OTIS Mock AIME 2025 Report

Solutions, results, and commentary

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Contents

1 Summary

The OTIS Mock AIME aired from December 19, 2024 to January 20, 2025. A total of 83 students submitted the AIME I officially and are listed in statistics (with another 44 unofficial submissions). A total of 116 students submitted the AIME II officially and are listed in statistics (with another 42 unofficial submissions).

§1.1 Top scores

Congratulations to the top scores:

§1.1.1 AIME I

14 points Yonghao Jin

13 points Michael Zhao, person1324.990, tenth

12 points Advaith Tumiki, Eric Dai, lethan3, sami

§1.1.2 AIME II

13 points channing421

12 points Dylan Frake, Jacob Khohayting, Lifewrath, Lilypad, Nathan Liu, person1324.990, Shihan Kanungo, Vihaan Gupta

11 points Andrew, Aryan Raj, bryant, Chenghao Hu, GRAPESUPREMACY, Hansen Shieh, Hyun-Jin Kim, Jefferson Zhou, Joey Zheng, Khio, Kyle Wu, Lucas Pavlov, Mingyue Yang, Oz, Scrge, sus, suspiciouscow, Sylvia Lee, tetra_scheme, TimmysG, YourJFForever

§1.2 Editorial notes

§1.2.1 Running two tests

The big decision this year was that we tried to run thirty problems instead of fifteen by having two parallel tests with different design goals. Last year we had the following passage in the report:

> *Ultimately, in choosing the problems, I had to balance picking the most interesting problems against skewing the difficulty higher than a "real" AIME. In the end, I ended up with a draft which I felt was perhaps 2.5 problems or so harder than an average AIME, which is pretty close to the limit that I'd be willing to push the difficulty bar.*
>
> *There was also an observation from several of the editors that the test was perhaps too* clean*: the problems that were popular tended to rely on having one or two key ideas, after which there was often not too much calculation required. The actual AIME requires a bit more grinding. Again, this was a design concession I made with the goal of showing my favorite problems rather than trying to emulate the real AIME as much as possible.*

So this year, we eventually ended up doing thirty problems.

This was a ton of editing work for me, and next year if we try this again I'm going to move the entire schedule up by on the order of two months, because the last week before the test took off was really miserable.

I am grateful we had enough problems to pull this off, as I was on the fence beforehand. It helped that we had problems left over from the previous year that didn't get used that we were able to bring in to fill in gaps that we needed.

§1.2.2 The Probase experiment

We continued to use Probase, but this year we had people opt in to be "serious testsolvers" (meaning they had to work with a timer and couldn't see the statement of a problem before they tried it) versus casual testsolvers who could just peruse the database as they saw fit. A snapshot of this data is again in Section 4.1.

Thanks to Howard for doing all the software stack stuff.

§1.2.3 Favorites

Evan's personal favorites on the test are I.12, I.13, I.14.

§1.2.4 Links to problems on Art of Problem Solving

Contest collections links:

- OTIS Mock AIME 2025 I: `https://aops.com/community/c4180955`
- OTIS Mock AIME 2025 II: `https://aops.com/community/c4180956`
- All years of OTIS Mock AIME: `https://aops.com/community/c4180954`

2 Solutions

§2.1 Answer key for AIME I

P#	Description	Author	Answer
1	Angry penguins	Jack Whitney-Epstein	119
2	Similar triangles	Haofang Zhu	567
3	$k - n/2$	Luis Fonseca	405
4	Dodecagon	Alansha Jiang	081
5	$\lvert x + 2y + 3z \rvert$	Benjamin Fu	112
6	Green and pink pencils	Andy Liu	013
7	$\triangle ABC$	Amogh Akella	135
8	$ab + bc + cd + de$	Ryan Tang	870
9	New-prime	Jiahe Liu	312
10	8×8 grid	Tanishq Pauskar	224
11	XA, XB, XC	Amogh Akella	071
12	System of equations	Benjamin Fu	108
13	BC^2	Tanishq Pauskar	369
14	$1/315$	Seongjin Shim	903
15	2000 cards	Abel George Mathew	401

§2.2 Answer key for AIME II

P#	Description	Author	Answer
1	$AM = BM$	James Stewart	112
2	$62n - 336$	Andy Liu	352
3	$8^{x+2} - 2^{x+6}$	James Stewart	064
4	$x^{\lfloor x \rfloor} + \lfloor x \rfloor^x$	Shining Sun	590
5	Rosa the Otter	Sohil Rathi	026
6	$f\colon \{\dots\} \to \{-1, 0, 1\}$	Benjamin Fu	730
7	Sheet of paper	Vikram Sarkar	954
8	$MD \perp AP$	Joshua Liu	315
9	$P(mi) = P(ni)$	Jack Whitney-Epstein	365
10	pqr/m	Karn Chutinan	071
11	Informatics contest	Jiahe Liu	570
12	Sphere	Kishore	062
13	$(6 + 239k)^{717}$	Catherine Xu & Ritwin Narra	711
14	$I_B I_C E_B E_C$	Jason Lee & Arjun Suresh	191
15	4×4 card grid	Neil Kolekar	064

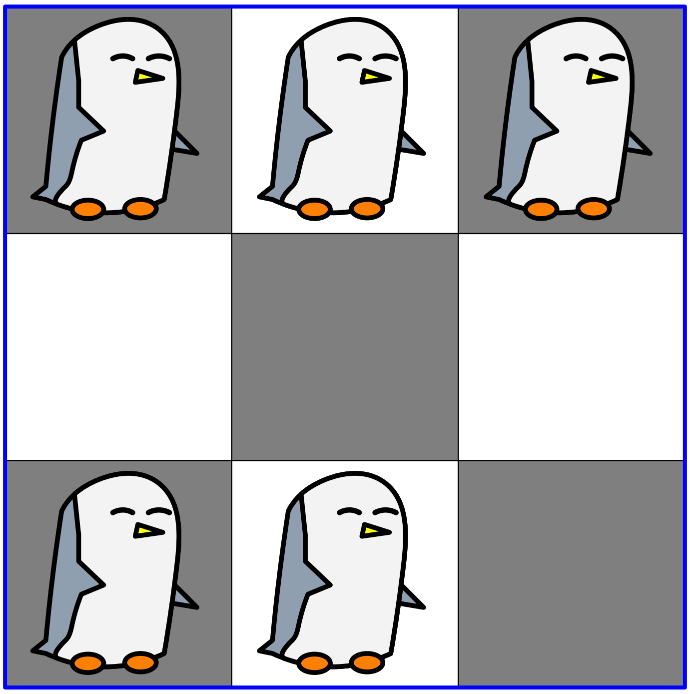
§2.3 Full solutions to AIME I

Problem 1. In a 3×3 grid, each cell is empty or contains a penguin. Two penguins are *angry* at each other if they occupy diagonally adjacent cells. Compute the number of ways to fill the grid so that none of the penguins are angry.

¶ **Answer.** 119

¶ **Problem author(s).** Jack Whitney-Epstein

The key idea is to color the grid in a checkerboard pattern, and notice that two penguins can only be angry at each other if they occupy squares of the same color. Therefore, we can consider the black squares and white squares independently! Let the corner squares be black.



We consider these placements in turn.

Black squares If there is a penguin in the center square, the 4 corners must all be empty, and if the center is empty, we can freely choose each of the 4 corners, yielding $2^4 = 16$ more ways. There are 17 ways to fill the black squares total.

White squares There can be either 0, 1, or 2 penguins in total; these cases give 1, 4, and 2 ways respectively for a total of $1 + 4 + 2 = 7$.

So, the answer is $7 \cdot 17 = \boxed{119}$.

Problem 2. Convex quadrilateral $ABCD$ has $AD = 72$, $\angle ABC = \angle ACD = 90^\circ$ and $\angle BAC = \angle CAD = 30^\circ$. Let M be the midpoint of AD and let N be the midpoint of BM. Compute CN^2.

¶ Answer. 567

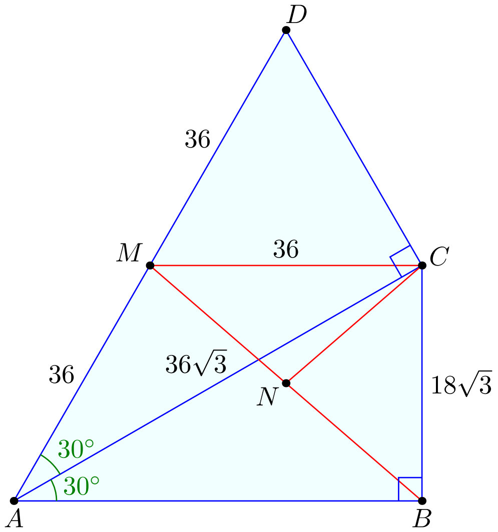
¶ Problem author(s). Haofang Zhu

The angle conditions imply that triangles $\triangle ABC$ and $\triangle ACD$ are 30°-60°-90°, as shown in the figure below. As such, it follows then that $AC = \frac{\sqrt{3}}{2} \cdot AD = 36\sqrt{3}$ so $CB = \frac{1}{2} \cdot AC = 18\sqrt{3}$.

Note that since M is the midpoint of the hypotenuse of $\triangle ADC$, it follows that $MC = MD = MA = \frac{AD}{2} = 36$. Similarly, $CN = NM = NB$, as

$$\angle MCA = 30^\circ \text{ and } \angle ACB = 60^\circ \implies \angle MCB = 90^\circ$$

implies N is the midpoint of the hypotenuse of right triangle $\triangle BCM$.



Since N is the midpoint of BM, we get

$$CN^2 = BN^2 = \frac{1}{4}BM^2 = \frac{1}{4}(CM^2 + CB^2) = \frac{1}{4}\left(36^2 + \left(18\sqrt{3}\right)^2\right) = \boxed{567}.$$

.

Problem 3. Compute the smallest integer $k > 1$ such that there are exactly 10 even integers $n \geq 2$ for which $k - n/2$ is divisible by n.

¶ Answer. 405

¶ Problem author(s). Luis Fonseca

Replace n with $2m$ in what follows. Then $k - m$ is divisible by $2m$ when it equals $a \cdot 2m$ for some integer a; i.e. we require $k = (2a+1) \cdot m$ for some integer a. In other words, k needs to be an odd multiple of m.

Thus, if $k = 2^e \cdot b$ for some integer $e \geq 0$ and odd integer b, then the number of m for which k is an odd multiple is given by the number of divisors of b.

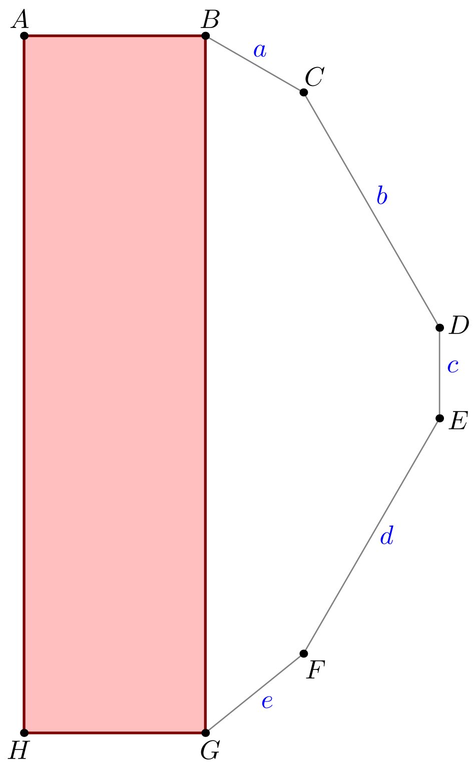
We thus can assume $e = 0$ and try to minimize b; we need to choose the minimal odd integer b with 10 positive divisors. Recall that the number of factors of a number is the product of one more than the exponents. Since $10 = 2 \cdot 5$, this means that either $k = p^9$ or $k = p^4 \cdot q$ for some primes p and q. In the first case $k \geq 3^9$ and in the second case $k \geq 3^4 \cdot 5$; the smaller of these is $3^4 \cdot 5 = \boxed{405}$.

Problem 4. In the Cartesian plane, let $A = (0, 10+12\sqrt{3})$, $B = (8, 10+12\sqrt{3})$, $G = (8, 0)$ and $H = (0, 0)$. Compute the number of ways to draw an equiangular dodecagon $\mathcal{P}$ in the Cartesian plane such that all side lengths of $\mathcal{P}$ are positive integers and line segments AB and GH are both sides of $\mathcal{P}$.

¶ **Answer.** 081

¶ **Problem author(s).** Alansha Jiang

Name the dodecagon $ABCDEFGHIJKL$ in the obvious way. Notice that the problem is symmetric, so we only need to find the number of ways for half of the dodecagon to be constructed since the two halves are independent. That is, we'll compute the number of ways to place the points C, D, E, F and then square it.



Define $a = BC$, $b = CD$, $c = DE$, $d = EF$, $e = FG$. Using the fact that the exterior angles of an equiangular dodecagon are 30°, we have

$$\frac{\sqrt{3}}{2}a + \frac{1}{2}b - \frac{1}{2}d - \frac{\sqrt{3}}{2}e = 0$$
$$\frac{1}{2}a + \frac{\sqrt{3}}{2}b + c + \frac{\sqrt{3}}{2}d + \frac{1}{2}e = 10 + 12\sqrt{3}.$$

Since the side lengths must be integers and $\sqrt{3}$ is irrational, the only way the first equation can be satisfied is if $a = e$ and $b = d$. Substituting into the second equation, we now have

$$a + b\sqrt{3} + c = 10 + 12\sqrt{3}.$$

Therefore $b = d = 12$, and $a + c = 10$, giving us 9 possible positive integer solutions for (a, c).

Thus, our final answer is $9^2 = \boxed{81}$.

Problem 5. Let x, y, and z be complex numbers satisfying

$$|x+z| = |y+z| = |x-y| = 4.$$

Compute $|x+2y+3z|^2$.

¶ **Answer.** 112

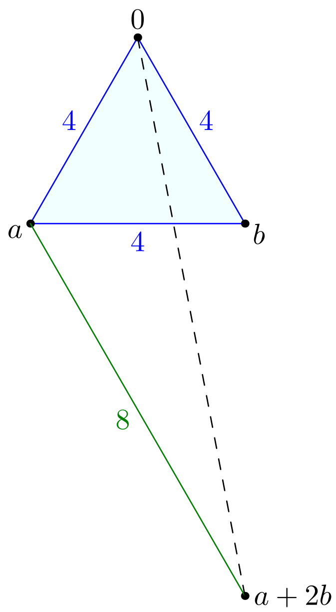
¶ **Problem author(s).** Benjamin Fu

Make the substitution $a = x+z$ and $b = y+z$, such that the given condition becomes

$$|a| = |b| = |a-b| = 4.$$

This means 0, a, and b form an equilateral triangle of side length 4 in the complex plane. We wish to find

$$|x+2y+3z|^2 = |a+2b|^2.$$



Considering the complex numbers 0, a, and $a+2b$ as vertices of a triangle. Then the vector joining a to $a+2b$ is $2b$, which has length 8, as in the figure above. So by the Law of Cosines we have

$$|a+2b|^2 = 4^2 + 8^2 - 2(4)(8)\cos 120^\circ = \boxed{112}.$$

> **Remark.** The problem has a lot of degrees of freedom in it; for example, one can arbitrarily set $z = 0$ and still find (x, y) satisfying the condition.

Problem 6. There are 2025 green pencils on a table. Every minute, Elphaba removes two randomly chosen pencils on the table. Right after that, Glinda adds back one pink pencil. After 2023 minutes, the probability that at least one of the two pencils remaining on the table is green is $\frac{m}{n}$ where m and n are relatively prime positive integers. Compute the remainder when $m+n$ is divided by 1000.

¶ Answer. 013

¶ Problem author(s). Andy Liu

First, let's consider a particular green pencil G_1. The probability that G_1 survives all the way to the end is given by

$$\begin{aligned}\frac{2023}{2025}\cdot\frac{2022}{2024}\cdot\frac{2021}{2023}\cdot\dots\cdot\frac{2}{4}\cdot\frac{1}{3} &= \frac{1}{2025}\cdot\frac{1}{2024}\cdot 2 \\ &= \frac{1}{1012\cdot 2025}.\end{aligned}$$

However, note that at the end of the process, the number of green pencils remaining must be either 0 or 1. In other words, if G_2, ..., G_{2025} denote the remaining pencils, the event of G_i surviving to the end is disjoint from any other such event. Hence, the probability is just

$$\underbrace{\frac{1}{1012\cdot 2025}+\dots+\frac{1}{1012\cdot 2025}}_{2025\text{times}} = \frac{1}{1012}.$$

The remainder when $1+1012$ is divided by 1000 is $\boxed{013}$.

Problem 7. Let ABC be a triangle with $AB = 5$, $BC = 13$, and $CA = 12$. Points D, E, and F are on segments BC, CA, and AB such that DEF is an isosceles right triangle with hypotenuse EF. Suppose that $BF = 3$. Then the length of CE can be written as $\frac{m}{n}$ for relatively prime positive integers m and n. Compute $m + n$.

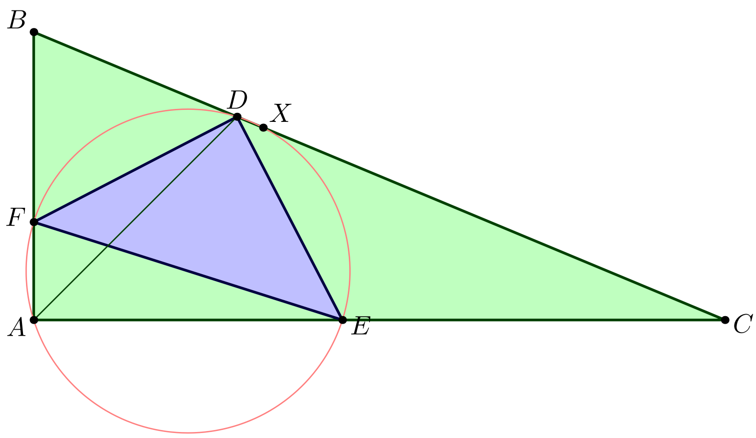
¶ **Answer.** 135

¶ **Problem author(s).** Amogh Akella

As $5^2 + 12^2 = 13^2$, we know $\angle BAC = 90^\circ$. And quadrilateral $AEDF$ is cyclic because $\angle EAF = \angle EDF = 90^\circ$. Hence, it follows that

$$\angle DAE = \angle DFE = 45^\circ$$

which implies that D lies on the angle bisector of $\angle BAC$.



From here we show two solutions.

¶ **First solution using power of a point (Jason Lee).** Suppose the circumcircle of $ADEF$ meets BC again at X. Then by power of a point and angle bisector theorem, we have

$$BF + CE = \frac{BD}{BA} \cdot BX + \frac{CD}{CA} \cdot CX = \frac{13}{5+12}(BX + CX) = \frac{13^2}{5+12}.$$

Since $BF = 3$ it follows $CE = \frac{13^2}{5+12} - 3 = \frac{118}{17}$. This gives answer $118 + 17 = \boxed{135}$.

¶ **Second solution using the length of the angle bisector.** We start by calculating the length of AD. By the angle bisector theorem, we know $\frac{BD}{DC} = \frac{BA}{AC} = \frac{5}{12}$. Then the height from D to line AC has length

$$\text{height}(D, AC) = \frac{12}{17} \cdot AB = \frac{60}{17}$$

and hence

$$AD = \frac{60\sqrt{2}}{17}.$$

(Alternatively, AD could be calculated using Stewart's theorem with $BD = \frac{BC \cdot AB}{AB+AC} = \frac{65}{17}$.)

Now we use Ptolemy's theorem on cyclic quadrilateral $AEDF$, which gives

$$AE \cdot DF + AF \cdot DE = AD \cdot EF.$$

Since $DE = DF = \frac{EF}{\sqrt{2}}$, dividing out by $DE = EF$ implies

$$AE + AF = AD \cdot \sqrt{2} = \frac{120}{17}.$$

It is given in the problem that $BF = 3$, so $AF = 2$ and hence $AE = \frac{86}{17}$. Therefore, we obtain $CE = 12 - \frac{86}{17} = \frac{118}{17}$ again.

Problem 8. Compute the maximum possible value of $ab+bc+cd+de$ over all choices of positive integers a, b, c, d, e satisfying $a+b+c+d+e=60$.

¶ Answer. 870

¶ Problem author(s). Ryan Tang

We show three solutions.

¶ First solution by inequalities. Let S denote the desired sum. The idea is to write

$$S := ab+bc+cd+de = (a+c+e)(b+d) - (ad+be).$$

Note that

- We have
$$(a+c+e)(b+d) = (a+c+e)(60-(a+c+e)) = 900 - ((a+c+e)-30)^2.$$
- We have $ad+be \ge (a+d-1)+(b+e-1) = 58-c$.

Hence, we get an estimate of

$$S \le 842 + \left(c - ((a+c+e)-30)^2\right).$$

Now we consider cases on c.

- If $c \le 28$, we get $S \le 842 + c \le 870$. Equality can indeed occur, for example if $a=e=1$, $c=28$, and $b+d=30$.
- Now assume $c > 28$. Then $a+c+e > 30$, and from $a, e \ge 1$ we can make the estimate
$$(a+c+e-30)^2 \ge (c-28)^2.$$
Thus, we have
$$S \le 842 + \left(c-(c-28)^2\right).$$
For every value of $c > 28$, it is easy to check
$$c-(c-28)^2 \le 28$$
so we have $c \le 870$ in this case too. And equality can indeed occur, for example if $a=e=1$, $c=29$, and $b+d=29$.

Hence we get a maximum of $\boxed{870}$ in both cases.

¶ Second short inequality solution. Another way to use the expression for S is by writing

$$\begin{aligned} ab+bc+cd+de &= (a+c+e)(b+d) - (ad+be) \\ &\le (a+c+e)(b+d) - (d+b) = (a+c+e-1)(b+d). \end{aligned}$$

Since $(a+c+e-1)+(b+d) = 59$, we get $29 \cdot 30 = 870$ as the bound again.

¶ Third solution by smoothing. Assume without loss of generality, that $a \geq e$, due to symmetry. We use smoothing, i.e. we modify (a, b, c, d, e) to make the sum maximal.

Claim — We can assume that $d = 1$.

Proof. If not, apply the operation

$$(b, d) \mapsto (b + 1, d - 1)$$

to increase the sum (since $a + c \geq c + e$) until this holds. □

As such, we want to maximize $ab+bc+c+e$ given $a+b+c+d = 59$ and $a \geq e$.

Claim — We can also assume that $a = e = 1$.

Proof. Note that $ab+b(c+1)+(c+1)+(e-1) \geq ab+bc+c+e$. Hence increasing c and decreasing e increases S. Similarly, increasing c and decreasing a increases S as well. □

Hence, the problem reduces to maximizing

$$b + bc + c + 1 = (b + 1)(c + 1)$$

subject to $b+c = 57$. By AM-GM, this is maximized when $b+1$ and $c+1$ are the closest to each other, or when $\{b, c\} = \{28, 29\}$, giving $(b + 1)(c + 1) = 29 \cdot 30 = \boxed{870}$.

Problem 9. Winston forgot the definition of a prime number. He instead defines a *New-prime* recursively as follows:

- 1 is not New-prime.
- A positive integer $n > 1$ is New-prime if and only if n cannot be expressed as the product of exactly two (not necessarily distinct) New-prime positive integers.

Compute the number of positive integers dividing 5005^4 which are New-primes.

¶ **Answer.** 312

¶ **Problem author(s).** Jiahe Liu

The main claim is:

> **Claim** — A positive integer is New-prime if and only if the sum of the exponents of its prime factors is odd.

Proof. By induction on $n \geq 1$, with the base case given. Consider any n. If the sum of the exponents of n is odd, then clearly it is not the product of two numbers whose sum of exponents is odd. Otherwise, if the sum of exponents of n is even, consider any prime $p \mid n$. Then, p and $\frac{n}{p}$ are both New-prime, since $\frac{n}{p}$ has an odd sum of exponents and is less than n. The induction applies, so this condition for New-primes works. □

Hence we need the number of divisors of $5005^4 = 5^4 \cdot 7^4 \cdot 11^4 \cdot 13^4$ with an odd exponent sum. This can be computed by using a roots of unity filter on

$$P(X) := (1 + X + X^2 + X^3 + X^4)^4$$

and looking for the sum of the coefficients of the terms with odd degree; that is

$$\frac{P(1) - P(-1)}{2} = \frac{625 - 1}{2} = \boxed{312}.$$

Problem 10. An 8×8 grid of unit squares is drawn; it thus has 144 unit edges. Let N be the number of ways to color each of the 144 unit edges one of six colors (red, orange, yellow, green, blue, or purple) such that every unit square is surrounded by exactly 3 different colors. Then N can be written as a prime factorization $p_1^{e_1} \dots p_k^{e_k}$ where $p_1 < \dots < p_k$ are primes and e_i are positive integers. Compute $e_1 + \dots + e_k$.

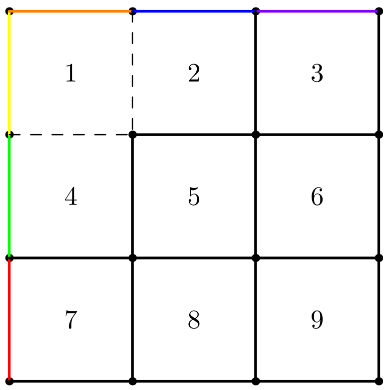
¶ Answer. 224

¶ Problem author(s). Tanishq Pauskar

We consider the case of a general $n \times n$ grid of unit squares which has $2n \cdot (n+1) = 2n^2 + 2n$ edges and show that the answer is

$$N = 6^{2n} \cdot 20^{n^2}.$$

We describe a process for coloring in the board. First, color the $2n$ boundary edges in an arbitrary way. Now consider the following diagram and suppose we want to color the two dashed lines.



Claim — Given the color of two known edges of a square, there are exactly 20 ways to color the other two uncolored edges.

Proof. There are 2 cases:

Case 1 If both of the known edges are the same color, then there are 5 ways to color the first edge and 4 for the second edge, for a total of $4 \cdot 5 = 20$.

Case 2 If both of the two known edges are different colors, then there are the following cases:

(i) If one of the colors of the uncolored edges is the same color as a known edge, then there are $2 \cdot 2 = 4$ ways to choose which known edge and uncolored edge. The remaining edge must be a different color, for $4 \cdot 4 = 16$ ways.

(ii) If neither of the uncolored edges has the same color as known edges, then they must be one of the 4 colors that don't appear.

This sums up to $4 + 16 = 20$ ways. □

We can apply this claim for the n^2 squares in order of the labels in the grid which gives us $6^{2n} \cdot 20^{n^2}$ total ways.

Now, plugging in $n = 8$ gives us $2^{144} \cdot 3^{16} \cdot 5^{64}$ ways, for an answer of $144 + 16 + 54 = \boxed{224}$.

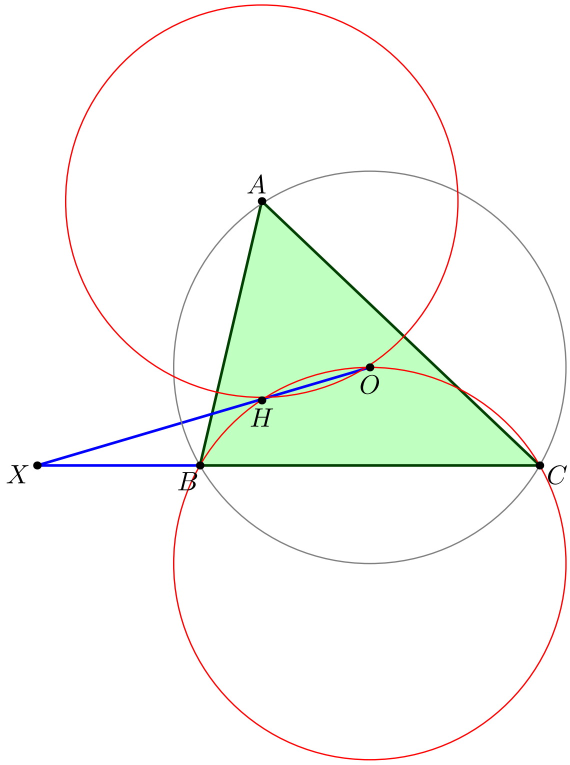
Problem 11. Let ABC be an acute non-equilateral triangle with $\angle BAC = 60^\circ$. The Euler line of triangle ABC intersects side BC at point X such that B lies between X and C. Given that $XA = 49$ and $XB = 23$, compute XC.

(The *Euler line* of a non-equilateral triangle refers to the line through its circumcenter, centroid, and orthocenter.)

¶ Answer. 071

¶ Problem author(s). Amogh Akella

Let H and O denote the orthocenter and circumcenter. We show the existence of the two red circles below from $\angle A = 60^\circ$ in two claims (both of which are standard).



Claim — In any triangle with $\angle A = 60^\circ$, we have $AH = AO = R$, where R is the circumradius. Hence one can draw a circle centered at A through both H and O.

Proof. We present a proof with trigonometry, although there are several others. Observe that $\angle ABH = 90^\circ - \angle BAC = 30^\circ$ and $\angle BAH = 90^\circ - \angle ABC$. Therefore, $\angle AHB = 180^\circ - \angle ABH - \angle BAH = 60^\circ + \angle ABC = 180^\circ - \angle ACB$. By the Law of Sines on $\triangle ABH$, we have

$$\frac{AH}{\sin \angle ABH} = \frac{AB}{\sin \angle AHB} \implies 2AH = \frac{AB}{\sin \angle ACB}.$$

But the extended law of sines says the right-hand side equals $2R$, as needed. □

Remark. In general, for any triangle, it turns out that $AH = 2R\cos A$, where $R = AO$ is the circumradius. In the special case $\angle A = 60^\circ$ we thus get $AH = AO$.

Claim — In any triangle with $\angle A = 60^\circ$, the points B, O, H, C are cyclic.

Proof. Since $\angle BHC = 180^\circ - \angle A = 120^\circ$ and $\angle BOC = 2\angle A = 120^\circ$. $\square$

Now on to the main proof. The point X lies on the radical axis of both circles we just mentioned, so their powers are equal and we get

$$XA^2 - R^2 = XB \cdot XC = XB(XB + BC).$$

Since $XA = 49$, $XB = 23$, $BC = 2R\cos 120^\circ = \sqrt{3}R$, we obtain

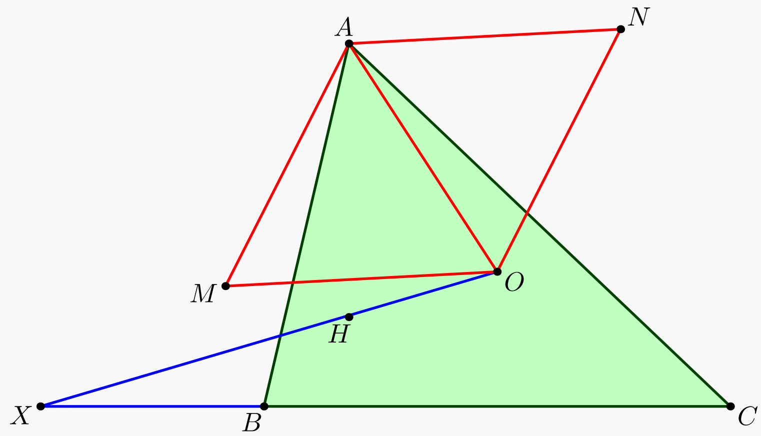
$$49^2 - R^2 = 23(23 + R\sqrt{3}) \implies R = 16\sqrt{3}.$$

Thus $BC = \sqrt{3} \cdot 16\sqrt{3} = 48$ and so $XC = 23 + 48 = \boxed{71}$.

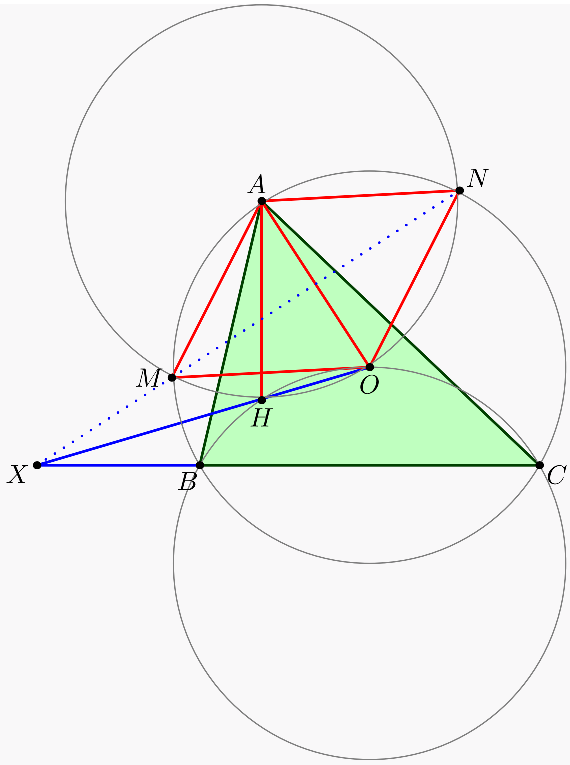
Remark. In fact, it turns out that we have a similarity of isosceles triangles

$$\triangle HAO \sim \triangle AXO.$$

To see why, let M and N be points such that $\triangle AMO$ and $\triangle ANO$ are equilateral.



Because $AO = MO = BO = CO = NO$, we conclude that $MBCN$ is a cyclic quadrilateral with center O. Similarly, because $AM = AN = AO = AH$, we conclude that $MHON$ is a cyclic quadrilateral with center A.



By the Radical Axis Theorem on $(MBCN)$, $(MHON)$, and $(BHOC)$, we see that MN, OH, and BC are concurrent. Therefore, MN passes through point X. Because MN is the perpendicular bisector of AO, we conclude that $AX = XO$.

Finally, we have shown that $\triangle AOH$ and $\triangle AXO$ are both isosceles, and they both share the base angle $\angle AOH$. Hence, it follows that they are similar.

Problem 12. There exists a unique tuple of positive real numbers (a, b, c, d) satisfying

$$\begin{aligned}(49 + ab)(a + b) &= 81a + 25b \\ (81 + bc)(b + c) &= 121b + 49c \\ (121 + cd)(c + d) &= 169c + 81d \\ a + b + c + d &= 12.\end{aligned}$$

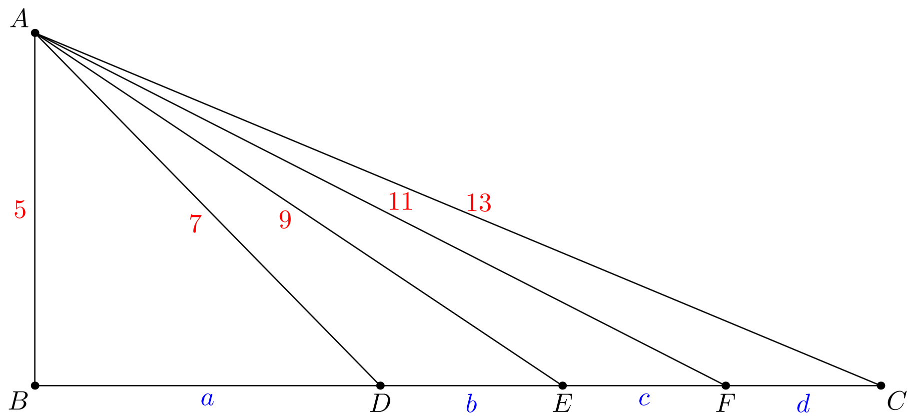
Given that $d = m - \sqrt{n}$ for positive integers m and n, compute $m + n$.

¶ Answer. 108

¶ Problem author(s). Benjamin Fu

We show two solutions.

¶ Geometry solution (from author). The first three given equations can be interpreted as Stewart's Theorem, as shown in the diagram below with triangle $\triangle ABC$ and cevians AD, AE, and AF. Note that we have lengths $BD = a$, $DE = b$, $EF = c$, and $FC = d$, and the distances from A to B, D, E, F, and C are 5, 7, 9, 11, and 13, respectively.



Furthermore, since $a + b + c + d = 12$, we know that $\triangle ABC$ is a right triangle as $5^2 + 12^2 = 13^2$. Using the Pythagorean Theorem, we can compute

$$d = 12 - \sqrt{11^2 - 5^2} = 12 - \sqrt{96},$$

so the answer is $12 + 96 = \boxed{108}$.

¶ Algebra solution. We write the equations as

$$\begin{aligned}a + b &= \tfrac{32}{b} - \tfrac{24}{a} \\ b + c &= \tfrac{40}{c} - \tfrac{32}{b} \\ c + d &= \tfrac{48}{d} - \tfrac{40}{c}.\end{aligned}$$

Upon noticing a and $\frac{24}{a}$, the trick is to make the ansatz

$$a = \sqrt{t + 24} - \sqrt{t}$$

for some real number t, so that

$$a + \frac{24}{a} = (\sqrt{t+24} - \sqrt{t}) + (\sqrt{t+24} + \sqrt{t}) = 2\sqrt{t+24}.$$

Our strategy is to rewrite b, c, d in terms of t, and then finally use $a+b+c+d = 12$ to find t.

Then

$$\begin{aligned}
\frac{32}{b} - b = a + \frac{24}{a} = 2\sqrt{t+24} &\implies b = \sqrt{t+56} - \sqrt{t+24}. \\
\frac{40}{c} - c = b + \frac{32}{b} = 2\sqrt{t+56} &\implies c = \sqrt{t+96} - \sqrt{t+56}. \\
\frac{48}{d} - d = c + \frac{40}{c} = 2\sqrt{t+96} &\implies d = \sqrt{t+144} - \sqrt{t+96}.
\end{aligned}$$

Then putting everything together gives

$$12 = a + b + c + d = \sqrt{t+144} - \sqrt{t}.$$

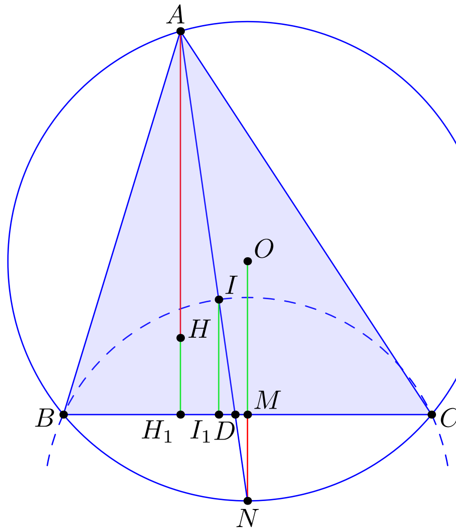
Hence $t = 0$, so $d = 12 - \sqrt{96}$ as claimed.

Problem 13. Let ABC be an acute triangle. Suppose the distances from its circumcenter, incenter, and orthocenter to side BC are 8, 6, and 4, respectively. Compute BC^2.

¶ **Answer.** 369

¶ **Problem author(s).** Tanishq Pauskar

Let O, I, H denote the usual triangle centers, and let M, I_1, H_1 be their projections onto line BC (so $BM = MC$ while $r := II_1$ is the inradius). Let the $\angle A$ bisector meet BC and (ABC) at D and N, respectively.



We recall the two well-known equations (whose proofs we postpone to a later remark)

$$AH = 2OM, \qquad NI^2 = ND \cdot NA.$$

This lets us get the length of the A-altitude:

$$h := AH_1 = AH + HH_1 = 2OM + HH_1 = 2 \cdot 8 + 4 = 20.$$

Claim — We have $MN = \frac{9}{2}$.

Proof. Note that $\triangle NMD \sim \triangle II_1D \sim \triangle AH_1D$, which gives ratios

$$\frac{ND}{MN} = \frac{NI}{MN + r} = \frac{NA}{MN + h}.$$

Since $NI^2 = ND \cdot NA$, it thus follows that

$$(MN + r)^2 = MN(MN + h).$$

We hence have $(MN + 12)^2 = MN(MN + 40)$ which solves to $MN = \frac{9}{2}$. □

Since the circumradius of ABC is $ON = OM + MN = 8 + \frac{9}{2} = \frac{25}{2}$, we finish with

$$BC^2 = 4\left(BO^2 - MO^2\right) = 4\left(\left(\frac{25}{2}\right)^2 - 8^2\right) = \boxed{369}.$$

Remark (Jason Lee)**.** Here is another way to get the length of MN. Let $a = BC$, $b = CA$, $c = AB$ and $s = \frac{1}{2}(a+b+c)$ as usual. Since the area of $[ABC] = \frac{1}{2}ah = rs$, we conclude that

$$\frac{a}{s} = \frac{2r}{h} = \frac{2 \cdot 6}{20} = \frac{3}{5} \iff \frac{a}{b+c} = \frac{3}{7}.$$

We can use this to compute all the ratios along line AN by noting that

$$\begin{aligned} \frac{AI}{ID} &= \frac{AB}{BD} = \frac{AC}{CD} = \frac{AB+AC}{BD+CD} = \frac{b+c}{a} = \frac{7}{3} \\ \frac{AI}{AN} &= \frac{s-a}{\frac{(s-a)+s}{2}} = \frac{b+c-a}{b+c} = \frac{4}{7} \end{aligned}$$

where for $\frac{AI}{ID}$ we used the angle bisector theorem on $\triangle ABD$ and $\triangle ACD$, and for $\frac{AI}{AN}$ we considered the homothety mapping the incircle to the A-excircle. Hence, we have

$$AI : ID : DN = 28 : 12 : 9.$$

Considering the projection of these ratios along the A-altitude gives

$$\frac{h}{MN} = \frac{AD}{DN} = \frac{28+12}{9}$$

so $MN = \frac{9}{40}h = \frac{9}{2}$.

Remark (Proofs of earlier lemmas)**.** We mention an outline of the proofs for the lemmas $AH = 2OM$ and $NI^2 = ND \cdot NA$.

For the first, it is well known that the reflection of H over M is the antipode of A, or A', and that the reflection of A over O is A'. Thus, there exists a homothety centered at A' that maps segment OM to AH. Alternatively, one can simply use complex numbers where $O = 0$, $H = a+b+c$, $A = a$ and $M = \frac{b+c}{2}$, so $H - A = 2(M - O)$ is clear.

The second follows from the incenter-excenter lemma together with the shooting lemma in the form $\triangle BDN \sim \triangle NBA$. See EGMO, section 4.7

Problem 14. There is a unique triplet of integers (a, b, c) such that $0 < a < b < c < 1000$ and

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{1}{315}.$$

Compute a.

¶ Answer. 903

¶ Problem author(s). Seongjin Shim

The key observation is that

$$\frac{1}{2} \cdot \frac{1}{315} = \frac{3}{42 \cdot 45} = \frac{1}{42} - \frac{1}{45}.$$

Then one can write the right-hand side as the sum of three Egyptian fractions as follows:

$$\begin{aligned} \frac{1}{2} \cdot \frac{1}{315} = \frac{1}{42} - \frac{1}{45} &= \left(\frac{1}{42} - \frac{1}{43}\right) + \left(\frac{1}{43} - \frac{1}{44}\right) + \left(\frac{1}{44} - \frac{1}{45}\right) \\ &= \frac{1}{42 \cdot 43} + \frac{1}{43 \cdot 44} + \frac{1}{44 \cdot 45}. \end{aligned}$$

Doubling everything gives

$$\frac{1}{315} = \frac{2}{42 \cdot 43} + \frac{2}{43 \cdot 44} + \frac{2}{44 \cdot 45}.$$

Hence one can take $a = \frac{42 \cdot 43}{2}$, $b = \frac{43 \cdot 44}{2}$, $c = \frac{44 \cdot 45}{2}$. So $a = \boxed{903}$.

Problem 15. Alice has a deck of 2000 cards, numbered 1 through 2000. Alice chooses an integer $1 \le n < 1000$ and deals Cheshire a random subset of $2n-1$ of the cards without repetition. Cheshire wins if the cards dealt contain any n consecutively numbered cards. Compute the value of n Alice should choose to minimize Cheshire's chances of winning.

¶ **Answer.** 401

¶ **Problem author(s).** Abel George Mathew

Fix a positive integer $n < 1000$. A subset of $2n-1$ cards is called *good* if it contains a collection of n consecutively numbered cards. To enumerate the good subsets, we categorize them into buckets $B_1, B_2, \dots, B_{2000}$.

Let B_k be the set of all subsets S of $2n-1$ cards such that:

$$k-1 \notin S \quad \text{and} \quad k, k+1, \dots, k+n-1 \in S.$$

Note that every element of B_k is a good subset, and every good subset is an element of some bucket B_k. Furthermore, every subset is part of at most 1 bucket because a subset of $2n-1$ cards cannot contain 2 disjoint collections of n consecutively numbered cards.

Therefore the probability of Cheshire winning is given by:

$$P_n = \frac{|B_1| + |B_2| + \dots + |B_{2000}|}{\binom{2000}{2n-1}}$$

By the definitions of B_k we also have that:

$$B_k = \begin{cases} \binom{2000-n}{n-1} & \text{if } k = 1 \\ \binom{2000-n-1}{n-1} & \text{if } 2 \le k \le 2000-n \\ 0 & \text{if } k > 2000-n \end{cases}$$

Plugging this in gives us a closed formula for P_n:

$$\begin{aligned} P_n &= \frac{\binom{2000-n}{n-1} + (2000-n) \cdot \binom{2000-n-1}{n-1}}{\binom{2000}{2n-1}} \\ &= \left(\frac{(2000-n)!}{(n-1)! \cdot (2000-2n+1)!} + \frac{(2000-n) \cdot (2000-n-1)!}{(n-1)! \cdot (2000-2n)!} \right) \Big/ \binom{2000}{2n-1} \\ &= \frac{(2000-n)! \cdot (2000-2n+2)}{(n-1)! \cdot (2000-2n+1)!} \Big/ \frac{2000!}{(2n-1)! \cdot (2000-2n+1)!} \\ &= \frac{(2000-n)! \cdot (2000-2n+2) \cdot (2n-1)!}{(n-1)! \cdot 2000!}. \end{aligned}$$

so we want to minimize P_n.

For $n = 1, \dots, 999$, we may also define the following ratio:

$$\begin{aligned} r_n &= \frac{P_{n+1}}{P_n} \\ &= \frac{(2000-n-1)!}{(2000-n)!} \cdot \frac{2000-2n}{2000-2n+2} \cdot \frac{(2n+1)!}{(2n-1)!} \cdot \frac{(n-1)!}{n!} \end{aligned}$$

$$= \frac{(2000-2n)\cdot 2n \cdot (2n+1)}{(2000-n)\cdot(2000-2n+2)\cdot n}$$
$$= \frac{2(C-2n)(2n+1)}{(C-n)(C-2n+2)}$$

where $C = 2000$.

As such,

$$\begin{aligned} P_{n+1} > P_n &\iff r_n > 1 \\ &\iff 2(C-2n)(2n+1) > (C-n)(C-2n+2) \\ &\iff (C-n)(C-2n+2) - 2(C-2n)(2n+1) < 0 \\ &\iff 10n^2 - (7C-2)n + C^2 < 0 \\ &\iff x_1 < n < x_2 \end{aligned}$$

where $x_1 < x_2$ are the 2 roots of the quadratic polynomial:

$$Q(n) = 10n^2 - (7C-2)n + C^2.$$

Claim — We have $400 < x_1 < 401$ and $999 < x_2 < 1000$.

Proof. By the quadratic formula, the roots are

$$\frac{7C - 2 \pm \sqrt{49C^2 - 28C + 4 - 40C^2}}{20} = \frac{7C - 2 \pm \sqrt{9C^2 - 28C + 4}}{20}.$$

Since $3C - 5 < \sqrt{9C^2 - 28C + 4} < 3C - 4$, we can compute x_1 and x_2 to the nearest integer:

$$\begin{aligned} 400.10 = \frac{4}{20}C + \frac{2}{20} < x_1 < \frac{4}{20}C + \frac{3}{20} = 400.15 \\ 999.65 = \frac{10}{20}C - \frac{7}{20} < x_2 < \frac{10}{20}C - \frac{6}{20} = 999.70 \end{aligned}$$

□

Hence

$$\begin{cases} P_{n+1} > P_n & \text{if } 401 \le n \le 999 \\ P_{n+1} < P_n & \text{if } n \le 400 \end{cases}$$

Therefore $n = \boxed{401}$ minimizes P_n, and we're done.

Remark (Author notes).

- In fact, $n = \lfloor \frac{1}{5}C \rfloor + 1$ is the answer for all sufficiently large C.
- A more intuitive solution path would be to calculate r_n by combinatorially relating P_n and P_{n-1}, instead of deriving a closed formula for P_n. I haven't been able to find such a solution yet.
- A computationally complex combo problem is oddly satisfying, since most "bashy" AIME problems seem to involve some form of analytic geometry.

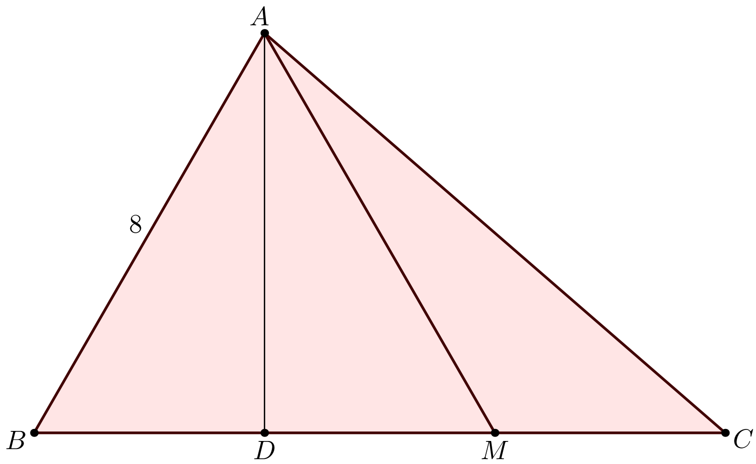
§2.4 Full solutions to AIME II

Problem 1. Let ABC be a triangle with $\angle B = 60^\circ$ and $AB = 8$. Let D be the foot of the altitude from A to BC, and let M be the midpoint of CD. If $AM = BM$, compute AC^2.

¶ Answer. 112

¶ Problem author(s). James Stewart

Since $\angle ABD = 60^\circ$ and $\angle ADB = 90^\circ$, it follows that $\triangle ABD$ is a 30°-60°-90° triangle.



This implies that $BD = 4$ and $AD = 4\sqrt{3}$. In addition, since $AM = BM$ and $\angle ABM = 60^\circ$, it follows that $\triangle ABM$ is equilateral. Thus $BM = AB = 8$, so it follows that

$$DM = BM - BD = 4.$$

Since M is the midpoint of CD, we get that $CM = DM = 4$, so $CD = 8$. Now, we can use the Pythagorean Theorem on right triangle $\triangle ADC$ to get

$$AC^2 = (4\sqrt{3})^2 + 8^2 = 64 + 48 = \boxed{112}.$$

Problem 2. Let P denote the product of all positive integers n such that the least common multiple of $2n$ and n^2 is $62n - 336$. Compute the remainder when P is divided by 1000.

¶ Answer. 352

¶ Problem author(s). Andy Liu

Note that when n is even, $\operatorname{lcm}(2n, n^2) = n^2$, but when n is odd, $\operatorname{lcm}(2n, n^2) = 2n^2$. We consider these two cases separately.

- If n is even, then $\operatorname{lcm}(2n, n^2) = n^2 = 62n - 336$. We can rearrange this to get
$$n^2 - 62n + 336 = 0$$
which factors as
$$(n-6)(n-56) = 0$$
so $n = 6$ and $n = 56$; and these both work because they are even.
- If n is odd, then $\operatorname{lcm}(2n, n^2) = 2n^2 = 62n - 336$. This implies that
$$2n^2 - 62n + 336 = 0 \implies n^2 - 31n + 168 = 0.$$
Factor to get
$$(n-7)(n-24) = 0.$$
However, $n = 24$ is even, so the only solution in this case is $n = 7$, which works.

Multiplying all of the solutions gives $6 \cdot 7 \cdot 56$, which is 2352. Thus, $\boxed{352}$ is our answer.

Problem 3. Let x be the unique positive real number satisfying

$$2^x + 32^x = 8^x + 16^x.$$

Compute $8^{x+2} - 2^{x+6}$.

¶ Answer. 064

¶ Problem author(s). James Stewart

Let $y = 2^x$. Note that our desired extraction $8^{x+2} - 2^{x+6}$ can be written as $64y^3 - 64y = 64(y^3 - y)$. We can then write our given equation as

$$y + y^5 = y^3 + y^4.$$

Note that $y = 0, 1$ satisfy this equation so this factors as

$$y(y-1)(y^3 - y - 1) = 0.$$

Since $x > 0$, it follows that $y > 1$ and thus $y^3 - y = 1$. As such, the answer is $64(y^3 - y) = 64 \cdot 1 = \boxed{64}$ as desired.

Problem 4. Compute the number of integers less than 1000 which can be written in the form

$$x^{\lfloor x \rfloor} + \lfloor x \rfloor^x$$

for some positive real number x.

¶ Answer. 590

¶ Problem author(s). Shining Sun

Let $S = x^{\lfloor x \rfloor} + \lfloor x \rfloor^x$. We will use casework on $n := \lfloor x \rfloor \geq 0$.

In the edge case $n = 0$ (and $x \neq 0$), the sum has fixed value $S = 0^x + x^0 = 1$, which gives 1 value.

Now for a fixed choice of $n \geq 1$, the value of S is strictly monotonically increasing in the interval $n \leq x < n + 1$. Hence the value of S will range across

$$2n^n \leq S < (n+1)^n + n^{n+1}$$

for $n > 0$. In particular, if $n \geq 5$ then $S > 1000$. For $1 \leq n \leq 4$, we see the values of S that appear are

$$\begin{aligned} n = 1 &\implies 2 \leq S < 3 \\ n = 2 &\implies 8 \leq S < 17 \\ n = 3 &\implies 54 \leq S < 145 \\ n = 4 &\implies 512 \leq S < 1649. \end{aligned}$$

Hence the total count is $1 + 1 + 9 + 91 + 488 = \boxed{590}$.

Problem 5. Rosa the otter is stacking 53 blocks in a tower. For $n \geq 1$, after successfully placing the previous $n-1$ blocks, the probability that placing the n^{th} block causes the whole tower to topple is $\frac{1}{54-n}$. Compute the expected number of blocks placed successfully before the block that causes the tower to topple.

¶ **Answer.** 026

¶ **Problem author(s).** Sohil Rathi

Note that the tower must collapse after successfully placing n blocks for some $1 \leq n \leq 53$ (as the last block guarantees a collapse). So the probability of the tower collapsing on placing the n^{th} block is

$$\left(1-\frac{1}{53}\right)\left(1-\frac{1}{52}\right)\cdot \dots \cdot \left(1-\frac{1}{55-n}\right)\cdot\frac{1}{54-n} = \frac{52}{53}\cdot\frac{51}{52}\cdots\frac{54-n}{55-n}\cdot\frac{1}{54-n} = \frac{1}{53}.$$

As such, each $1 \leq n \leq 53$ is equally likely. Thus, the expected number of blocks is thus

$$\frac{0}{53}+\frac{1}{53}+\dots+\frac{52}{53} = \frac{52\cdot 53}{2\cdot 53} = \boxed{26}.$$

Problem 6. Compute the number of functions $f\colon \{1,2,\dots,15\} \to \{-1,0,1\}$ such that $f(ab) = f(a)f(b)$ holds whenever a and b are positive integers with $ab \le 15$.

¶ **Answer.** 730

¶ **Problem author(s).** Benjamin Fu

Since $f(1)^2 = f(1)$, we have either $f(1) = 0$ or $f(1) = 1$.

- If $f(1) = 0$ then we deduce $f \equiv 0$, which certainly works.
- If $f(1) = 1$, then the function is uniquely determined by its values on $f(2)$, $f(3)$, $f(5)$, $f(7)$, $f(11)$, $f(13)$. There are thus $3^6 = 729$ functions in this case.

Hence the answer $1 + 729 = \boxed{730}$.

Problem 7. Vikram has a sheet of paper with all the numbers from 1 to 1000 written on it in a row. He then removes every multiple of 6 or 7. In doing so, the remaining numbers are split up into contiguous runs of consecutive numbers, such as $\{1, 2, 3, 4, 5\}$, $\{25, 26, 27\}$, or $\{13\}$. The average length of a run can be written as $\frac{p}{q}$ for relatively prime positive integers p and q. Compute $p + q$.

¶ Answer. 954

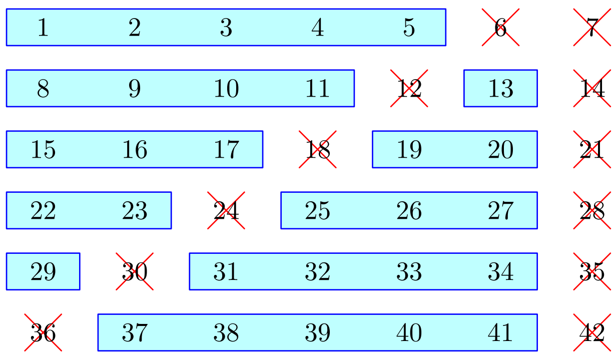
¶ Problem author(s). Vikram Sarkar

The multiples of 42 are all removed, and the removal pattern is periodic after that. To that end, let's consider a group of 42 such consecutive numbers ending with a multiple of 42.

Claim — Among any group of numbers of the form $\{42k+1, 42k+2, \dots, 42k+42\}$, there are 30 integers remaining and they form 10 contiguous runs.

Proof. We assume $k = 0$ without loss of generality. By the principle of Inclusion-Exclusion, the number of removed numbers is equal to $R = \frac{42}{6} + \frac{42}{7} - \frac{42}{42} = 12$. There are exactly two consecutive pairs of removed numbers, namely $(6, 7)$ and $(35, 36)$, and no run occurs between them. So in total there are $R - 2 = 10$ contiguous runs,

Alternatively, one can draw the figure by hand for all 42 numbers.



which can be seen to have the desired number of runs and numbers. □

Suppose momentarily we extended our sheet of paper to be from 1 to $1008 = 42 \cdot 24$. The claim above implies that there are $24 \cdot 30 = 720$ numbers which form $24 \cdot 10$ runs.

If we look then at the extra numbers we added, the number 1001 is divisible by 7, the number 1002 is divisible by 6, the number 1008 is divisible by both, while $\{1003, \dots, 1007\}$ remain and gives one extra run. Hence for the original sheet of paper,

- We have $24 \cdot 30 - 5 = 715$ numbers;
- We have $24 \cdot 10 - 1 = 239$ runs.

Thus, the requested average is $\frac{715}{239}$, and $715 + 239 = \boxed{954}$.

Problem 8. Let ABC be an equilateral triangle with side length 600, and let P be a point on the circumcircle of ABC such that $AP = 630$ and $PB > PC$. Let M be the midpoint of BC. Point D is chosen on line BP such that $MD \perp AP$. Compute PD.

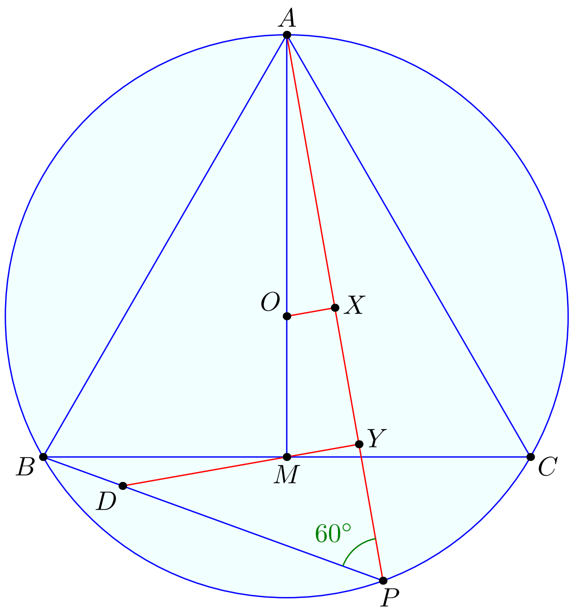
¶ Answer. 315

¶ Problem author(s). Joshua Liu

We will prove that $PD = \frac{1}{2}AP$, so the answer is 315.

Let O denote the center of ABC and let the feet of the altitude from O and M to AP be X and Y respectively. Then X is the midpoint of AP, and also (from $AO : OM = 2 : 1$) we get $AX : XY = 2 : 1$. Putting this all together we get

$$AX : XY : YP = 2 : 1 : 1 \implies \frac{PY}{PA} = \frac{1}{4}.$$



On the other hand $\triangle DYP$ is right with $\angle DYP = 90°$ and $\angle APB = \angle ACB = 60°$. As such,

$$DP = 2PY = \frac{AP}{2} = 315.$$

Problem 9. Let N denote the number of polynomials $P(x)$ of degree 3 and with leading coefficient 1 such that

- Every coefficient of $P(x)$ is an integer with absolute value at most 10;
- There exist two distinct integers m and n such that $P(mi) = P(ni)$. (Here $i = \sqrt{-1}$.)

Compute the remainder when N is divided by 1000.

¶ Answer. 365

¶ Problem author(s). Jack Whitney-Epstein

Let $P(x) = x^3 + ax^2 + bx + c$. Then

$$P(mi) = -m^3 i - am^2 + bmi + c = -am^2 + c + i(-m^3 + bm),$$

so equating the real and imaginary parts of $P(mi) = P(ni)$ gives the following necessary and sufficient:

$$-am^2 = -an^2$$
$$-m^3 + bm = -n^3 + bn.$$

Notice that these equations are independent of c, so we should count the number of pairs (a, b) that work and then multiply by 21 across the choices $c \in \{-10, -9, \dots, 10\}$.

To further simplify we factor the two equations and use $m \neq n$ to rewrite them as

$$a(m^2 - n^2) = 0 \implies a = 0 \text{ or } m = -n$$
$$(m - n)\left[b - (m^2 + mn + n^2)\right] = 0 \implies b = m^2 + mn + n^2.$$

We consider two cases:

- If $a \neq 0$, then since $m \neq n$, we get that $m = -n$, and that both are nonzero, so $b = n^2$. As such, b is a perfect square, which gives three choices of $b \in \{1, 4, 9\}$. There are 20 nonzero choices of a, giving $3 \cdot 20 = 60$ cases here.
- If $a = 0$, then we need $b = m^2 + mn + n^2$. We know $b \geq 0$ as $m^2 + mn + n^2 = \left(m + \frac{1}{2}n\right)^2 + \frac{3}{4}n^2 \geq 0$. However, equality can never occur (since $m + \frac{1}{2}n = n = 0 \iff (m, n) = (0, 0)$ which isn't allowed).

  We now assert that the possible values of b are exactly $b \in \{1, 3, 4, 7, 9\}$. The constructions giving these values are:

  $$1 = 1^2 + 1 \cdot 0 + 0^2$$
  $$3 = 2^2 + 2 \cdot (-1) + (-1)^2$$
  $$4 = 2^2 + 2 \cdot 0 + 0^2$$
  $$7 = 2^2 + 2 \cdot 1 + 1^2$$
  $$9 = 3^2 + 3 \cdot 0 + 0^2.$$

  We need to rule out $b \in \{2, 5, 6, 8, 10\}$. It suffices to prove the following lemma:

Lemma

Let $p \in \{2,5\}$. If p divides m^2+mn+n^2, then $p \mid m,n$.

Proof. If m or n is divisible by p this follows. Else, we get that $q = \frac{m}{n}$ is a root of $q^2+q+1 \equiv 0 \pmod p$.

It can be checked directly for $p = 2, 5$, but holds for general $p \equiv 2 \pmod 3$ by considering the expression $\frac{q^3-1}{q-1}$. □

This implies that the exponent of p in m^2+mn+n^2 should be even. This is enough to rule out the cases we just described.

Hence, we get 5 choices of b.

Thus there are $3 \cdot 20 + 5 = 65$ choices of (a,b), and hence $N = 65 \cdot 21 = 1365$. The requested remainder is $\boxed{365}$.

Problem 10. Let S be the set of positive integers that are divisible by either 14 or 34 (or both), but not by any prime that doesn't divide 14 or 34. (For example, $14 \cdot 34 \in S$, but $14 \cdot 3 \cdot 4 \notin S$.) Let $d(s)$ denote the number of positive integers dividing s. Suppose that

$$\sum_{s \in S} \frac{d(s)}{s} = \frac{pqr}{m}$$

for some primes p, q, r and a positive integer m. Compute $p+q+r$.

¶ **Answer.** 071

¶ **Problem author(s).** Karn Chutinan

Throughout the solution we will use the fact that $d(ab) = d(a) \cdot d(b)$ holds for all relatively prime a and b. In particular, for primes p, define

$$f(p) \coloneqq \sum_{n \geq 0} \frac{d(p^n)}{p^n}.$$

Claim — We have

$$\sum_{s \in S} \frac{d(s)}{s} = (f(2)-1)\,(f(7)f(17)-1)\,.$$

Proof. Notice that

$$f(2) = \sum_{a \geq 0} \frac{d(2^a)}{2^a}$$

$$f(7)f(17) = \sum_{b,c \geq 0} \frac{d(7^b 17^c)}{7^b 17^c}.$$

Hence if we subtract one we get

$$f(2) - 1 = \sum_{a \geq 1} \frac{d(2^a)}{2^a}$$

$$f(7)f(17) - 1 = \sum_{\substack{b,c \geq 0 \\ (b,c) \neq (0,0)}} \frac{d(7^b 17^c)}{7^b 17^c}.$$

Thus, the product of these is equals to

$$\sum_{\substack{a \geq 1 \\ b,c \geq 0 \\ (b,c) \neq (0,0)}} \frac{d(2^a 7^b 17^c)}{2^a 7^b 17^c}.$$

This matches the description of S, since S consists of those elements of the form $2^a \cdot 7^b \cdot 17^c$ where $a \geq 1$, $b \geq 0$, $c \geq 0$, and $(b,c) \neq (0,0)$. □

On the other hand, f can be evaluated easily.

Claim — We have that $f(p) = \left(\frac{1}{1-\frac{1}{p}}\right)^2$ for prime p.

Proof. Since $d(p^n) = n+1$, we have

$$f(p) = \sum_{n\ge 0} \frac{d(p^n)}{p^n} = \sum_{n\ge 0} \frac{n+1}{p^n} = \sum_{n\ge 0}\sum_{k\le n} \frac{1}{p^n}.$$

Switching the order of summation, we have

$$\sum_{k\ge 0}\sum_{n\ge k} \frac{1}{p^n} = \sum_{k\ge 0} \frac{1}{p^k} \cdot \left(\sum_{n\ge 0} \frac{1}{p^n}\right) = \left(\frac{1}{1-\frac{1}{p}}\right)^2. \qquad \square$$

Putting this together gives

$$\begin{aligned} \sum_{s\in S} \frac{d(s)}{s} &= \left(\frac{2^2}{1} - 1\right)\left(\frac{7^2}{6^2}\cdot\frac{17^2}{16^2} - 1\right) \\ &= \frac{3\cdot(119^2 - 96^2)}{96^2} = \frac{23\cdot 215}{32\cdot 96} = \frac{5\cdot 23\cdot 43}{32\cdot 96}. \end{aligned}$$

Thus, the answer is $5 + 23 + 43 = \boxed{071}$.

Problem 11. At an informatics competition each student earns a score in $\{0, 1, \dots, 100\}$ on each of six problems, and their total score is the sum of the six scores (out of 600). Given two students A and B, we write $A \succ B$ if there are at least five problems on which A scored strictly higher than B.

Compute the smallest integer c such that the following statement is true: for every integer $n \geq 2$, given students $A_1, \dots, A_n$ satisfying $A_1 \succ A_2 \succ \dots \succ A_n$, the total score of A_n is always at most c points more than the total score of A_1.

¶ **Answer.** 570

¶ **Problem author(s).** Jiahe Liu

The answer is $600 - 2(1+2+3+4+5) = \boxed{570}$. We prove this by providing a construction and then showing it's best possible.

¶ **Construction** The example with seven students

$$\begin{aligned}(5,4,3,2,1,0) &\succ (4,3,2,1,0,100) \succ (3,2,1,0,100,99) \\ &\succ (2,1,0,100,99,98) \succ (1,0,100,99,98,97) \\ &\succ (0,100,99,98,97,96) \succ (100,99,98,97,96,95)\end{aligned}$$

is a construction for $c = 570$, since the first student has total score $5+4+3+2+1 = 15$ and the last student has total score $100+99+98+97+96+95 = 585$.

¶ **Bound** We now show that no higher value can be achieved. The two main claims are that if the score of A_1 is too low, or the score of A_n is too high, then $n \leq 5$. We state and prove these claims in turn:

Claim — If the total score of A_1 is less than 15, then there cannot exist students A_2, A_3, A_4, A_5, A_6 such that

$$A_1 \succ A_2 \succ A_3 \succ A_4 \succ \dots \succ A_6.$$

In other words, we have $n \leq 5$.

Proof. Assume for contradiction such students existed. For $i > 1$, we say a problem P is a *shallow victory* for A_i if A_i scored higher than A_{i-1} on P (hence each of $A_2, \dots, A_6$ found at most one shallow victory). Then for each i, let S_i denote the sum of the scores of A_i across all problems which weren't shallow victories for any of A_2 through A_i (for $i = 1$, S_1 is just the total score of A_1).

Note that S_i decreases by at least the number of problems that haven't been shallow victories minus one, so

$$S_{i+1} \leq S_i - (6 - i).$$

Since $S_1 \leq 14$, by repeating this we get that $S_6 \leq -1$, which is impossible. □

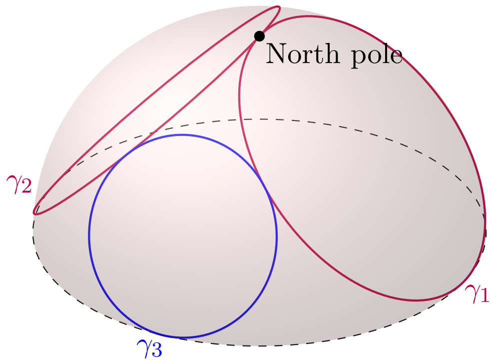
Claim — If A_n has total score exceeding $600 - (0 + 1 + 2 + 3 + 4 + 5) = 585$, a symmetric argument proves that $n \le 5$ as well.

Proof. The proof is basically the same as the previous claim. Indeed, one can see this directly from the previous claim by letting B_i be a student scoring $100 - x$ on each problem if A_i scored x; then $A_1 \succ \cdots \succ A_n$ is the same as $B_n \succ B_{n-1} \succ \cdots \succ B_1$, and A_n having total score greater than 585 is the same as B_n having total score at most 15. □

Suppose we're in a case where $n \le 5$. Then the difference between the total scores of A_{i+1} and A_i is always at most 95; hence the difference between A_1 and A_n is at most is at most $4 \cdot 95 = 380$.

Now suppose $n \ge 6$. The two claims imply that the difference is at most $585 - 15 = 570$, matching the construction above (which uses $n = 7$). The proof is complete.

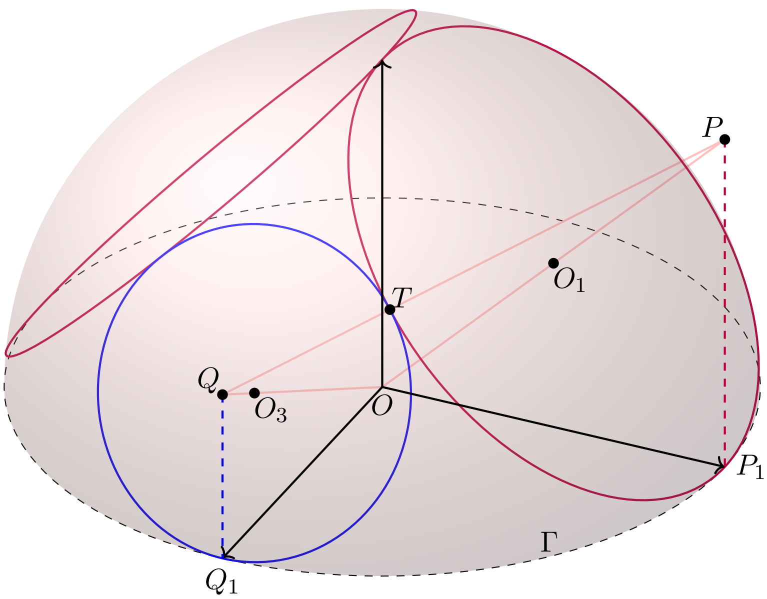
Problem 12. Let γ_1, γ_2, and γ_3 be circles drawn on the surface of a hemisphere with radius 10. Each circle is tangent to the base of the hemisphere and pairwise tangent to one another. Additionally, γ_1 and γ_2 are congruent and tangent to each other at the north pole of the hemisphere, the point of the hemisphere farthest from the base. Compute the greatest integer less than the area of γ_3 (here, the area of γ_3 is taken with respect to the plane containing γ_3).



¶ **Answer.** 062

¶ **Problem author(s).** Kishore

We show two approaches to this. Let Γ be the base of the hemisphere, and let T, P_1, Q_1 be the common tangency points of the pairs $(\gamma_1, \gamma_3), (\gamma_1, \Gamma), (\gamma_3, \Gamma)$ respectively. Let O, O_1, O_2, O_3 be the centers of Γ, γ_1, γ_2, γ_3 respectively.



First proof, by author. Let P be a point such that the tangent lines from P to the hemisphere touch along the circle γ_1; define Q similarly for γ_3. In addition, let the north pole of the hemisphere be point N.

By basic tangent properties, we have $\angle PNO = \angle PP_1O = 90^\circ$. We also have $OP_1 = ON$ and $\angle NOP_1 = 90^\circ$, so PP_1ON is a square. Thus, $PP_1 = OP_1 = OQ_1 = 10$. The Pythagorean theorem in three dimensions lets us write

$$PQ^2 = (PP_1 - QQ_1)^2 + OP_1^2 + OQ_1^2 = (10 - QQ_1)^2 + 200.$$

However, since P, Q, T both lie in the tangent plane of T and plane OO_1O_2, the points P, Q, T are collinear as well, so

$$PQ^2 = (PT + TQ)^2 = (PP_1 + QQ_1)^2 = (10 + QQ_1)^2.$$

Hence $(10 + QQ_1)^2 = (10 - QQ_1)^2 + 200$ gives $QQ_1 = 5$, so $\tan \angle QOQ_1 = \frac{1}{2}$. □

Second proof, by Andrew Lin. Let $\theta = \angle O_3OQ_1$. Note that for γ_1, γ_3 to be tangent we must have

$$\angle O_1OO_3 = \angle O_1OT + \angle TOO_3 = \angle O_1OP_1 + \angle O_3OQ_1 = 45^\circ + \theta$$

must hold.

Impose coordinates over $\mathbb{R}^3$ so that $Q_1 = (10, 0, 0)$, $P_1 = (0, 10, 0)$ and the north pole is at $(0, 0, 10)$. The unit vectors in the directions of OO_1 and OO_3 are $\left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and $(\cos(\theta), 0, \sin(\theta))$. Then by dot products, it follows that

$$\cos(45^\circ + \theta) = \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \cdot (\cos(\theta), 0, \sin(\theta)) = \frac{\sin(\theta)}{\sqrt{2}}.$$

By the cosine addition formula, this rewrites as

$$\frac{\cos(\theta) - \sin(\theta)}{\sqrt{2}} = \frac{\sin(\theta)}{\sqrt{2}} \implies \tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} = \frac{1}{2}.$$ □

Consider right triangle O_3OQ_1 now. From $OQ_1 = 10$ and $\tan \angle O_3OQ_1 = \frac{1}{2}$. Thus $O_3Q = 2\sqrt{5}$, and the area of γ is $(2\sqrt{5})^2\pi = 20\pi$. Finally, $\lfloor 20\pi \rfloor = \boxed{62}$.

Problem 13. Compute the largest positive integer m such that 2^m divides

$$\sum_{k=0}^{717}(-1)^k\binom{717}{k}(6+239k)^{717}.$$

¶ **Answer.** 711

¶ **Problem author(s).** Catherine Xu & Ritwin Narra

We recall the following general lemma.

Lemma

Let $P(x)$ be a degree-d polynomial with leading coefficient c. Then

$$\sum_{k=0}^{d}(-1)^{n-k}\binom{d}{k}P(k) = c\cdot d!.$$

The lemma is proved by considering the finite differences of $P(x)$.

Let S denote the sum in the problem. Applying this to our problem by setting $P(k)=(239k+6)^{717}$, we have

$$S := \sum_{k=0}^{717}(-1)^k\binom{717}{k}P(k) = 717!\cdot 239^{717}$$

since P has degree 717 and leading coefficient 239^{717}.

We seek to compute

$$m = \nu_2(S) = \nu_2(717!\cdot 239^{717}) = \nu_2(717!).$$

To finish we invoke Legendre's formula in the form

$$\begin{aligned}\nu_2(m) = \nu_2(717!) &= \sum_{i=1}^{\infty}\left\lfloor \frac{n}{2^i}\right\rfloor \\ &= \left\lfloor \frac{717}{2}\right\rfloor + \left\lfloor \frac{717}{4}\right\rfloor + \cdots \\ &= 358+179+89+44+22+11+5+2+1 = \boxed{711}.\end{aligned}$$

(An equivalent form of Legendre's formula in this case is to write 717 in binary as $717 = 1011000111_2$, which has sum of binary digits 6. Then $\nu_2(717!) = 717-6 = 711$.)

Problem 14. The incircle of ABC is tangent to BC at D. Let the internal bisectors of $\angle BAD$ and $\angle BDA$ meet at I_B and their external bisectors at E_B, and define I_C and E_C similarly. Suppose that $I_BI_C = 1$, $E_BE_C = 6$, and the area of quadrilateral $I_BI_CE_BE_C$ is 7. The area of triangle ABC can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute $m + n$.

¶ **Answer.** 191

¶ **Problem author(s).** Jason Lee & Arjun Suresh

Let $a := BC$, $b := CA$, $c := AB$ and $s := \frac{1}{2}(a+b+c)$. Let $d = AD$. Hence $BD = s - b$ and $CD = s - c$.

We start with the following claim.

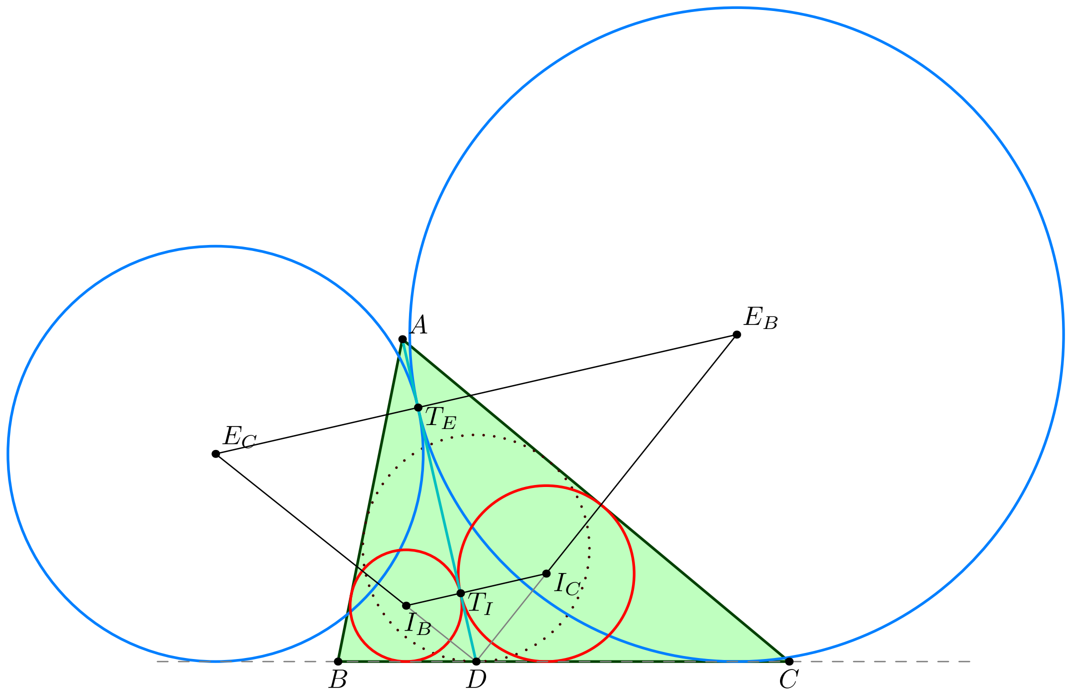
Claim — The incircles of $\triangle ABD$ and $\triangle ACD$ (centered at I_B and I_C) are tangent at a point T_I on line AD. Similarly, the excircles (opposite D, centered at E_B and E_C) are tangent at a point T_E on line AD.

Moreover, T_I and T_E are reflections across the midpoint of AD.

Proof. The length of the tangents from A along line AD are given by

$$\frac{c + d + (s - b)}{2} - (s - b) = \frac{b + d + (s - c)}{2} - (s - c)$$

and hence the tangency points coincide at the point $\frac{d+(s-a)}{2}$ from A. The extouch coincidence follows in the same way by the classical symmetry; they touch at the point $\frac{d+(s-a)}{2}$ from D. □



Claim — Quadrilateral $I_BI_CE_BE_C$ is a trapezoid whose midline is the perpendicular bisector of $\overline{AD}$.

Proof. Follows directly from the previous claim. □

Henceforth denote $t_A := AT_I = \frac{d+(s-a)}{2}$ and $t_D := DT_I = \frac{d-(s-a)}{2}$, and let h denote the height of the trapezoid, i.e. $h = T_IT_E$. Using the given area conditions, we can solve for h:

$$h = [I_BI_CE_BE_C] / \left(\frac{I_BI_C + E_BE_C}{2} \right) = \frac{7}{\frac{1+6}{2}} = 2.$$

Now, the homothety at D mapping

$$\overline{I_BT_II_C} \to \overline{E_CT_EE_B}$$

has ratio 6, so we conclude

$$t_D = \frac{h}{5} = \frac{2}{5}, \qquad t_A = 6t_D = \frac{12}{5}, \qquad d = 7t_D = \frac{14}{5}.$$

Next, we recover the inradii of the two smaller incircles. Consider $\triangle I_BDI_C$, which is right-angled with $\angle D = 90^\circ$. Letting r_B and r_C we know that

$$\begin{aligned} 1 &= I_BI_C = r_B + r_C \\ \frac{2}{5} &= t_D = \sqrt{r_Br_C}. \end{aligned}$$

This means that r_B, r_C are the roots of $x^2 - x + \frac{4}{25}$, and solving gives $r_B = \frac{1}{5}$ and $r_C = \frac{4}{5}$.

We now move on to extracting the quantities needed for triangle ABC. We compute the height of the A-altitude h_A, and the inradius r:

$$\begin{aligned} h_A &= d \cdot \sin\angle ADB = d \cdot 2\sin\angle I_BI_CD \cos\angle I_BI_CD \\ &= d \cdot 2 \cdot \frac{1}{\sqrt{5}} \cdot \frac{2}{\sqrt{5}} = \frac{4}{5}d = \frac{56}{25}, \\ r &= (s-a)\tan\frac{A}{2} = (s-a)\tan(\angle I_BAT_I + \angle I_CAT_I) \\ &= (s-a)\frac{\tan\angle I_BAT_I + \tan\angle I_CAT_I}{1 - \tan\angle I_BAT_I \tan\angle I_CAT_I} \\ &= (s-a)\frac{\frac{r_B}{t_A} + \frac{r_C}{t_A}}{1 - \frac{r_B}{t_A}\cdot\frac{r_C}{t_A}} = 2 \cdot \frac{\frac{1}{12} + \frac{4}{12}}{1 - \frac{1}{12}\cdot\frac{4}{12}} = \frac{6}{7}. \end{aligned}$$

Now, note that $[ABC] = rs = \frac{1}{2}a \cdot h_A$, so substituting it follows that

$$\frac{6}{7}s = \frac{1}{2}a \cdot \frac{56}{25} \implies s = \frac{98}{75} \cdot a.$$

Hence, combining this with $s - a = 2$, we find

$$a = 2 \cdot \frac{75}{23}, \qquad s = 2 \cdot \frac{98}{23}.$$

The requested area is thus

$$[ABC] = rs = \frac{6}{7} \cdot 2 \cdot \frac{98}{23} = \frac{168}{23},$$

so the answer is the sum $168 + 23 = \boxed{191}$.

Problem 15. The Queen of Hearts has a special deck of 16 playing cards and a 4×4 square grid. Each card has one of four different ranks and one of four different suits, with each combination occurring exactly once. She wishes to place the cards in the grid, with one card in each cell, such that any cards in adjacent cells share either a rank or a suit. Compute the remainder when the number of ways to fill the grid is divided by 1000.

¶ Answer. 064

¶ Problem author(s). Neil Kolekar

We'll use a, b, c, d for the four suits and 1, 2, 3, 4 for the four ranks. Hence we need to fill the grid with these 16 cards which come in four ranks and four suits, each pair once from a_1 to d_4, such that adjacent cards either share a rank or share a suit.

In fact, even for a 2×2 sub-square, there are not that many possible shapes that can appear. We claim that, up to relabeling the suits and ranks, only four "kinds" of shapes could appear, which we abbreviate with the following names:

Flush $\left[\begin{smallmatrix} a_1 & a_2 \\ a_3 & a_4 \end{smallmatrix}\right]$ (i.e. all suits the same)

Four-of-a-kind $\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$ (i.e. all ranks the same)

Horizontal-pairs $\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$ (i.e. two ranks and two suits, same ranks horizontally)

Vertical-pairs $\left[\begin{smallmatrix} a_1 & a_2 \\ b_1 & b_2 \end{smallmatrix}\right]$ (i.e. two ranks and two suits, same ranks vertically).

The proof that these are the only four cases is straightforward: assume WLOG the northwest card is a_1. Then the northeast card either shares a rank (so WLOG it is b_1) or a suit (so WLOG it is a_2). Similarly, the southwest card either sharks a rank or a suit. In all $2^2 = 4$ cases, there is exactly one way to finish the 2×2 grid and it leads to the four cases above.

Now, as for the main 4×4 grid, we divide it into four 2×2 subgrids, say $\begin{bmatrix} G_1 & G_2 \\ G_3 & G_4 \end{bmatrix}$. The main claim is that **in fact all G_i are the same type!** We prove this for each of the four types in turn, enumerating as we go.

> **Claim** — If G_1 is a flush, then every G_i is; there are $4 \cdot 4! \cdot 18$ such grids.

Proof. By symmetry, we'll assume WLOG that G_1 is a flush for suit a, and label the ranks as in the left table below. (This accounts for the $4 \cdot 4!$.) Then there are no more cards with suit a, so the two cells just to the south of G_1 must form horizontal-pairs, say for suit b (this adds a factor of 3). Similarly the two eastern cells should form vertical-pairs, say for suit c (this adds a factor of 2). This gives the right table below.

a_1	a_2	*	*
a_3	a_4	*	*
*	*	*	*
*	*	*	*

a_1	a_2	c_2	*
a_3	a_4	c_4	*
b_3	b_4	d_4	*
*	*	*	*

In the center 2×2 we can now place d_4 (colored green). As all cards of rank 4 have been placed, and there is one in each G_i, so every G_i is a flush.

For the count, once we are in the grid above, there are only 3 ways to finish the grid; once the southeast cell is filled with one of $\{d_1, d_2, d_3\}$, the rest of the grid is uniquely determined. This gives $4 \cdot 4! \cdot (3 \cdot 2 \cdot 3)$ for the count, as needed. □

Claim — If G_1 is a four-of-a-kind, then every G_i is; there are $4 \cdot 4! \cdot 18$ such grids.

Proof. Identical to previous claim with suits and ranks flipped. □

Claim — If G_1 is vertical-pairs, then every G_i is; there are $(4 \cdot 3)^2 \cdot 16$ such grids.

Proof. From the previous claims we may assume every G_i is either vertical-pairs or horizontal-pairs. Suppose G_1 is vertical-pairs with the labeling shown below; this accounts for a factor of $(4 \cdot 3)^2$. Then the two cards to the right of a_2 and b_2 (marked red below) must have the same rank. Meanwhile, the cards below b_1 and b_2 must have the same suit. Consider these cases:

a_1	a_2	a_3	a_4
b_1	b_2	b_3	b_4
$*$	$*$	$*$	$*$
$*$	$*$	$*$	$*$

Case I

a_1	a_2	c_2	c_i
b_1	b_2	d_2	d_i
b_3	b_4	d_4	d_j
x_3	x_4	y_4	y_j

Case II

- In Case I, we assume that the 2×2 square between G_1 and G_2 was also vertical-pairs. There is no distinction between 3 and 4 and we pick rank 3 for a factor of 2. Then there is only one way to finish G_2 and then the third and fourth row must be (c_1, c_2, c_3, c_4) and (d_1, d_2, d_3, d_4) in some order. So overall, Case I leads to $2 \cdot 2 = 4$ ways to complete.

- Now consider the 2×2 square between G_1 and G_3. If they formed vertical pairs, say (c_1, c_2), then the same logic as before lets us complete the grid; we find that this ends us back up in Case I!

- So we finally consider Case II, where the 2×2 square between G_1 and G_3 is a flush and the 2×2 square between G_1 and G_2 is a four-of-a-kind. By swapping c versus d, or 3 versus 4, we have $2^2 = 4$ ways to place the red cards above. This then places d_4 as shown in green above. We then fill the grey cells as shown; we have a choice between $\{x, y\} = \{a, c\}$ and $\{i, j\} = \{1, 3\}$. Of these four possibilities, all of them work except for $j = 1$ and $x = a$ (in which case the card a_1 reappears, for example). This gives a total of $2^2 \cdot 3 = 12$ for Case II.

In total across Case I and Case II we get $4 + 12 = 16$, as needed. □

Claim — If G_1 is horizontal-pairs, then every G_i is; there are $(4 \cdot 3)^2 \cdot 16$ such grids.

Proof. Identical to previous claim with suits and ranks flipped. □

Summing up, we get a grand total of

$$2(4 \cdot 4! \cdot 18 + (4 \cdot 3)^2 \cdot 16) = 8064.$$

possible labellings. The answer is $\boxed{064}$.

3 Statistics

§3.1 Total score statistics

There were 82 official submissions to AIME I and 115 official submissions to AIME II.

AIME I score	Freq	**AIME II** score	Freq
Total score = 0	0	Total score = 0	0
Total score = 1	2	Total score = 1	0
Total score = 2	2	Total score = 2	0
Total score = 3	2	Total score = 3	0
Total score = 4	3	Total score = 4	3
Total score = 5	3	Total score = 5	7
Total score = 6	10	Total score = 6	6
Total score = 7	10	Total score = 7	12
Total score = 8	12	Total score = 8	16
Total score = 9	16	Total score = 9	21
Total score = 10	8	Total score = 10	20
Total score = 11	6	Total score = 11	21
Total score = 12	4	Total score = 12	8
Total score = 13	3	Total score = 13	1
Total score = 14	1	Total score = 14	0
Total score = 15	0	Total score = 15	0

§3.2 Number of correct answers per problem

P#	#Correct	% Correct	Description
I.1	68	81.9%	3×3 grid with penguins
I.2	74	89.2%	Quadrilateral $ABCD$
I.3	70	84.3%	$k - n/2$
I.4	62	74.7%	Dodecagon
I.5	73	88.0%	$\lvert x + 2y + 3z \rvert$
I.6	43	51.8%	Green and pink pencils
I.7	61	73.5%	ABC
I.8	58	69.9%	$ab + bc + cd + de$
I.9	57	68.7%	New-prime
I.10	18	21.7%	8×8 grid
I.11	13	15.7%	XA, XB, XC
I.12	17	20.5%	System of equations
I.13	14	16.9%	BC^2
I.14	24	28.9%	$1/315$
I.15	4	4.8%	2000 cards

P#	#Correct	% Correct	Description
II.1	107	93.0%	$AM = BM$
II.2	107	93.0%	$62n - 336$
II.3	113	98.3%	$8^{x+2} - 2^{x+6}$
II.4	78	67.8%	$x^{\lfloor x \rfloor} + \lfloor x \rfloor^x$
II.5	104	90.4%	Rosa the Otter
II.6	85	73.9%	$f\colon \{\dots\} \to \{-1, 0, 1\}$
II.7	97	84.3%	Sheet of paper
II.8	92	80.0%	$MD \perp AP$
II.9	56	48.7%	$P(mi) = P(ni)$
II.10	90	78.3%	pqr/m
II.11	45	39.1%	Informatics contest
II.12	17	14.8%	Sphere
II.13	24	20.9%	$(6 + 239k)^{717}$
II.14	4	3.5%	$I_B I_C E_B E_C$
II.15	5	4.3%	4×4 card grid

4 Behind the scenes

§4.1 Testsolving statistics

Serious testsolvers had a time limit and unlimited answer checks. We record

- the fastest solve time (in seconds) among testsolvers who got the answer in one try;
- the median solve time among testsolvers who got the answer in one try (if the number of such solves is even, we take the larger of the two times);
- the number of testsolvers that got the correct answer on the first try;
- the number that eventually got the correct answer before the timer expired (but may have required multiple tries). The timer was set as follows: the author was asked to estimate the difficulty of each problem on a scale of one light bulb to five light bulbs, where n light bulbs was a problem suitable for problem $3n$ on the AIME. Then solvers got $5(n+1)$ minutes for their testsolving session. (However, they could give up early if they didn't want to wait for the full time.)

Here's the data for AIME I.

P#	Description	Fastest	Median	Correct	Finished
I.1	Penguins	0:07	1:20	30	50
I.2	Quad $ABCD$	1:20	3:21	21	28
I.3	$k-n/2$	0:07	1:47	30	38
I.4	Dodecagon	2:08	3:22	15	18
I.5	$\lvert x+2y+3z\rvert$	0:22	3:20	19	32
I.6	Colored pencils	1:39	5:16	9	16
I.7	Triangle $\triangle ABC$	2:16	5:14	21	27
I.8	$ab+bc+cd+de$	0:41	4:51	4	12
I.9	New-prime	0:20	4:44	19	24
I.10	8×8 grid	2:33	9:30	2	4
I.11	XA, XB, XC	7:11	16:46	4	4
I.12	System of equations	2:39	16:23	4	4
I.13	BC^2	5:15	24:52	2	4
I.14	1/315	0:07	11:41	8	9
I.15	2000 cards	-	-	0	1

Here's the data for AIME II.

P#	Description	Fastest	Median	Correct	Finished
II.1	$AM = BM$	0:35	1:12	39	45
II.2	$62n - 336$	0:08	1:56	29	39
II.3	$8^{x+2} - 2^{x+6}$	0:34	1:47	24	28
II.4	$x^{\lfloor x \rfloor} + \lfloor x \rfloor^x$	0:06	3:53	10	21
II.5	Rosa the Otter	0:29	1:42	18	34
II.6	$f\colon \{\dots\} \to \{-1,0,1\}$	0:22	1:18	11	30
II.7	Sheet of paper	1:46	4:12	13	22
II.8	$MD \perp AP$	3:22	8:22	18	23
II.9	$P(mi) = P(ni)$	7:36	8:48	6	11
II.10	pqr/m	2:12	6:47	12	15
II.11	Informatics contest	1:01	2:48	18	30
II.12	Sphere	6:44	14:41	2	2
II.13	$(6 + 239k)^{717}$	0:57	6:47	6	8
II.14	$I_B I_C E_B E_C$	10:10	10:10	1	2
II.15	4×4 card grid	5:38	13:34	6	6

This data is mostly for comedic value than anything else (e.g., "holy crap someone got this in x seconds WTF??"). You shouldn't take it too seriously because, e.g., the version the testsolvers worked on may not even look remotely like the final version, etc. Problems went through a lot of changes and so on.

§4.2 "Raw" versions of problems (pre-editing)

I.1 In a 3×3 grid, each cell is empty or contains a penguin. Two penguins are *angry* if they occupy diagonally adjacent cells. Find the number of ways to fill the grid so that no penguins are angry.

I.2 Right triangle ABC has a right angle at B and $\angle BAC = 30^\circ$. Let D be the point such that $\triangle ABC \sim \triangle ACD$ and the two triangles' interiors do not intersect. Let M be the midpoint of AD and N be the midpoint of BM. Given that $AB = 24$, find CN^2.

I.3 Let n, $k > 1$ be positive integers and let k be *n-neutral* for even n if k has a remainder of $\frac{n}{2}$ when divided by n. What is the smallest k such that there are exactly 10 values of n that make k *n-neutral*?

I.4 Let S be the set of equiangular dodecagons that lie on the Cartesian plane such that the line segment with endpoints at $(2, 5 + 6\sqrt{3})$ and $(4, 5 + 6\sqrt{3})$, and the line segment with endpoints at $(4, -5 - 6\sqrt{3})$ and $(2, -5 - 6\sqrt{3})$ are sides of the dodecagon. How many dodecagons in S have side lengths that are all positive integers?

I.5 Let x, y, and z be complex numbers satisfying

$$|x + z| = |y + z| = |x - y| = 4.$$

Compute $|x + 2y + 3z|^2$.

I.6 There are 2025 identical pencils on a table. Every minute, Nathan removes two pencils from the table. Right after that, Ethan adds back one pencil identical to the rest from his collection. After 2023 minutes, the probability that at least one of the two pencils on the table was originally there can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find the remainder when n is divided by 1000.

I.7 Let ABC be a triangle with $AB = 5$, $BC = 13$, and $CA = 12$. Points D, E, and F are on segments BC, CA, and AB such that DEF is an isosceles right triangle with hypotenuse EF and $BF = 3$. The value of CE can be written as $\frac{x}{y}$ for relatively prime positive integers x and y. Compute $x + y$.

I.8 Given that $a_1, a_2, \cdots, a_5$ is a sequence of positive integers and $a_1+a_2+\cdots+a_5 = 100$, find the remainder when the maximum value of $a_1a_2 + a_2a_3 + \cdots + a_4a_5$ is divided by 1000.

I.9 Alice forgot the definition of a prime number. Thus, we define positive integers to be Alice-prime in the following way:

1 is not Alice-prime. Then, a positive integer $n > 1$ is Alice-prime if and only if n cannot be expressed as the product of exactly two (not necessarily distinct) Alice-prime positive integers.

Find the number of Alice-prime divisors of 210^4.

I.10 Let n be the number of ways to color each unit edge of a 4×4 square grid one of six colors (red, orange, yellow, green, blue, purple) such that every unit cell is surrounded by exactly 3 colors. Find the remainder when number of divisors of n is divided by 1000

I.11 Let ABC be a triangle with $\angle BAC = 60^\circ$. The Euler line of ABC intersects side BC at point X. Given that $XA = 49$ and $XB = 23$, find XC.

I.12 There exists a unique tuple of positive real numbers (a, b, c, d) satisfying the following equations:

$$\begin{aligned} (49 + ab)(a + b) &= 81a + 25b \\ (81 + bc)(b + c) &= 121b + 49c \\ (121 + cd)(c + d) &= 169c + 81d \\ a + b + c + d &= 12. \end{aligned}$$

Given that $d = m - \sqrt{n}$ for positive integers m and n, compute $m + n$.

I.13 Let ABC be an acute triangle so that the distances from its circumcenter, orthocenter, and incenter to segment BC are 6, 3, and 5 respectively. If the area of ABC can be expressed as $m\sqrt{n}$ for positive integers m and n where n is square-free, find $m + n$

I.14 Positive integers $a < b < c < 1000$ satisfy

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{1}{315}.$$

Compute $c - a$.

I.15 Alice has a deck of 2000 cards, numbered 1 through 2000. Upon choosing a positive integer $n < 1000$, she deals Bob a randomly chosen hand of $2n - 1$ cards. Bob wins if his hand contains a collection of n consecutively numbered cards.

Which n should Alice choose to minimize Bob's chances of winning?

II.1 Let $\triangle ABC$ be a triangle with $\angle B = 60^\circ$ and $AB = 8$. Let D be the foot of the altitude from A to BC, and let M be the midpoint of CD. If $AM = BM$, find AC^2.

II.2 Find the remainder when the product of all postive integers n such that $\operatorname{lcm}(2n, n^2) = 62n - 336$ is divided by 1000

II.3 Let x be a positive real number satisfying

$$2^x + 32^x = 8^x + 16^x.$$

Compute the value of

$$8^{x+2} - 2^{x+6}.$$

II.4 How many integers $1 \le y \le 1000$ satisfy the property that $x^{\lfloor x \rfloor} = y - x$ has no real solutions x?

II.5 An expert jenga player is stacking 53 Jenga Blocks. For $n \ge 1$, the probability that placing the nth block causes the whole tower to topple is $\frac{1}{54-n}$ assuming the tower still stands after placing $n-1$ blocks. What is the expected number of blocks placed before the block that causes the tower to topple?

II.6 How many unique functions $f : \{1, 2, \dots, 15\} \mapsto \{-1, 0, 1\}$ exist such that $f(ab) = f(a)f(b)$ for all positive integers a and b where $ab \le 15$?

II.7 Vikram has a sheet of paper with all the numbers from 1 to 1000 written on it. He then colors every multiple of 6 or 7 with red, and colors every other number blue. In doing so, the blue numbers are split up into contiguous runs of consecutive numbers. If the average length of such a run is $\frac{p}{q}$ for relatively prime positive integers p and q, find $p + q$.

Note: A run may have length 1.

II.8 Let ABC be an equilateral triangle with length 25, and let P be a point on (ABC) such that $AP = 28$ and $PB > PC$. If the line that is perpendicular to $\overline{AP}$ and passes through the midpoint of $\overline{BC}$ intersects $\overline{BP}$ at D, find PD.

II.9 Let $i = \sqrt{-1}$. A polynomial $P(x)$ is called *freaky* if:

▪ $P(x)$ has leading coefficient 1 and degree 3 ▪ Each coefficient of $P(x)$ is an integer between -10 and 10, inclusive ▪ $P(mi) = P(ni)$ for some distinct nonzero integers m, n.

Find the remainder when the number of *freaky* polynomials is divided by 1000.

II.10 Let S be the set of positive integers that are divisible by 14 or 34, but not by any prime that doesn't divide 14 or 34. For example, $14 \cdot 34 \in S$, but $14 \cdot 3 \cdot 4 \notin S$. Let $\sigma_0(n)$ denote the number of positive divisors of n. If

$$3 + \sum_{s \in S} \frac{\sigma_0(s)}{s} = \frac{p}{q},$$

for positive coprime integers p and q, compute $\sqrt{p} + \sqrt{3q}$.

II.11 In a group of 2024 students, each student assigned an integer score between 0 and 100 in each of three categories, with the total score being the sum of the three. We say a student A beats a student B if A scored strictly higher than B in at least two categories. Then, we say that a student A eventually beats a student B if there exists students $C_1, C_2, \dots, C_n$ such that A beats C_1, C_i beats C_{i+1} for $1 \le i \le n-1$, and C_n beats B. If student A with total score a eventually beats student B with total score b, find the maximum possible value of $b - a$.

II.12 Let γ_1, γ_2, and γ_3 be circles on the surface of a hemisphere with radius 10. Each circle is tangent to the base of the hemisphere and pairwise tangent to one another. Additionally, γ_1 and γ_2 are congruent and tangent to each other at the north pole of the hemisphere. If the area of γ_3 is A, find $\lfloor A \rfloor$.

II.13 Given that

$$S = \sum_{k=0}^{717} (-1)^k \binom{717}{k} (6 + 239k)^{717},$$

find $\nu_2(S)$.

II.14 Triangle ABC has an incircle tangent to BC at D. In $\triangle ABD$, let O_B, H_B, I_B, and Θ_B be the circumcenter, orthocenter, incenter, and B-excenter (where the bisectors of $\angle ABC$ and $\angle ADC$ meet); in $\triangle ACD$, define O_C, H_C, I_C, Θ_C similarly. Suppose that $O_BO_C = 16$, $H_BH_C = \sqrt{n}$, $I_BI_C = 2$, and $\Theta_B\Theta_C = 38$. Find n.

II.15 A 4×4 grid is to be filled with the numbers $1, 2, \dots, 16$, so that each number is used exactly once. A filled grid called *spectacular* if for any numbers a and b in adjacent cells, at least one of the following is true: (i) $4 \mid (a-b)$; (ii) $\lceil a/4 \rceil = \lceil b/4 \rceil$. Let N be the number of spectacular filled grids. Find the remainder when N is divided by 1000.

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