides with the circumcenter of $M_aM_bM_c$ (equivalently the nine-point center of $\triangle ABC$). By symmetry S_b and S_c are too.
- If S is the circumcenter of $\triangle M_aM_bM_c$ (equivalently the nine-point center of $\triangle ABC$) then S_a coincides with the de Longchamps point of $\triangle M_aM_bM_c$ (equivalently orthocenter of $\triangle ABC$). By symmetry S_b and S_c are too.
- If S is either of the intersections of the Euler line with γ, then $S = S_a = S_b = S_c$ (as $S = X = Y = Z$).

This concludes the proof. □



We now use Trig Ceva to carry over the concurrence. By sine law,

$$\frac{\sin \angle M_cAX}{\sin \angle AM_cX} = \frac{M_cX}{AX}$$

and a similar relation for M_b gives that

$$\frac{\sin \angle M_cAX}{\sin \angle M_bAX} = \frac{\sin \angle AM_cX}{\sin \angle AM_bX} \cdot \frac{M_cX}{M_bX} = \frac{\sin \angle AM_cX}{\sin \angle AM_bX} \cdot \frac{\sin \angle XM_aM_c}{\sin \angle XM_aM_b}.$$

Thus multiplying cyclically gives

$$\prod_{\text{cyc}} \frac{\sin \angle M_c AX}{\sin \angle M_b AX} = \prod_{\text{cyc}} \frac{\sin \angle AM_c X}{\sin \angle AM_b X} \prod_{\text{cyc}} \frac{\sin \angle XM_a M_c}{\sin \angle XM_a M_b}.$$

The latter product on the right-hand side equals 1 by Trig Ceva on $\triangle M_a M_b M_c$ with cevians $\overline{M_a X}$, $\overline{M_b Y}$, $\overline{M_c Z}$. The former product also equals 1 by Trig Ceva for the concurrence in the previous claim (and the fact that $\angle AM_c X = \angle H_c H_a X$). Hence the left-hand side equals 1, implying the result.

Third solution by moving points (Gopal Goel) In this solution, we will instead use barycentric coordinates with resect to $\triangle ABC$ to bound the degrees suitably, and then verify for seven distinct choices of S.

We let R denote the radius of $\triangle ABC$, and N the nine-point center.

First, imagine solving for X in the following way. Suppose $\vec{X} = (1 - t_a)\vec{M}_a + t_a \vec{S}$. Then, using the dot product (with $|\vec{v}|^2 = \vec{v} \cdot \vec{v}$ in general)

$$\begin{aligned}
\frac{1}{4}R^2 &= \left| \vec{X} - \vec{N} \right|^2 \\
&= \left| t_a(\vec{S} - \vec{M}_a) + \vec{M}_a - \vec{N} \right|^2 \\
&= \left| t_a(\vec{S} - \vec{M}_a) \right|^2 + 2t_a \left(\vec{S} - \vec{M}_a \right) \cdot \left(\vec{M}_a - \vec{N} \right) + \left| \vec{M}_a - \vec{N} \right|^2 \\
&= t_a^2 \left| (\vec{S} - \vec{M}_a) \right|^2 + 2t_a \left(\vec{S} - \vec{M}_a \right) \cdot \left(\vec{M}_a - \vec{N} \right) + \frac{1}{4}R^2
\end{aligned}$$

Since $t_a \neq 0$ we may solve to obtain

$$t_a = -\frac{2(\vec{M}_a - \vec{N}) \cdot (\vec{S} - \vec{M}_a)}{\left| \vec{S} - \vec{M}_a \right|^2}.$$

Now imagine S varies along the Euler line, meaning there should exist linear functions $\alpha, \beta, \gamma \colon \mathbb{R} \to \mathbb{R}$ such that

$$S = (\alpha(s), \beta(s), \gamma(s)) \qquad s \in \mathbb{R}$$

with $\alpha(s) + \beta(s) + \gamma(s) = 1$. Thus $t_a = \frac{f_a}{g_a} = \frac{f_a(s)}{g_a(s)}$ is the quotient of a linear function $f_a(s)$ and a quadratic function $g_a(s)$.

So we may write:

$$\begin{aligned}
X &= (1 - t_a) \left(0, \frac{1}{2}, \frac{1}{2} \right) + t_a (\alpha, \beta, \gamma) \\
&= \left(t_a \alpha, \frac{1}{2}(1 - t_a) + t_a \beta, \frac{1}{2}(1 - t_a) + t_a \gamma \right) \\
&= (2f_a \alpha : g_a - f_a + 2f_a \beta : g_a - f_a + 2f_a \gamma).
\end{aligned}$$

Thus the coordinates of X are quadratic polynomials in s when written in this way.

In a similar way, the coordinates of Y and Z should be quadratic polynomials in s. The Ceva concurrence condition

$$\prod_{\text{cyc}} \frac{g_a - f_a + 2f_a \beta}{g_a - f_a + 2f_a \gamma} = 1$$

is thus a polynomial in s of degree at most six. Our goal is to verify it is identically zero, thus it suffices to check seven positions of S.

- If S is the circumcenter of $\triangle M_aM_bM_c$ (equivalently the nine-point center of $\triangle ABC$) then $\overline{AX}$, $\overline{BY}$, $\overline{CZ}$ are altitudes of $\triangle ABC$.
- If S is the centroid of $\triangle M_aM_bM_c$ (equivalently the centroid of $\triangle ABC$), then $\overline{AX}$, $\overline{BY}$, $\overline{CZ}$ are medians of $\triangle ABC$.
- If S is either of the intersections of the Euler line with γ, then $S = X = Y = Z$ and all cevians concur at S.
- If S lies on the $\overline{M_aM_b}$, then $Y = M_a$, $X = M_c$, and thus $\overline{AX} \cap \overline{BY} = C$, which is of course concurrent with $\overline{CZ}$ (regardless of Z). Similarly if S lies on the other sides of $\triangle M_aM_bM_c$.

Thus we are also done.

Fourth solution using Pascal (official one) We give a different proof of the claim that $\overline{XH_a}$, $\overline{YH_b}$, $\overline{ZH_c}$ are concurrent (and then proceed as in the end of the second solution).

Let H denote the orthocenter, N the nine-point center, and moreover let N_a, N_b, N_c denote the midpoints of $\overline{AH}$, $\overline{BH}$, $\overline{CH}$, which also lie on the nine-point circle (and are the antipodes of M_a, M_b, M_c).

- By Pascal's theorem on $M_bN_bH_bM_cN_cH_c$, the point $P = \overline{M_cH_b} \cap \overline{M_bH_c}$ is collinear with $N = \overline{M_bN_b} \cap \overline{M_cN_c}$, and $H = \overline{N_bH_b} \cap \overline{N_cH_c}$. So P lies on the Euler line.
- By Pascal's theorem on $M_bYH_bM_cZH_c$, the point $\overline{YH_b} \cap \overline{ZH_c}$ is collinear with $S = \overline{M_bY} \cap \overline{M_cZ}$ and $P = \overline{M_bH_c} \cap \overline{M_cH_b}$. Hence YH_b and ZH_c meet on the Euler line, as needed.