OTIS Mock AIME 2026 Report

Solutions, results, and commentary

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Contents

1 Summary **3**
1.1 Top scores . . . 3
1.2 Editorial notes . . . 3
1.2.1 One problem set . . . 3
1.2.2 Probase . . . 3
1.2.3 Taylor series . . . 3
1.2.4 Errata . . . 3
1.2.5 Links to problems on Art of Problem Solving . . . 4

2 Solutions **5**
2.1 Answer key . . . 5
2.2 Full solutions . . . 6

3 Statistics **31**
3.1 Total score statistics . . . 31
3.2 Number of correct answers per problem . . . 31

4 Behind the scenes **32**
4.1 Testsolving statistics . . . 32
4.2 "Raw" versions of problems (pre-editing) . . . 32

5 Acknowledgments **35**

1 Summary

The OTIS Mock AIME aired from December 18, 2025 to January 20, 2026. There were a total of 153 submissions, of which 108 were under contest conditions.

§1.1 Top scores

Congratulations to the top scores:

14 points Arush Krisp, Carey Li, David Fox, Fagye, Patrick Sun, Ryan Tang, tenth

13 points Brayden Choi, Chenghao Hu, Joey Zheng, Vivdax

12 points Aryan, Hongming Allan Zhao, Kyle Liao, Niranjan, Vihaan Gupta, gouyanei, stead_axu, vockey

§1.2 Editorial notes

§1.2.1 One problem set

This year, we only chose to run one set of 15 problems. We were lucky last year to have a bumper crop, but this year I didn't feel like we had the same kind of explosive output as last year. I was also generally hosed, and didn't have time to edit a full 30 problems (which was a ton of work last year).

§1.2.2 Probase

We continued to use Probase for our solving system.

§1.2.3 Taylor series

The most controversial problem on the test is likely the 11th problem, due to its mandatory knowledge of the Taylor series

$$e^x = \sum_0^\infty \frac{x^n}{n!}.$$

Including it was a bit of an experiment. Some people hated it. Some people loved it. No surprise there, actually.

I feel like in general, there is too much fear in the math contest community of venturing outside what people consider "fair game" (which is not even actually defined anywhere, and instead just extrapolated from past years as if it's some sort of oral tradition). In the words of Jobu Tupaki, "*right is a small box invented by people who are afraid*".

§1.2.4 Errata

Much to my annoyance, we gave a set of numbers initially for the 13th problem that were impossible. In the originally aired version of the problem, the perimeter was given to be 3000 instead of 2048.

We updated this at Sun Dec 21 03:57:36 AM UTC 2025 after realizing that for perimeter 3000 the quadrilateral cannot actually exist (solving for a and d will give at least one negative value). However, in this impossible situation, most contestants still end up with the same numerical answer after finding the intended solution.

For scoring, we consider both the stale answer and the valid one as correct.

§1.2.5 Links to problems on Art of Problem Solving

Contest collections links:

- OTIS Mock AIME 2026: `https://aops.com/community/c4686568`
- All years of OTIS Mock AIME: `https://aops.com/community/c4180954`

2 Solutions

§2.1 Answer key

P#	Description	Author	Answer
1	$3n$ and $4n$	Oron Wang	108
2	10 pairwise coprime	Vincent Pirozzo	212
3	Product of altitudes	James Stewart	324
4	Rectangle counting	Neil Kolekar	876
5	Parabola	Benjamin Song	448
6	$\sum(a!b!c!)^{-1}$	James Stewart	041
7	$i < 3u$	Jiahe Liu	701
8	$\cos^3$	Tane Park	073
9	$\sum_n f(n, 100)$	Tane Park	455
10	$\angle B = \angle C = \angle D$	Joshua Liu	573
11	log sum	Aatmik Krishna	325
12	Harry Otter	Ashwin Shekhar	757
13	45° cyclic	Jack Whitney-Epstein	011
14	13-gon labels	Tanishq Pauskar	768
15	$\sqrt{2^{15}} \mod 127^4$	Royce Yao	157

§2.2 Full solutions

Problem 1. Compute the smallest three-digit positive integer n such that the decimal digits of $3n$ and $4n$ are permutations of each other.

¶ Answer. 108

¶ Problem author(s). Oron Wang

¶ First solution. Since $3n$ and $4n$ have the same sum of decimal digits, we need to have $3n \equiv 4n \pmod 9$, or $n \equiv 0 \pmod 9$. The smallest multiple of 9 exceeding 100 is $\boxed{108}$, which works since

$$3 \times 108 = 324 \text{ and } 4 \times 108 = 432.$$

¶ Second solution. As above, note that 108 works. Now, for all $0 \le i < 8$,

$$3(100 + i) = \overline{3(3 \cdot i)} \text{ and } 4(100 + i) = \overline{4(4 \cdot i)}$$

since $3i < 100$ and $4i < 100$ for small i in this range. Thus, $4i$ must contain an occurrence of the digit 3. However, this is clearly impossible for all $0 \le i < 8$, so no value less than $\boxed{108}$ can work.

Problem 2. Compute the sum of all positive integers n for which $\operatorname{lcm}(1, \dots, n)$ can be written as the product of 10 distinct pairwise coprime positive integers less than or equal to n.

¶ **Answer.** 212

¶ **Problem author(s).** Vincent Pirozzo

In what follows, $\pi(n)$ is the number of primes in $\{1, \dots, n\}$.

Claim — Suppose $\operatorname{lcm}(1, \dots, n)$ has been written as the product of a set S of 10 pairwise coprime integers as in the problem statement.

- If $1 \in S$, there $\pi(n) = 9$.
- If $1 \notin S$, there $\pi(n) = 10$.

Proof. Let T be S with the element 1 removed, if $1 \in S$ (otherwise, $T = S$). We will show $\pi(n) = |T|$.

If $|T| > \pi(n)$, then by the Pigeonhole principle at least two elements of T share a prime factor, which is impossible.

If $|T| < \pi(n)$, then some $t \in T$ is divisible by two different primes $p < q < n$. Since t is the only element of T divisible by either p or q, it follows that t is divisible by both $p^{\lfloor \log_p n \rfloor}$ and $q^{\lfloor \log_q n \rfloor}$ (the largest powers of p and q in $\{1, \dots, n\}$). However, we would then get

$$t \geq \operatorname{lcm}(p^{\lfloor \log_p n \rfloor}, q^{\lfloor \log_q n \rfloor}) > p^{\lfloor \log_p n \rfloor} \cdot p^1 > n$$

which is also a contradiction. $\square$

Conversely, if $9 \leq \pi(n) \leq 10$, then such a set S exists:

- If $\pi(n) = 9$, then one chooses S to consist of 1 and the largest powers of 2, 3, ..., 23 at most n.
- If $\pi(n) = 10$, then one chooses S to consist of the largest powers of 2, 3, ..., 23, 29 at most n.

Hence, the requested n are exactly those n for which $9 \leq \pi(n) \leq 10$. Since 23 is the 9th prime and 31 is the 11th prime, we get

$$23 + 24 + \cdots + 30 = \boxed{212}.$$

Problem 3. A triangle with area 90 is inscribed in a circle of radius 50. Compute the product of the lengths of the three altitudes of the triangle.

¶ Answer. 324

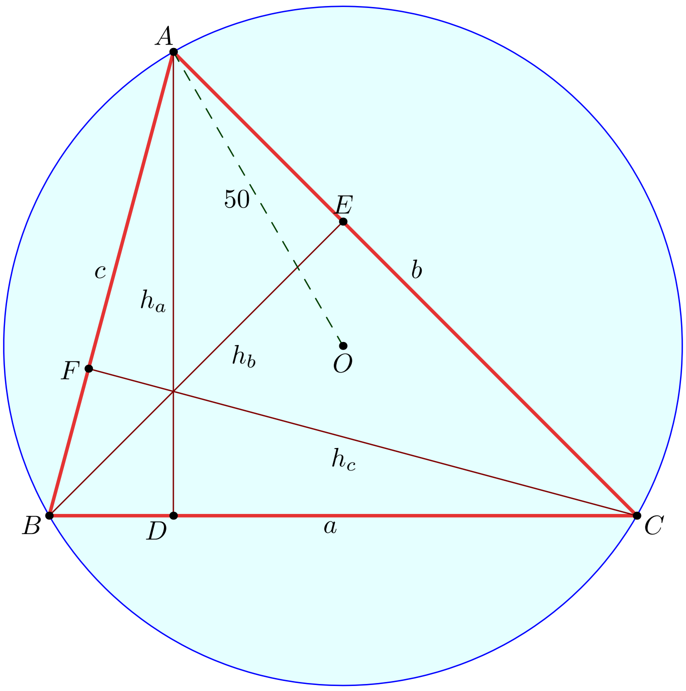
¶ Problem author(s). James Stewart

Let $a = BC$, $b = CA$, $c = AB$ and let h_a, h_b, h_c be the lengths of the three altitudes. We know the area of ABC obeys the formula

$$[ABC] = \frac{1}{2}ah_a = \frac{1}{2}bh_b = \frac{1}{2}ch_c.$$

Thus,

$$[ABC]^3 = \frac{1}{8}(abc)(h_ah_bh_c).$$



However, we also have the formula for $[ABC]$ in terms of the circumradius $R = 50$:

$$[ABC] = \frac{abc}{4R}.$$

Hence, dividing the equations gives

$$\begin{aligned} [ABC]^2 &= \frac{\frac{1}{8}(abc)(h_ah_bh_c)}{\frac{abc}{4R}} \\ &= \frac{1}{2}R(h_ah_bh_c) \\ \implies h_ah_bh_c &= \frac{2[ABC]^2}{R} = \frac{2(90)^2}{50} = \boxed{324}. \end{aligned}$$

Problem 4. Compute the number of rectangles $\mathcal{R}$ that can be drawn in the 8×8 grid below such that

- the edges of $\mathcal{R}$ lie along the gridlines;
- at least one cell of $\mathcal{R}$ is labeled with a multiple of 9.

For example, one such rectangle would be formed by taking the cells labeled 9, 10, 11, 17, 18, 19.

0	1	2	3	4	5	6	7
8	9	10	11	12	13	14	15
16	17	18	19	20	21	22	23
24	25	26	27	28	29	30	31
32	33	34	35	36	37	38	39
40	41	42	43	44	45	46	47
48	49	50	51	52	53	54	55
56	57	58	59	60	61	62	63

¶ **Answer.** 876

¶ **Problem author(s).** Neil Kolekar

¶ **Solution (Vincent Pirozzo)** We will use complementary counting. Any rectangle is defined by picking two vertical segments and two horizontal segments both of which span the whole square, so there are

$$\binom{9}{2}^2 = 36^2 = 1296$$

of them. Thus, we need to count how many rectangles do *not* contain a multiple of 9.

Notice that in the grid given, all multiples of 9 lie on the top-left to bottom-right diagonal. Coloring them in black as shown below splits the remaining cells into two staircases each with $1 + 2 + \cdots + 7 = 28$ cells.

0	1	2	3	4	5	6	7
8	9	10	11	12	13	14	15
16	17	18	19	20	21	22	23
24	25	26	27	28	29	30	31
32	33	34	35	36	37	38	39
40	41	42	43	44	45	46	47
48	49	50	51	52	53	54	55
56	57	58	59	60	61	62	63

We focus on just the bottom staircase to start:

Claim — The number of rectangles contained within the bottom staircase is $\binom{8+2}{4} = 210$.

Proof. Index the cells by coordinates (i, j) with $1 \le i, j \le 8$, with $(1, 1)$ in the bottom left. Observe that there are $i \cdot j$ rectangles with (i, j) at the top right corner, since the bottom-left corner has first coordinate in $[1, i]$ and second coordinate in $[1, j]$. Therefore, the number of rectangles in this region is

$$\begin{aligned}
\sum_{\substack{i,j \ge 1 \\ i+j \le 8}} ij &= \sum_{i=1}^{7} \sum_{j=1}^{8-i} ij \\
&= \sum_{i=1}^{7} i \cdot \frac{(8-i)(9-i)}{2} = \sum_{i=1}^{7} i \binom{9-i}{2} \\
&= 1 \cdot \binom{8}{2} + 2 \cdot \binom{7}{2} + 3 \cdot \binom{6}{2} \dots + 7 \cdot \binom{2}{2} \\
&= \sum_{j=2}^{8} \sum_{i=2}^{j} \binom{i}{2} \\
&= \binom{9}{3} + \binom{8}{3} + \dots + \binom{3}{3} \\
&= \binom{10}{4} = 210.
\end{aligned}$$

□

Similarly, there are 210 rectangles that lie in the top staircase. This gives us the final answer of

$$1296 - 210 - 210 = \boxed{876}.$$

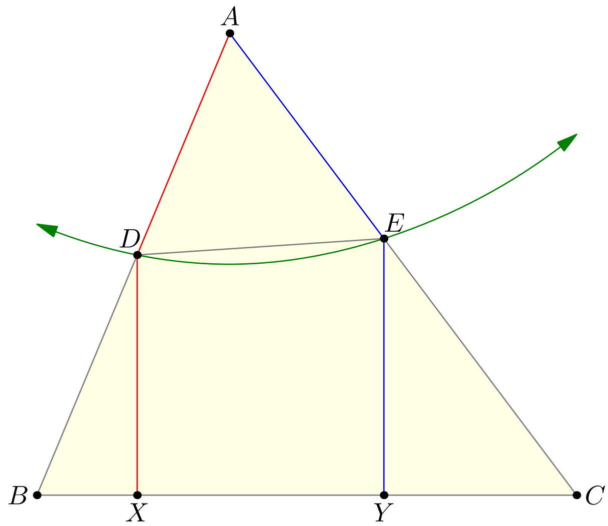
Problem 5. Triangle ABC has $AB = 65$, $BC = 70$, $CA = 75$. Let $\mathcal{P}$ be a parabola with focus A and directrix BC, and let $\mathcal{P}$ intersect segments AB and AC at D and E respectively. Compute the area of triangle ADE.

¶ Answer. 448

¶ Problem author(s). Benjamin Song

Since triangle ABC is a multiple of the well-known 13-14-15 triangle, we have $\sin\angle B = \frac{12}{13}$ and $\sin\angle C = \frac{4}{5}$ (say, by the law of cosines) as well as $[ABC] = 5^2 \cdot \sqrt{21(21-13)(21-14)(21-15)} = 5^2 \cdot 84$.

Let X and Y be the feet of the perpendiculars from D and E to BC, and by the definition of a parabola, we have that $DA = DX$ and $EA = EY$.



We compute each of AD and AE separately.

Claim — We have $AD = \frac{156}{5}$.

Proof. Write

$$AD = XD = \frac{BD}{\sin B} = \frac{AB - AD}{\sin B} = \frac{12}{13}(65 - AD) \implies AD = \frac{156}{5}.$$

□

Claim — We have $AE = \frac{100}{3}$.

Proof. In the same way, write

$$AE = YE = \frac{CE}{\sin C} = \frac{AC - AE}{\sin\angle C} = \frac{4}{5}(75 - EA) \implies AE = \frac{100}{3}.$$

□

Then, we can compute the area of $[ADE]$ by noting $\frac{[ADE]}{[ABC]} = \frac{AD \cdot AE}{AB \cdot AC}$. We get

$$[ADE] = \frac{AD \cdot AE}{AB \cdot AC} \cdot [ABC] = \frac{\frac{156}{5} \cdot \frac{100}{3}}{65 \cdot 75} \cdot (5^2 \cdot 84) = \boxed{448}.$$

Problem 6. Let k, p, q be positive integers such that $\gcd(pq, 3) = 1$ and

$$\sum_{\substack{a+b+c=81\\a,b,c\ge0}} \frac{1}{a!b!c!} = \frac{3^k p}{q}.$$

Compute k. Here the summation is over all triples (a, b, c) of nonnegative integers with sum 81.

¶ Answer. 041

¶ Problem author(s). James Stewart

Let S be the summation that we wish to evaluate, and note that

$$81! \cdot S = \sum_{\substack{a+b+c=81\\a,b,c\ge0}} \frac{81!}{a!b!c!}.$$

We give a combinatorial interpretation of the LHS:

Claim —

$$\sum_{\substack{a+b+c=81\\a,b,c\ge0}} \frac{81!}{a!b!c!} = 3^{81}.$$

Proof. The main idea is that $\frac{81!}{a!b!c!}$ is the number of ways to arrange 81 total fruits consisting of a indistinguishable apples, b indistinguishable bananas, and c indistinguishable pears in a row (such that $81 = a + b + c$.) Yet the sum counts this over all possible a, b, c, so this really counts the number of ways to arrange some number of apples, bananas, and pears in a row without restrictions on the number of apples, bananas, and pears.

We can count this in a different way by noticing that there are 3 choices of fruit for each position in the row, so the number of ways to make such an arrangement is 3^{81}. □

This means that

$$81! \cdot S = 3^{81} \implies S = \frac{3^{81}}{81!}.$$

Let $\nu_p(n)$ be defined in the usual way. The problem asks for $\nu_3(S)$, which is

$$81 - \nu_3(81!) = 81 - (27 + 9 + 3 + 1) = \boxed{041}.$$

Remark. The formula for $\nu_3(81!)$ follows by noting in $\{1, \dots, 81\}$ there are 27 multiples of 3, 9 multiples of 9, 3 multiples of 27 and 1 multiple of 81. By the same principle, Legendre's formula generally states that

$$\nu_p(n!) = \sum_{k\ge1}^{\infty} \left\lfloor \frac{n}{p^k} \right\rfloor$$

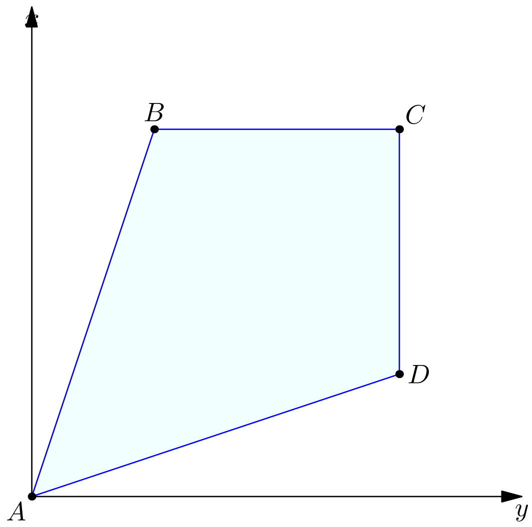
for every prime p and integer $n \ge 1$.

Problem 7. A pair of nonnegative integers (i, u) is said to be compatible if $i < 3u$ and $u < 3i$. Let N be the number of compatible pairs of integers (i, u) for $0 \le i, u < 300$. Compute the remainder when N is divided by 1000.

¶ **Answer.** 701

¶ **Problem author(s).** Jiahe Liu

The problem is asking for the number of lattice points strictly inside the quadrilateral bounded by the lines $y = 3x$, $x = 3y$, $x = 300$, and $y = 300$ which has vertices $A = (0, 0)$, $B = (100, 300)$, $C = (300, 300)$, and $D = (300, 100)$, shown below. Note that this quadrilateral has area 60000.



Our strategy is to use Pick's theorem. To that end, we compute the number of lattice points on the boundary.

Claim — There are 600 lattice points on the boundary of $ABCD$.

Proof. Consider each of the four sides.

- Note that the lattice points on segment AB are of the form $(x, 3x)$ for $0 \le x \le 100$, so there are 101 of them.
- Similarly, there are 101 lattice points on segment AD.
- The lattice points on segment BC are of the form $(x, 300)$ for $100 \le x \le 300$, so there are 201 of them.
- Similarly, there are 201 lattice points on line CD.

Of course, the four vertices A, B, C, and D appear twice in the above count. The number of lattice points on the border of the quadrilateral is

$$101 + 201 + 101 + 201 - 4 = 600.$$ □

By Pick's Theorem, we have

$$[ABCD] = N + \frac{B}{2} - 1 \implies 60000 = N + \frac{600}{2} - 1 \implies N = 59701.$$

This yields the answer $\boxed{701}$.

Problem 8. Let p and q be relatively prime integers (not necessarily positive) satisfying

$$\frac{p}{q} = \cos^3(20^\circ)\cos^3(140^\circ) + \cos^3(140^\circ)\cos^3(260^\circ) + \cos^3(260^\circ)\cos^3(20^\circ).$$

Compute $p^2 + q^2$.

¶ Answer. 073

¶ Problem author(s). Tane Park

¶ First solution (author). The basic strategy is to create a cubic polynomial whose roots are $\cos^3(20^\circ)$, $\cos^3(140^\circ)$, $\cos^3(260^\circ)$. We do this as follows:

Claim — Suppose $\theta \in \{20^\circ, 140^\circ, 260^\circ\}$ and $x = \cos\theta$. Then

$$64x^9 - 24x^6 - 24x^3 - 1/8 = 0.$$

Proof. For these θ, we have

$$\frac{1}{2} = \cos(3\theta) = 4\cos^3\theta - 3\cos\theta = 4x^3 - 3x$$

by the triple-angle formula. Thus

$$\left(4x^3 - \frac{1}{2}\right)^3 = (3x)^3 = 27x^3$$

which rearranges to the cubic claimed. □

It follows that the polynomial

$$p(T) := 64T^3 - 24T^2 - 24T - 1/8$$

has roots at each of $t_1 := \cos^3(20^\circ)$, $t_2 := \cos^3(140^\circ)$, $t_3 := \cos^3(260^\circ)$. Since $\deg P = 3$ these are in fact all of the roots, and by Vieta's formulas,

$$t_1t_2 + t_2t_3 + t_3t_1 = -\frac{24}{64} = -\frac{3}{8}.$$

Hence $p^2 + q^2 = 73$.

¶ Alternate solution via complex numbers (Vincent Pirozzo). Notice that by the cosine product-to-sum formula, we have

$$\begin{aligned}
\cos 20^\circ \cos 140^\circ &= \frac{\cos 160^\circ - \cos 60^\circ}{2} = \frac{\cos 160^\circ - \frac{1}{2}}{2} \\
\cos 140^\circ \cos 260^\circ &= \frac{\cos 400^\circ - \cos 60^\circ}{2} = \frac{\cos 40^\circ - \frac{1}{2}}{2} \\
\cos 260^\circ \cos 20^\circ &= \frac{\cos 280^\circ - \cos 60^\circ}{2} = \frac{\cos 280^\circ - \frac{1}{2}}{2}.
\end{aligned}$$

Now, we must compute

$$\frac{p}{q} = \left(\frac{\cos 160^\circ - \frac{1}{2}}{2}\right)^3 + \left(\frac{\cos 40^\circ - \frac{1}{2}}{2}\right)^3 + \left(\frac{\cos 280^\circ - \frac{1}{2}}{2}\right)^3.$$

Let

$$S := \{40^\circ, 160^\circ, 280^\circ\}.$$

Then the requested quantity is

$$\begin{aligned}
\frac{p}{q} &= \sum_{\theta \in S} \left(\frac{\cos\theta - \frac{1}{2}}{2}\right)^3 \\
&= \sum_{\theta \in S} \frac{1}{8}\left(\cos^3\theta - \frac{3}{2}\cos^2\theta + \frac{3}{4}\cos\theta - \frac{1}{8}\right) \\
&= \frac{1}{8}\sum_{\theta \in S}\cos^3\theta - \frac{3}{16}\sum_{\theta \in S}\cos^2\theta + \frac{3}{32}\sum_{\theta \in S}\cos\theta - \frac{3}{64} \\
&= \frac{1}{8}\sum_{\theta \in S}\frac{\cos(3\theta) + 3\cos\theta}{4} - \frac{3}{16}\sum_{\theta \in S}\frac{\cos(2\theta) + 1}{2} + \frac{3}{32}\sum_{\theta \in S}\cos\theta - \frac{3}{64} \\
&= \frac{1}{32}\sum_{\theta \in S}\cos(3\theta) - \frac{3}{32}\sum_{\theta \in S}\cos(2\theta) + \frac{3}{16}\sum_{\theta \in S}\cos\theta - \frac{21}{64}
\end{aligned}$$

due to the double and triple angle formulas.

On the other hand, for all angles t we have the identity

$$\cos(t) + \cos(t + 120^\circ) + \cos(t + 240^\circ) = 0. \tag{2.1}$$

(Indeed, (2.1) follows by taking the real part of $e^{it}(1 + \omega + \omega^2) = 0$ for $\omega = e^{2\pi i/3}$.) From (2.1) we deduce

$$\sum_{\theta \in S}\cos\theta = 0 \qquad \sum_{\theta \in S}\cos 2\theta = 0$$

while

$$\sum_{\theta \in S}\cos 3\theta = -\frac{1}{2} - \frac{1}{2} - \frac{1}{2} = -\frac{3}{2}.$$

Hence, the sum above collapses into simply

$$\frac{p}{q} = \frac{1}{32} \cdot \left(-\frac{3}{2}\right) - \frac{21}{64} = -\frac{3}{8}.$$

So $p^2 + q^2 = 73$.

Problem 9. A function f taking pairs of nonnegative integers to nonnegative integers satisfies $f(0,0)=0$ and

$$f(x,y)=f(y,x)=f(x+y,y) \text{ and } f(x,x)=f(x^2-1,x^3-1)+x$$

for all integers $x\geq 1$ and $y\geq 0$. Compute the remainder when

$$f(1,100)+f(2,100)+\cdots+f(100,100)$$

is divided by 1000.

¶ **Answer.** 455

¶ **Problem author(s).** Tane Park

We begin by characterizing the function f.

Claim — For all pairs of positive integers x and y,

$$f(x,y)=\binom{\gcd(x,y)+1}{2}$$

Proof. We first establish the key identity

$$f(x,y)=f(\gcd(x,y),\gcd(x,y)). \tag{2.2}$$

Indeed, the recursion $f(x,y)=f(y,x)=f(x+y,y)$ is the Euclidean algorithm. Put another way, when $x>y$ we have $f(x-y,y)=f(x,y)$ and $\gcd(x-y,y)=\gcd(x,y)$. Hence, by applying the Euclidean algorithm inductively, (2.2) follows.

Now, observe that since $\gcd(n^2-1,n^3-1)=n-1$ for all positive integers n,

$$f(n,n)=f(n^2-1,n^3-1)+n=f(n-1,n-1)+n$$

which together with $f(0,0)=0$ implies

$$f(n,n)=\binom{n+1}{2}$$

for all positive integers n. Combining with (2.2), the claim is proved. □

To compute the requested sum, we note that $100=4\cdot 25$ while 4 and 25 are relatively prime. Since $\gcd(i,100)=\gcd(i,4)\gcd(i,25)$, by Chinese Remainder Theorem we get the identities

$$\sum_{i=1}^{100}\gcd(i,100)=\sum_{i=1}^{4}\gcd(i,4)\cdot\sum_{i=1}^{25}\gcd(i,25)$$
$$\sum_{i=1}^{100}\gcd(i,100)^2=\sum_{i=1}^{4}\gcd(i,4)^2\cdot\sum_{i=1}^{25}\gcd(i,25)^2.$$

We may now compute:

$$\begin{aligned}
\sum_{i=1}^{100} f(i,100) &= \sum_{i=1}^{100} \binom{\gcd(i,100)+1}{2} \\
&= \frac{1}{2}\left(\sum_{i=1}^{100} \gcd(i,100)^2 + \sum_{i=1}^{100} \gcd(i,100)\right) \\
&= \frac{1}{2}\left(\sum_{i=1}^{4} \gcd(i,4)^2 \cdot \sum_{i=1}^{25} \gcd(i,25)^2 + \sum_{i=1}^{4} \gcd(i,4) \cdot \sum_{i=1}^{25} \gcd(i,25)\right) \\
&= \frac{1}{2}((2\cdot 1^2 + 1\cdot 2^2 + 1 \cdot 4^2)\cdot(20\cdot 1^2 + 4\cdot 5^2 + 1\cdot 25^2) \\
&+ (2\cdot 1 + 1\cdot 2 + 1\cdot 4)\cdot(20\cdot 1 + 4\cdot 5 + 1\cdot 25)) \\
&= \frac{1}{2}(22\cdot 745 + 8\cdot 65) \\
&= 8455
\end{aligned}$$

so the answer is $\boxed{455}$.

Problem 10. Quadrilateral $ABCD$ with area k satisfies $\angle B = \angle C = \angle D$. Suppose $BC = 29$, $AD = 41$, and the distance from A to CD is 40. Compute $\lfloor k \rfloor$.

¶ Answer. 573

¶ Problem author(s). Joshua Liu

Denote by M the foot of the altitude from A to side CD, and by D' the reflection of D over M. Observe that $\triangle AD'M \cong \triangle ADM$ due to reflection, and in fact both right triangles have side lengths 9, 40, 41 (by Pythagoras).

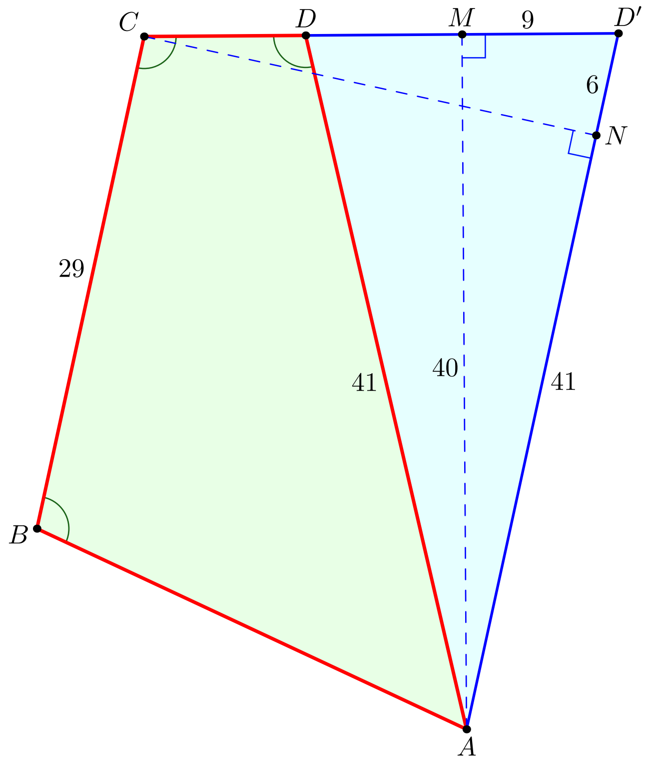
Claim — Quadrilateral $ABCD'$ is an isosceles trapezoid with $AB = CD'$.

Proof. Note that

$$\measuredangle DD'A = \measuredangle ADD' = \measuredangle CBA = \measuredangle DCB$$

implying both that $ABCD'$ is cyclic and $AD' \parallel BC$. This is enough to show $ABCD'$ is an isosceles trapezoid with $AB = CD'$. □

We will now compute $[ABCD']$.



Claim — We have $[ABCD'] = \frac{2800}{3}$.

Proof. Denote by N the foot of the altitude from C to line AD'. Observe that if N' is the foot of the altitude from B to AD' then $\triangle BAN' \cong \triangle CD'N$ and $BCNN'$ is a rectangle which implies that,

$$D'N = \frac{AD' - BC}{2} = \frac{41 - 29}{2} = 6.$$

Furthermore, $\triangle NCD' \sim \triangle MAD'$. This allows us to compute

$$CN = \frac{40}{9} \cdot ND' = \frac{40}{9} \cdot 6 = \frac{80}{3}.$$

Thus, the area of the trapezoid is,

$$[ABCD'] = \frac{CN(AD' + BC)}{2} = \frac{80}{3} \cdot \frac{70}{2} = \frac{2800}{3}. \qquad \square$$

To finish, write

$$[ABCD] = [ABCD'] - 2[ADD'] = \frac{2800}{3} - 40 \cdot 9 = \frac{1720}{3}$$

so the answer is $\left\lfloor \frac{1720}{3} \right\rfloor = \boxed{573}$.

Problem 11. Let $e \approx 2.718$ denote the base of the natural logarithm ln. Compute the remainder when

$$\frac{1}{e} \sum_{i \ge 1} \sum_{j \ge 1} \sum_{k=1}^{2026} \frac{\binom{i-1}{j-1} \ln(k)^j}{j(i-1)!}$$

is divided by 1000.

¶ Answer. 325

¶ Problem author(s). Aatmik Krishna

¶ First solution (author). For any positive integer n, set

$$S_n = \sum_{i \ge 1} \sum_{j \ge 1} \sum_{k=1}^{n} \frac{\binom{i-1}{j-1} \ln(k)^j}{j\,(i-1)!}$$

we wish to evaluate $\frac{1}{e} \cdot S_{2026}$. We know that $\frac{1}{j}\binom{i-1}{j-1} = \frac{1}{i}\binom{i}{j}$. Hence,

$$S = \sum_{i \ge 1} \sum_{j \ge 1} \sum_{k=1}^{n} \frac{\binom{i}{j} \ln(k)^j}{i!} = \sum_{k=1}^{n} \sum_{i \ge 1} \sum_{j \ge 1} \frac{\binom{i}{j} \ln(k)^j}{i!}.$$

Now, we know that

$$\sum_{j \ge 1} \binom{i}{j} \ln(k)^j = (1 + \ln(k))^i - 1$$

so

$$\begin{aligned} S_n = \sum_{k=1}^{n} \sum_{i \ge 1} \sum_{j \ge 1} \frac{\binom{i}{j} \ln(k)^j}{i!} &= \sum_{k=1}^{n} \sum_{i \ge 1} \frac{(1+\ln(k))^i - 1}{i!} \\ &= \sum_{k=1}^{n} \sum_{i \ge 1} \frac{(1+\ln(k))^i}{i!} - \sum_{k=1}^{n} \sum_{i \ge 1} \frac{1}{i!} \\ &= \sum_{k=1}^{n} e^{\ln(e \cdot k)} - \sum_{k=1}^{n} e \\ &= \left(\sum_{k=1}^{n} ek \right) - en \\ &= e \cdot \frac{n(n-1)}{2}. \end{aligned}$$

For $n = 2026$, we get $2025 \cdot 1013 \equiv 25 \cdot 13 \equiv \boxed{325} \pmod{1000}$.

¶ Second solution via swapping the order of summation (Vincent Pirozzo). We give another way to evaluate S_n. Rewrite the inner expression as

$$\frac{\frac{(i-1)!}{(j-1)!(i-j)!}\ln(k)^j}{j(i-1)!} = \frac{\ln(k)^j}{(i-j)!(j)!}.$$

Swapping the order of summation gives

$$\begin{aligned}\frac{1}{e}\sum_{k=1}^{2026}\left(\sum_{j\ge 1}\left(\sum_{i\ge j}\frac{\ln(k)^j}{(i-j)!(j)!}\right)\right) &= \frac{1}{e}\sum_{k=1}^{2026}\sum_{j\ge 1} e\frac{\ln(k)^j}{j!} \\ &= \sum_{k=1}^{2026}(e^{\ln k}-1) \\ &= \sum_{k=1}^{2026}(k-1)\end{aligned}$$

which is equivalent to $0+1+2+\cdots+2025$, congruent to $\boxed{325}$ modulo 1000.

Problem 12. Oscar the otter is reading the seven volumes of his favorite book series, Harry Otter. He has them all in a row in his bookshelf, but he wants to put them in chronological order. He wishes to do this by making *swaps*, where in one swap he will switch the positions of any two books in the row (these two books do *not* necessarily have to be adjacent). If the books in the row are in a random permutation of the chronological ordering, then the expected number of swaps Oscar needs for the books in the row to be in chronological order is $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute $m+n$.

¶ **Answer.** 757

¶ **Problem author(s).** Ashwin Shekhar

The solution given here is from Isaac Chan-Osborn.

Let $n = 7$ and let S_n denote the set of $n!$ possible permutations of $\{1, \dots, n\}$. Given a permutation $\sigma \in S_n$, we use the following characterization.

Lemma

The minimum number of swaps needed is $n - c(\sigma)$, where $c(\sigma)$ is the number of cycles of σ.

Remark. A more intuitive way to state this lemma is that, if a cycle has length ℓ, it takes $\ell - 1$ swaps to sort it out. More generally, if σ consists of a cycle decomposition with lengths $\ell_1, \dots, \ell_{c(\sigma)}$, the total number of moves needed will be

$$(\ell_1 - 1) + (\ell_2 - 1) + \cdots + (\ell_{c(\sigma)} - 1) = n - c(\sigma)$$

since $\sum_i \ell_i = n$ by definition.

Idea of proof. It is clear that $0 \le c(\sigma) \le n$. One may check that applying a transposition changes $c(\sigma)$ by either $+1$ or -1. Meanwhile, σ is the identity if and only if $c(\sigma) = n$. This shows that at least $n - c(\sigma)$ moves are necessary; but they are also sufficient because a cycle of length ℓ can be made into the identity with $\ell - 1$ transpositions. □

Remark. This lemma is far from new; see for example USA TST 2016 Problem 1. In modern language, it is phrased as saying that $n - c(\sigma)$ is the transposition distance of σ.

Claim — We have

$$\mathbb{E}[c(\sigma)] = \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n}$$

across $\sigma \in S_n$.

Proof. The idea is to use linearity of expectation as follows. Pick an integer $k \ge 1$ and distinct integers $t_1, \dots, t_k \in \{1, \dots, n\}$. We will calculate the probability that σ contains the cycle

$$t_1 \mapsto t_2 \mapsto \cdots \mapsto t_k \mapsto t_1$$

that is, $t_i = \sigma(t_{i-1})$ for all i with indices modulo k.

Indeed this is actually straightforward: fix an arbitrary book t_1. The probability that $\sigma(t_1) = t_2$ is $\frac{1}{n}$; the probability that $\sigma(t_2) = t_3$ is $\frac{1}{n-1}$, and so on, getting the probability that $\sigma(t_k) = t_1$ as $\frac{1}{n-k+1}$. So the probability that our cycle is present is exactly

$$\frac{1}{(n)(n-1)\dots(n-k+1)}.$$

On the other hand, the number of possible cycles of length k is $\binom{n}{k} \cdot (k-1)!$. So by linearity of expectation, the total contribution to $c(\sigma)$ from all the cycles of length k is

$$\begin{aligned}\binom{n}{k} \cdot (k-1)! \cdot \frac{1}{(n)(n-1)\dots(n-k+1)} &= \frac{n!}{k!(n-k)!} \cdot (k-1)! \cdot \frac{(n-k)!}{n!} \\ &= \frac{(n!)(n-k)!(k-1)!}{(k!)(n-k)!(n!)} \\ &= \frac{(k-1)!}{k!} \\ &= \frac{1}{k}.\end{aligned}$$

Summing over $1 \le k \le n$ yields the result. □

Remark. The special case $k = 1$ is more well-known in the math competition community; it states that the expected number of fixed points of a permutation is given by 1. See the first example of https://web.evanchen.cc/handouts/ProbabilisticMethod/ProbabilisticMethod.pdf.

Returning to the problem, when $n = 7$ this yields the answer

$$7 - \left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7}\right) = \frac{617}{140}.$$

Therefore, the desired answer is $617 + 140 = \boxed{757}$.

Problem 13. Let $ABCD$ be a cyclic quadrilateral whose perimeter is 2048 and area is $2026(1+\sqrt{2})$. Assume $\angle C = 45^\circ$. Given that the lengths BC and DC are both integers, compute $\min(BC, DC)$.

¶ **Answer.** 011

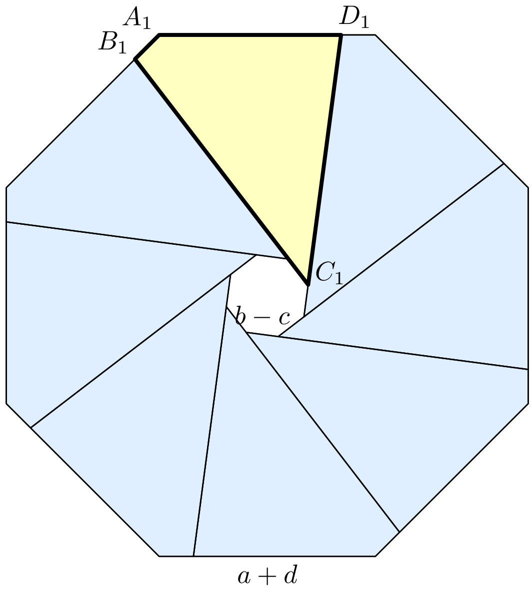
¶ **Problem author(s).** Jack Whitney-Epstein

Let $AB = a, BC = b, CD = c, DA = d$. We are going to prove the following formula for the area of $ABCD$:

Claim — We have

$$[ABCD] = \frac{1+\sqrt{2}}{4}\left((a+d)^2 - (b-c)^2\right).$$

First proof, by author. WLOG assume $c \ge b$. We make eight congruent copies $A_iB_iC_iD_i$ for $1 \le i \le 8$. Paste the copies $A_iB_iC_iD_i$ around each other, each rotated by 45°, with each quadrilateral's vertex D_i coinciding with the previous B_{i-1}. Notice that A_i, $B_i = D_{i+1}$, and A_{i+1} are collinear. This creates a regular octagon with an octagonal hole in the middle, as shown in the figure below.



The area of a regular octagon with side length s is $s^2 \cdot 2(1+\sqrt{2})$. Hence, we get

$$8[ABCD] = 2(1+\sqrt{2}) \cdot ((a+d)^2 - (c-b)^2)$$

which gives the desired formula. □

Second proof, Vincent Pirozzo. By the law of cosines we have

$$BD^2 = a^2 + d^2 + \sqrt{2}ad = b^2 + c^2 - \sqrt{2}bc$$
$$\implies (a^2+d^2) - (b^2+c^2) = -\sqrt{2}[ad+bc]$$

$$\implies (a+d)^2 - (b-c)^2 = (2-\sqrt{2})(ad+bc).$$

Then

$$\begin{aligned}[ABCD] &= [BAD] + [BCD] \\ &= \frac{1}{2} \cdot \frac{\sqrt{2}}{2} \cdot (ad+bc) \\ &= \frac{\sqrt{2}/4}{2-\sqrt{2}} \cdot [(a+d)^2 - (b-c)^2] \\ &= \frac{\sqrt{2}+1}{4}[(a+d)^2 - (b-c)^2]. \end{aligned}$$ □

To extract $\min(b,c)$, we now use the specific numbers given. Writing $(a+d)^2 - (b-c)^2$ as a difference of squares, we have

$$2^3 \cdot 1013 = 4 \cdot 2026 = (a+d+b-c)(a+d+c-b) = (2048-2b)(2048-2c).$$

By quadrilateral inequality, each factor on the right-hand side should be nonnegative (and at most 2048). Since 1013 is prime; this can only occur if

$$\{2048-2b, 2048-2c\} = \{4, 2026\}.$$

Hence $\{b,c\} = \{11, 1022\}$ and the answer is $\boxed{011}$.

> **Remark.** In the originally aired version of the problem, the perimeter was given to be 3000 instead of 2048. We updated this at Sun Dec 21 03:57:36 AM UTC 2025 after realizing that for perimeter 3000 the quadrilateral cannot actually exist (solving for a and d will give at least one negative value). The intended answer was $487 = \frac{3000-2026}{2}$ for the wrong version of the problem.

Problem 14. Let N be the number of ways to label the vertices of a regular 13-gon, each with an integer from 1 through 14 (repetitions allowed), so that no set of consecutive labels has a sum divisible by 15. Compute the remainder when N is divided by 1000.

¶ **Answer.** 768

¶ **Problem author(s).** Tanishq Pauskar

Let $0 \neq v_i \in \mathbb{Z}/15\mathbb{Z}$ denotes the label of the i^{th} vertex, viewed modulo 15. Confusingly, although the labels are always modulo 15, in this solution we break symmetry and do *not* take indices modulo 13.

For any $(i,j) \in \{1,\dots,13\}^2$ we define

$$S_{i,j} = \begin{cases} v_i + v_{i+1} + \dots + v_j & \text{if } i \le j \\ v_i + v_{i+1} + \dots + v_{15} + v_1 + \dots + v_j & \text{if } i > j \end{cases} \in \mathbb{Z}/15\mathbb{Z}.$$

Then the problem statement requires $S_{i,j}$ is always nonzero across all 13^2 choices of $(i,j) \in \{1,\dots,13\}^2$. (In particular, $S_{i,i} \neq 0$ is the requirement $v_i \neq 0$.) For each $1 \le k \le 13$, we also define the prefix sum

$$P_k := v_1 + \dots + v_k \in \mathbb{Z}/15\mathbb{Z}.$$

Let's now rewrite the condition in terms of prefix sums:

Claim — The following are equivalent:

- The assignment is valid, i.e. it satisfies $S_{i,j} \neq 0$ for all $(i,j) \in \{1,\dots,13\}^2$.
- We have $P_{13} \neq 0$ and for all $1 \le k < \ell \le 13$, we have $P_k \neq P_\ell$ and $P_\ell - P_k \neq P_{13}$.

Proof. We first rewrite all the conditions on $S_{i,j}$ in terms of $P_{i,j}$. When $i \le j$, we have

$$S_{i,j} \neq 0 \iff P_j - P_{i-1} \neq 0$$

where $P_0 = 0$ for convenience. When $i > j$, we instead have

$$\begin{aligned} S_{i,j} \neq 0 \pmod{15} &\iff P_{13} - P_{i-1} + P_j \neq 0 \pmod{15} \\ &\iff P_{i-1} - P_j \neq P_{13} \pmod{15}. \end{aligned}$$

We now carefully tally these $13^2 = 169$ into the conclusion of the claim:

- For the 12 pairs (i,j) with $i = 1$ and $1 \le j \le 12$, we get $P_j \neq 0$, which is $P_\ell - P_k \neq P_{13}$ for $1 \le k \le 12$ and $\ell = 13$.
- For the 1 pair $(i,j) = (1,13)$, we get $P_{13} \neq 0$.
- For the 12 pairs (i,j) with $i = j+1$ we get $P_{13} \neq 0$ again.
- For the 78 pairs (i,j) with $2 \le i \le j \le 13$, we $P_k \neq P_\ell$ for $(k,\ell) = (i-1,j)$, which spans $1 \le k < \ell \le 13$.

- For the 66 pairs (i,j) with $i \ge j+2$, we get $P_\ell - P_k \neq P_{13}$ for $(k,\ell) = (j, i-1)$, which spans $1 \le k < \ell \le 12$.

The translation thus gives precisely the $1+12+78+66 = 1+2\binom{13}{2}$ statements claimed. $\square$

Hence from now on we only work with the sequence of prefix sums $(P_1, P_2, \dots, P_{13})$. From here on, we consider cases based on whether

$$X \coloneqq P_{13}$$

is coprime to 15.

- **Case 1**: X is relatively prime to 15.

  Consider the 14 nonzero multiples of X:

  $$X,\ 2X,\ \dots,\ 14X \pmod{15}.$$

  We know we have 13 distinct prefix sums in the sequence $(P_1, \dots, P_{13})$. So exactly one of the nonzero multiples is omitted in this set, say tX.

  We further subdivide our cases based on t. For a fixed t, the remaining condition is now that $X, 2X, \dots, (t-1)X$ must appear in decreasing order, as do the indices of $(t+1)X, \dots, 14X$. Hence, the valid assignments correspond to just choosing which subset of indices i satisfy $P_i \in \{X, \dots, (t-1)X\}$. There are $\binom{12}{t-2}$ ways to do this.

  Hence, for each X, the total number of arrangements is

  $$\sum_{t=2}^{14} \binom{12}{t-2} = 2^{12}.$$

  Finally, since there are $\varphi(15) = 8$ choices of X, the total number of labellings in this case is $\varphi(15) \cdot 2^{12}$.

- **Case 2**: X is not relatively prime to 15.

  We contend this case cannot occur. Let $m = \gcd(15, X) \ge 3$. We consider the $m-1$ arithmetic progressions (modulo 15, each with $\frac{15}{m}$ elements):

  $$\begin{gathered} \{1, 1+X, 1+2X, \dots, 1-X\}, \\ \{2, 2+X, 1+2X, \dots, 2-X\}, \\ \vdots \\ \{m-1, m-1+X, 1+2X, \dots, (m-1)-X\}. \end{gathered}$$

  Since the sequence $(P_1, \dots, P_{13})$ omits exactly one nonzero residue, at least one of these arithmetic progressions must appear entirely. However, in that case the condition that $P_\ell - P_k \neq X$ for $k < \ell$ is plainly impossible to satisfy.

So, only the first case can occur and we obtain $N = \varphi(15) \cdot 2^{12} = 32768$, hence the remainder is $\boxed{768}$.

Problem 15. Let a be the smallest positive integer such that $a^2 - 2^{15}$ is divisible by 127^4. Compute the remainder when a is divided by 1000.

¶ Answer. 157

¶ Problem author(s). Royce Yao

Let $p = 127$ be a prime: we need to find a square root of

$$2^{15} = 256(p+1) = 256p + 256 \pmod{p^4}.$$

The basic motivation is to try to think of $\sqrt{2^{15}} = 16\sqrt{127+1}$ as a Taylor series. Specifically, if one knows the identity

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \dots$$

then $16\sqrt{1+p} = 16 + 8p - 2p^2 + p^3 + O(p^4)$ should be the desired value of a. To make this into a real proof would require working over p-adic numbers; hence we are content to give the following identity and a self-contained elementary derivation:

Claim — Let $F := 16 + 8p - 2p^2 + p^3$, where $p = 127$. Then we have

$$F^2 \equiv 256(p+1) \pmod{p^4}.$$

Proof. We need to guess a quadruple (c_0, c_1, c_2, c_3) of integers, say with $c_0 > 0$ such that $F = c_0 + c_1 p + c_2 p^2 + c_3 p^2$ satisfies $F^2 = 256(p+1) \pmod{p^4}$. Viewing F^2 as polynomial in p, it would be sufficient (not necessary) for the coefficients of 1, p, p^2, p^3 to match. We can thus recursively compute a working (c_0, c_1, c_2, c_3):

$$\begin{aligned} c_0^2 = 256 &\iff c_0 = 16 \\ 2 \cdot 8 \cdot 16 \cdot c_1 = 256 &\iff c_1 = 8 \\ 2 \cdot 16 c_2 + 8^2 = 0 &\iff c_2 = -2 \\ 2 \cdot 16 c_3 + 2 \cdot 8(-2) = 0 &\iff c_3 = 1. \end{aligned}$$

These are the coefficients in the claim. □

All that remains is to note that $16 + 8 \cdot p - 2 \cdot p^2 + p^3 < \frac{p^4}{2}$, so

$$a = 16 + 8p - 2p^2 + p^3 = 2017157.$$

(That's because the equation $x^2 \equiv 2^{15} \pmod{p^4}$ always has exactly one solution in $(0, p^4/2)$; actually in $\mathbb{Z}/p^4\mathbb{Z}$ the solutions are $\pm a$). The requested remainder is thus $\boxed{157}$.

3 Statistics

§3.1 Total score statistics

Score	Freq
Total score = 0	1
Total score = 1	1
Total score = 2	1
Total score = 3	3
Total score = 4	5
Total score = 5	8
Total score = 6	10
Total score = 7	4
Total score = 8	5
Total score = 9	18
Total score = 10	14
Total score = 11	18
Total score = 12	9
Total score = 13	4
Total score = 14	7
Total score = 15	0

§3.2 Number of correct answers per problem

P#	#Correct	% Correct	Description
1	100	92.59%	$3n$ and $4n$
2	71	65.74%	10 pairwise coprime
3	102	94.44%	Product of altitudes
4	79	73.15%	Rectangle counting
5	93	86.11%	Parabola
6	96	88.89%	$\sum(a!b!c!)^{-1}$
7	64	59.26%	$i < 3u$
8	73	67.59%	$\cos^3$
9	73	67.59%	$\sum_n f(n, 100)$
10	54	50.00%	$\angle B = \angle C = \angle D$
11	42	38.89%	log sum
12	54	50.00%	Harry Otter
13	31	28.70%	$45°$ cyclic
14	6	5.56%	13-gon labels
15	20	18.52%	$\sqrt{2^{15}} \mod 127^4$

4 Behind the scenes

§4.1 Testsolving statistics

Serious testsolvers had a time limit and unlimited answer checks. We record

- the fastest solve time (in seconds) among testsolvers who got the answer in one try;
- the median solve time among testsolvers who got the answer in one try (if the number of such solves is even, we take the larger of the two times);
- the number of testsolvers that got the correct answer on the first try;
- the number that eventually got the correct answer before the timer expired (but may have required multiple tries). The timer was set as follows: the author was asked to estimate the difficulty of each problem on a scale of one light bulb to five light bulbs, where n light bulbs was a problem suitable for problem $3n$ on the AIME. Then solvers got $5(n+1)$ minutes for their testsolving session. (However, they could give up early if they didn't want to wait for the full time.)

P#	Description	Fastest	Median	Correct	Finished
1	$3n$ and $4n$	0:18	1:32	24	27
2	10 pairwise coprime	0:55	1:25	7	16
3	Product of altitudes	0:20	1:11	31	37
4	Rectangle counting	1:36	2:29	8	16
5	Parabola	1:16	2:28	16	19
6	$\sum(a!b!c!)^{-1}$	0:29	1:23	18	28
7	$i < 3u$	1:09	3:30	8	13
8	$\cos^3$	0:56	3:19	13	19
9	$\sum_n f(n, 100)$	0:35	1:51	20	24
10	$\angle B = \angle C = \angle D$	2:45	12:01	6	6
11	log sum	2:25	11:23	6	9
12	Harry Otter	1:31	9:08	5	5
13	$45°$ cyclic	2:18	17:39	3	3
14	13-gon labels	—	—	0	1
15	$\sqrt{2^{15}} \mod 127^4$	5:57	7:08	4	5

This data is mostly for comedic value than anything else (e.g., "holy crap someone got this in x seconds WTF??"). You shouldn't take it too seriously because, e.g., the version the testsolvers worked on may not even look remotely like the final version, etc. Problems went through a lot of changes and so on.

§4.2 "Raw" versions of problems (pre-editing)

1. Let n be the smallest four-digit number such that $3n$ and $4n$ are permutations of each other. Find the remainder when n is divided by 1000.

2. For any positive integers $n \geq i$, say n is i-nteresting if $\operatorname{lcm}(1, 2, 3, \dots n)$ can be written as the product of exactly i distinct **relatively prime** positive integers less than or equal to n.

   Find the sum of all 10-nteresting positive integers.

3. Triangle $\triangle ABC$ inscribed in the unit circle has area $\frac{1}{20}$. The product of the lengths of its altitudes can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute the value of $m+n$.

4. Let G be a 6×6 grid of unit square cells. Find the number of subrectangles R of G such that R contains at least one cell on the diagonal of G containing the top-left and bottom-right cells of G.

5. Let triangle ABC have side lengths $AB = 13$, $BC = 14$, and $CA = 15$. Let $\mathcal{P}$ be a parabola with focus A and directrix BC, and let $\mathcal{P}$ intersect segments AB and AC at D and E respectively. If $AD + AE$ can be represented as $\frac{m}{n}$ in lowest terms, find m.

6. Compute
$$\nu_3\left(\sum_{a+b+c=81} \frac{1}{a!b!c!}\right),$$
   where the summation is over nonnegative integers a, b, c summing to 81. Note that $0! = 1$.

7. A pair of nonnegative integers (i, u) is said to be compatible if $i < 3u$ and $u < 3i$. Let N be the number of compatible pairs of integers (i, u) for $0 \leq i, u < 300$ (note the strictly less than 300). Find the remainder when N is divided by 1000.

8. Let N be equal to,
$$\cos^3(20^\circ)\cos^3(140^\circ) + \cos^3(140^\circ)\cos^3(260^\circ) + \cos^3(260^\circ)\cos^3(20^\circ).$$
   Given that $|N|$ can be represented in form $\frac{p}{q}$ where p and q are relatively prime positive integers, find the value of $p+q$

9. Otis the otter is marking every lattice point with an integer. Let $P(x, y)$ denote the number that he assigned for the point (x, y). It is known that for all integers x and y,
$$P(x+y, y) = P(y, x) = P(x, y),$$
$$P(x, x) = |x|.$$
   Find the remainder when the sum of all positive integers $n < 2026$ such that $P(n, 2026) = 2$ is divided by 1000.

10. Quadrilateral $ABCD$ satsifies $\angle B = \angle C = \angle D$. If $BC = 29$, $AD = 41$, and the distance from A to $\overline{CD}$ is 40, find the largest integer less than $[ABCD]$.

11. Calculate the remainder of the sum
$$\left\lfloor \frac{1}{2001} \sum_{i \geqslant 1} \sum_{j \geqslant 1} \sum_{k=1}^{2001} \frac{\binom{i-1}{j-1} \ln(k)^j}{j(i-1)!} \right\rfloor$$
    when divided by 1000.

12. Oscar the otter is reading the seven volumes of his favorite book series, Harry Otter. He has them all in a row in his bookshelf, but he wants to put them in chronological order. He wishes to do this by making *swaps*, where in one swap he will switch the positions of any two books in the row (these two books do *not* necessarily have to be adjacent). If the books in the row are in a random permutation of the chronological ordering, then the expected number of swaps Oscar needs for the books in the row to be in chronological order is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

13. In cyclic quadrilateral $ABCD$, $\angle BCD = 45^\circ$, $CD = BC + 6$, and $AB + AD = 40$. If the area of $ABCD$ can be expressed as $m + n\sqrt{p}$, where m and n are positive integers and p is not divisible by the square of any prime, compute $m+n+p$.

14. Let n be the number of ways to label the vertices of a regular 13-gon, each with an integer from 1-14 (repetitions allowed), so that no substring of consecutive vertices has a sum divisible by 15. Find the remainder when n is divided by 1000.

15. Let $p = 127$. Let a be the smallest positive integer such that $a^2 - 2^{15}$ is divisible by p^4. Compute the remainder when a is divided by 1000.

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Testsolvers

To be added later.