OTIS Mock AIME 2024 Report

Solutions, results, and extended commentary from your friendly neighborhood game master

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Contents

1 Summary

The OTIS Mock AIME aired from December 22, 2023 to January 15, 2024. A total of 92 students submitted officially and are listed in statistics (with another 62 unofficial submissions).

§1.1 Top scores

Congratulations to the top scores:

15 points Zaahir Ali

14 points thanosaops

13 points tenth

12 points AW

11 points qwertyasdf, Sam Zhang, S. Erat, Zongshu Wu

10 points Huashi, Kaden Wu, P_Groudon

§1.2 Editorial notes

We selected the 15 problems here from around 60 proposed problems. This year's process was pretty much "put wings on after takeoff"; I didn't have the idea to run the AIME until mid-November, and from there it was move fast and break things.

§1.2.1 The Probase experiment

The most interesting part of this development process was **Probase**, the system written by Howard Halim for aggregating and test solving proposals. Unlike other contests I've worked with where there's just a list of problems somewhere and solvers submit subjective ratings on the difficulty, we designed a system where every contributor could attempt any number of problem proposals in the entire pool, with the website tracking how many attempts and how many minutes it took before the solver got the correct three-digit answer (or gave up).

Moreover, problems were solved in isolation, one at a time, rather than in sets. In other words, any OTIS student could testsolve a proposal: they would be given a single problem, and had a timer on how long they had to submit answers for it, with immediate feedback. The testsolve ended once the correct answer was submitted or the time limit expired (the time limit varied by the author's suggested difficulty).

This gave an interesting source of data that was new to me; rather than subjective impressions based on text comments or ratings from people, you instead had "real" numerical data proving that, e.g., it was really possible to solve a certain problem in only 90 seconds, for instance, or that not a single person was able to solve a certain problem in less than 15 minutes. In fact, a big motivation for me to run this mock AIME is to see what kind of results a process like this would yield.

I think there are certainly some limits on how useful this data is, but I also think the data quality could be improved a lot in the future, since this was the first year we used Probase and we were just trying things out. A snapshot of this data Section 4.1, together with some commentary on why you shouldn't trust it too much.

Special thanks to Howard for doing all the software stack while the metaphorical ground was moving beneath his feet. The source code for the website is at https://github.com/howard36/probase.

§1.2.2 Concessions made during problem selection

The biggest issue I ran into in selecting the line-up was that the most popular problems tended to cluster up: there was a lot of geometry, and a lot of hard problems. We considered having two versions of the test (an AIME I and II), but found we had enough geometry and enough killer problems but nowhere near enough easy non-geometry to make that happen.

Ultimately, in choosing the problems, I had to balance picking the most interesting problems against skewing the difficulty higher than a "real" AIME. In the end, I ended up with a draft which I felt was perhaps 2.5 problems or so harder than an average AIME, which is pretty close to the limit that I'd be willing to push the difficulty bar.

There was also an observation from several of the editors that the test was perhaps too *clean*: the problems that were popular tended to rely on having one or two key ideas, after which there was often not too much calculation required. The actual AIME requires a bit more grinding. Again, this was a design concession I made with the goal of showing my favorite problems rather than trying to emulate the real AIME as much as possible.

§1.2.3 Favorites

Evan's personal favorites on the test are

- problem 7 (polynomial),
- problem 8 (1000 divides $\binom{n}{k}$),
- problem 12 (rubber bands),
- problem 14 (cardboard triangle),
- problem 15 (parabola).

§1.3 Contest analysis from Evan

Here is some commentary by Evan from a coach's perspective.

P1 Floors I initially thought this was a nice warm-up problem, but actually it turns out a lot of people got the answer wrong. In hindsight, I think I could have moved this later in the test.

P2 *BOATIS* This problem was fairly straightforward; you could optimize the solve time a bit if you saw certain symmetries like in the official solution, but eventually you'll get it.

P3 Perry the panda I think testers liked this problem mostly as a 1434 joke. I thought it was a good showcase of linearity of expectation, in the sense that you can use it to teach someone how the theorem works by comparing the "advanced solution" to the elementary one and illustrating how they're equivalent.

P4 a_{a_i} A fairly normal counting problem, where you think a bit about how to set up the cases and then execute it. This is standard fare for the AIME but a bit time-consuming.

P5 Pentagon $ABCDE$ This is a good practice problem for how to use power of a point well. (Alternatively, you can also eventually solve it by bashing Pythagorean theorem, but this takes a bit longer.)

P6 Floor equation This is actually pretty similar to problem 1, but more involved. I think it could have been pushed back a bit later, looking back.

P7 Divisible by $x^2 + x + 1$**.** I think this is a really great test of how well you understand polynomial division, with a fairly normal answer extraction (once you have shown the divisibility condition amounts to $a_2 + a_5 + a_8 = a_1 + a_4 + a_7 = a_0 + a_3 + a_6$, then you split into cases based on the common sum). So I think it's a nice fit for the AIME and a good example of how a problem can require multiple steps (get the condition then count it) in a natural way.

P8 $\binom{n}{k}$ I think this is a really nice ad-hoc problem. It has a clean statement and doesn't require a trick; instead it relies a lot on having good number sense instincts, rather than knowing a lot of standard theory.

We put this a bit later because it's easy to get wrong, but in hindsight I wish I had moved it a bit earlier because it's easy to jump into and fun to work on even if it takes a while. Nonetheless, the statement is so short I imagine a lot of people jumped to it anyway.

This would have been the perfect ARML tiebreaker problem.

P9 OH^2 I felt a bit bad placing this problem where it was, because I think you solve it really quickly if you have seen $H = a + b + c$ before, and otherwise it's tough and takes a lot more thinking. So this is a problem where students who have seen a lot have a major advantage.

P10 GCD/LCM This problem is fine content-wise, but is surprisingly pernicious for its placement. Looking back, I wish I had either placed it much later or (probably better) used an easier problem instead and moved other problems up.

P11 arcsin This problem went through a lot of changes during the development process, so we didn't have testsolving data on what became the final version. This turned out to bite us because we underestimated its difficulty.

I think what I've learned from this problem is that adding in a "boundary" condition like $a \le 2b$ and $b \le 2a$ is far more pernicious than it seems. That is, a lot of students were able to figure out that $\arcsin(a/x) + \arcsin(b/x) = 120^\circ$ implies $x^2 = \frac{4}{3}(a^2 - ab + b^2)$, but didn't notice the irreversibility of the logic, or really that when $a \gg b$ the original equation lacks any real solutions at all.

I had tried to account for this by deliberately rigging the numbers so that forgetting $a \le 2b$ and $b \le 2a$ would lead to an answer greater than 1000, however this guard-rail *still* turned out to be insufficient.

P12 Rubber bands This problem is tricky; but I think the beauty of this problem is that if you see the idea quickly, you can get it in almost no time at all. The fastest test-solve of this problem took only 3 minutes! I think problems like this are really nice for designing an exam to reward the best students, because the strongest contestants will be able to get the point *and* still have time to work out the other difficult problems.

P13 $n^{22}-1$ and $n^{40}-1$ I think this is a good test for orders. If you have spent a lot of time thinking about $a^m - 1$ expressions (which come up a lot in IMO-level contexts), it should be pretty clear how to proceed and you just need to carry it out.

But the nasty part is that the problem has a *lot* of moving parts, and even though it's not particularly casework-y, even really strong students might make a mistake somewhere along the line.

In other words, under exam conditions, it's really impressive to get this correct (i.e. not making a mistake) in the time limit. In particular, since the problem asks for "exactly one" to be divisible, the Venn diagram count should be $|A|+|B|-2|A\cap B|$; note the extra factor of 2. We saw a number of wrong answers where the student answered $|A|+|B|-|A\cap B|$.

P14 Cardboard triangle This is a problem you can eventually solve given enough hours. The hard part is getting it correct within 20 minutes; there are a lot of different approaches, and even with the "best" approach it's a bit of a time-sink. Picking a setup for this problem and then executing it is really challenging. So it wasn't a surprise to me when, looking through the statistics, a lot of the correct answers to this problem came from contestants who didn't answer many other problems.

P15 Parabola This problem was placed last because it requires working with a rotated conic, which is something that isn't covered well in either school or contest curriculums. If you don't know, for example, that a general conic is $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, you are kind of screwed.

However, if you *do* happen to have the necessary background, then actually this problem is a lot easier than its placement might suggest. (Whereas, say the cardboard triangle doesn't require anything besides just similar triangles.) Indeed, the fastest solve times by testsolvers on this problems are well under 10 minutes.

2 Solutions

§2.1 Answer key

P#	Description	Author	Answer
1	Floors	Skyler Mao	199
2	$BOATIS$	Skyler Mao	005
3	Perry the panda	Joshua Liu	434
4	a_{a_i}	Atticus Stewart	322
5	Pentagon $ABCDE$	Arjun Suresh and Tanishq Pauskar	648
6	Floor equation	Joshua Liu and Ashvin Sinha	506
7	Divisible by $x^2 + x + 1$	Atticus Stewart	831
8	$\binom{n}{k}$	Amogh Akella	132
9	OH^2	Jiahe Liu	432
10	GCD/LCM	Kenny Tran	026
11	arcsin	Neil Kolekar	640
12	Rubber bands	Ethan Lee	125
13	$n^{22} - 1$ and $n^{40} - 1$	Raymond Zhu	688
14	Cardboard triangle	Ethan Lee	759
15	Parabola	Wilbert Chu	046

§2.2 Links to problems on Art of Problem Solving

We posted the contest collection at:

https://aops.com/community/c3727730_2024_otis_mock_aime

§2.3 YouTube video presentation

A livestream of the solutions to all 15 problems can be found at:

https://youtu.be/e013kdQ153o

§2.4 Full solutions to all problems

Problem 1. Compute the number of real numbers x such that $0 < x \le 100$ and

$$x^2 = \lfloor x \rfloor \cdot \lceil x \rceil .$$

¶ **Answer.** 199

¶ **Problem author(s).** Skyler Mao

We consider two disjoint cases.

- If x is an integer, then it obviously works. Thus, we get 100 solutions for $x \in \{1, 2, \dots, 100\}$.
- Now suppose x is not an integer, so we may set $n = \lfloor x \rfloor$ and $n + 1 = \lceil x \rceil$. Then, the equation reads

  $$x^2 = n(n+1) \implies x = \sqrt{n(n+1)}.$$

  In other words, for each integer n, there is *at most* one non-integer value of x fulfilling the equation in the range $n < x < n + 1$, namely $x = \sqrt{n(n+1)}$.

  On the other hand, the inequality $n < \sqrt{n(n+1)} < n + 1$ is true for all $n \ge 1$. So each of the intervals $(1, 2)$, $(2, 3)$, ..., $(99, 100)$ does indeed have exactly one valid solution; namely, $x = \sqrt{n(n+1)}$ for the nth interval.

In total, we obtain $100 + 99 = \boxed{199}$ solutions.

Problem 2. Evan throws a dart at regular hexagon $BOATIS$, which lands at a uniformly random point inside the hexagon. The probability the dart lands in the interior of exactly one of the quadrilaterals $BOAT$ and $OTIS$ can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute $m+n$.

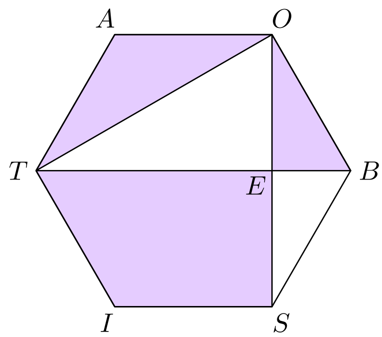
¶ **Answer.** 005

¶ **Problem author(s).** Skyler Mao

Define $E = \overline{OS} \cap \overline{TB}$, which by symmetry is also the midpoint of OS. Then we want to find the value of

$$\frac{[OAT] + [BOE] + [SITE]}{[BOATIS]}.$$

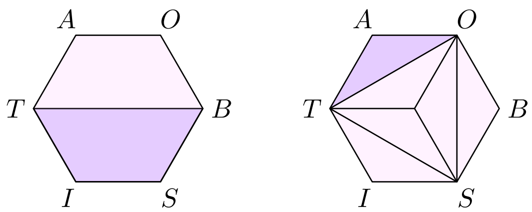
The numerator corresponds to the shaded region below.



Since $[BOE] = [BSE]$, it follows that

$$[OAT] + [BOE] + [SITE] = [OAT] + [BTIS].$$

However, in the following diagrams, we can see by symmetry that $[BTIS]$ is half of the hexagon while $[OAT]$ is one-sixth of the hexagon, respectively:



As such, it follows that the desired probability is $\frac{1}{2} + \frac{1}{6} = \frac{2}{3}$. This gives $2 + 3 = \boxed{5}$.

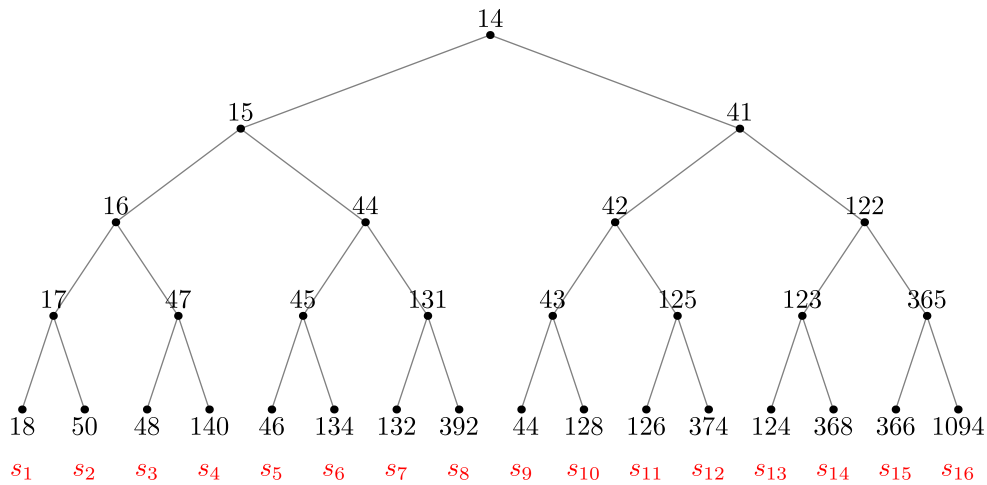
Problem 3. Perry the Panda is eating some bamboo over a five-day period from Monday to Friday (inclusive). On Monday, he eats 14 pieces of bamboo. Each following day, Perry eats either one less than three times the previous day or one more than the previous day, with equal probability. Compute the expected number of pieces of bamboo Perry has eaten throughout the week after the end of Friday.

¶ Answer. 434

¶ Problem author(s). Joshua Liu

We show two ways to write the solution; one elementary, and one using more advanced notation. These solutions are actually equivalent; that is, the second solution is a more opaque but succinct way of showing the first.

¶ Elementary solution. Consider the binary tree showing the 16 possible outcomes as shown below. (The student is NOT expected to actually compute any entries in the tree; however we wrote them out anyway just to make the figure easier to follow.)



Label the bottom entries of the tree by 1, 2, . . . , 16 from left to right and let s_i denote the sum of the entries from the root the i^{th} leaf. (For example, $s_1 = 14+15+16+17+18$ while $s_6 = 14 + 15 + 44 + 45 + 134$) The problem asks us to compute the value of

$$A = \frac{s_1 + s_2 + \cdots + s_{16}}{16}.$$

However, for this large sum it's easier to sum across the rows of the tree instead.

Claim — Imagine expanding the numerator of A so that it has $16 \cdot 5 = 80$ terms. Then in the numerator of A, a node of the tree in the k^{th} level from the top (for $k = 1, \dots, 5$) appears 2^{5-k} times among these 80 terms.

Proof. Each node is counted a number of times equal to the number of leaves below it. □

Claim — The sum of each row is four times that of the previous row.

Proof. The two direct descendants of a node labeled t are $t+1$ and $3t-1$ which have sum $4t$. □

Using these two key observations, the answer becomes

$$\begin{aligned} A &= \frac{2^4 \cdot 14 + 2^3 \cdot 4^1 \cdot 14 + 2^2 \cdot 4^2 \cdot 14 + 2^1 \cdot 4^3 \cdot 14 + 2^0 \cdot 4^4 \cdot 14}{2^4} \\ &= 14 \cdot (2^0 + 2^1 + 2^2 + 2^3 + 2^4) = \boxed{434}. \end{aligned}$$

¶ Advanced solution. Let X_i denote the random variable for the number of pieces of bamboo on the ith day, for $i = 1, \dots, 5$. (Thus, X_1 is always 14, while X_2 is either 15 or 41 with equal probability, etc.) And as usual, let $\mathbb{E}[\bullet]$ denote expected value. By linearity of expectation, the desired answer equals

$$\mathbb{E}[X_1 + \dots + X_5] = \mathbb{E}[X_1] + \dots + \mathbb{E}[X_5].$$

Claim — We have $\mathbb{E}[X_i] = 2\mathbb{E}[X_{i-1}]$ for $i = 2, \dots, 5$.

Proof. Armed with the magic that is linearity of expectation, the following proof is actually completely airtight:

$$\begin{aligned} \mathbb{E}[X_i] &= \frac{\mathbb{E}[X_{i-1}+1]}{2} + \frac{\mathbb{E}[3X_{i-1}-1]}{2} \\ &= \frac{(\mathbb{E}[X_{i-1}]+1) + (3\mathbb{E}[X_{i-1}]-1)}{2} \\ &= 2\mathbb{E}[X_{i-1}]. \end{aligned}$$ □

Hence, the expected pieces of bamboo eaten doubles every day. As such, our answer is (again by linearity of expectation)

$$\mathbb{E}[X_1] + \dots + \mathbb{E}[X_5] = 14(1 + 2 + 2^2 + 2^3 + 2^4) = 14 \cdot 31 = \boxed{434}.$$

Remark. When we compare the two solutions, we see that the elementary solution had two main “insights”: first to sum across the rows instead of by the branches, and then that the average value of each row is doubled compared to the preceding one. The advanced solution shows that *both* insights are actually each a special case of linearity of expectation. This demonstrates how powerful the linearity of expectation theory is: it “automagically” erases both main hurdles of the direct solution.

Problem 4. Let N denote the number of 7-tuples of positive integers $(a_1, \dots, a_7)$ such that for each $i = 1, \dots, 7$, we have

$$1 \le a_i \le 7 \quad \text{and} \quad a_{a_i} = a_i.$$

Compute the remainder when N is divided by 1000.

¶ Answer. 322

¶ Problem author(s). Atticus Stewart

Given a 7-tuple satisfying the given condition, let S be the set of all of integers appearing at least once. Then the condition is equivalent to the tuple satisfying $a_i = i$ for all $i \in S$, because this means that $a_{a_i} = i$, so $a_i = i$.

Now, we apply casework on $k = |S|$. There are $\binom{7}{k}$ ways to choose a subset of S with fixed cardinality k, and there are k^{7-k} ways to choose a_j for $j \notin S$. Thus,

$$\begin{aligned} N &= \sum_{k=1}^{7} \binom{7}{k} k^{7-k} \\ &= 7 \cdot 1 + 21 \cdot 32 + 35 \cdot 81 + 35 \cdot 64 + 21 \cdot 25 + 7 \cdot 6 + 1 \cdot 1 \\ &= 7 + 672 + 2835 + 2240 + 525 + 42 + 1 = 6322. \end{aligned}$$

This gives the answer $\boxed{322}$.

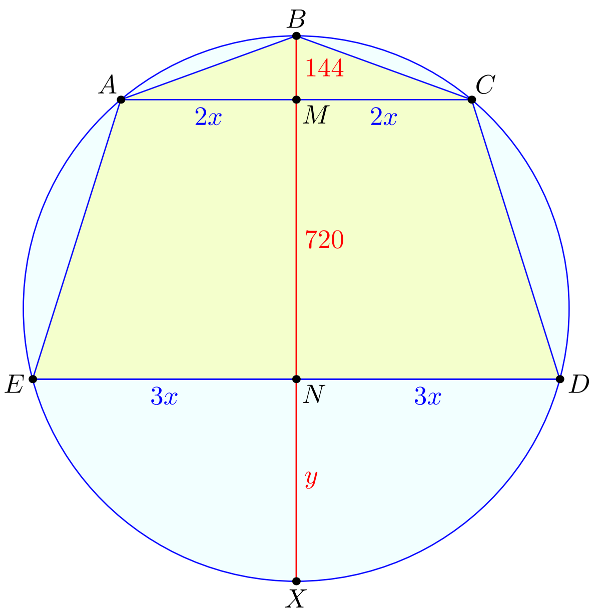
Problem 5. Convex pentagon $ABCDE$ is inscribed in circle ω such that $\frac{AC}{DE} = \frac{2}{3}$, $AE = CD$, and $AB = BC$. Suppose the distance from B to line AC is 144 and the distance from B to line DE is 864. Compute the radius of ω.

¶ **Answer.** 648

¶ **Problem author(s).** Arjun Suresh and Tanishq Pauskar

The condition $AE = CD$ implies that $ACDE$ is an isosceles trapezoid while the condition $AB = BC$ implies that B is the midpoint of arc AC (not containing D and E), as shown in the figure below. Note that this means the perpendicular bisectors of lines AC and DE coincide and pass through B.

Let M and N denote the midpoints of $\overline{AC}$ and $\overline{DE}$ (so that line BMN is perpendicular to the bases of the trapezoid $ACDE$). Let $AM = MC = 2x$, so $EN = ND = 3x$. Finally, let X be diametrically opposite from B and $NX = y$.



By power of a point on M and N, we obtain

$$144 \cdot (720 + y) = 4x^2$$
$$864 \cdot y = 9x^2.$$

Taking the quotient of these gives

$$\frac{4}{9} = \frac{144 \cdot (720 + y)}{864y} = \frac{720 + y}{6y} \implies y = 432.$$

Thus the diameter of ω is given by

$$BX = BM + MN + NX = 144 + 720 + y = 144 + 720 + 432 = 1296$$

and so the radius of ω is $\frac{1296}{2} = \boxed{648}$.

Problem 6. For each real number $k > 0$, let $S(k)$ denote the set of real numbers x satisfying

$$\lfloor x \rfloor \cdot (x - \lfloor x \rfloor) = kx.$$

The set of positive real numbers k such that $S(k)$ has exactly 24 elements is a half-open interval of length ℓ. Compute $1/\ell$.

¶ **Answer.** 506

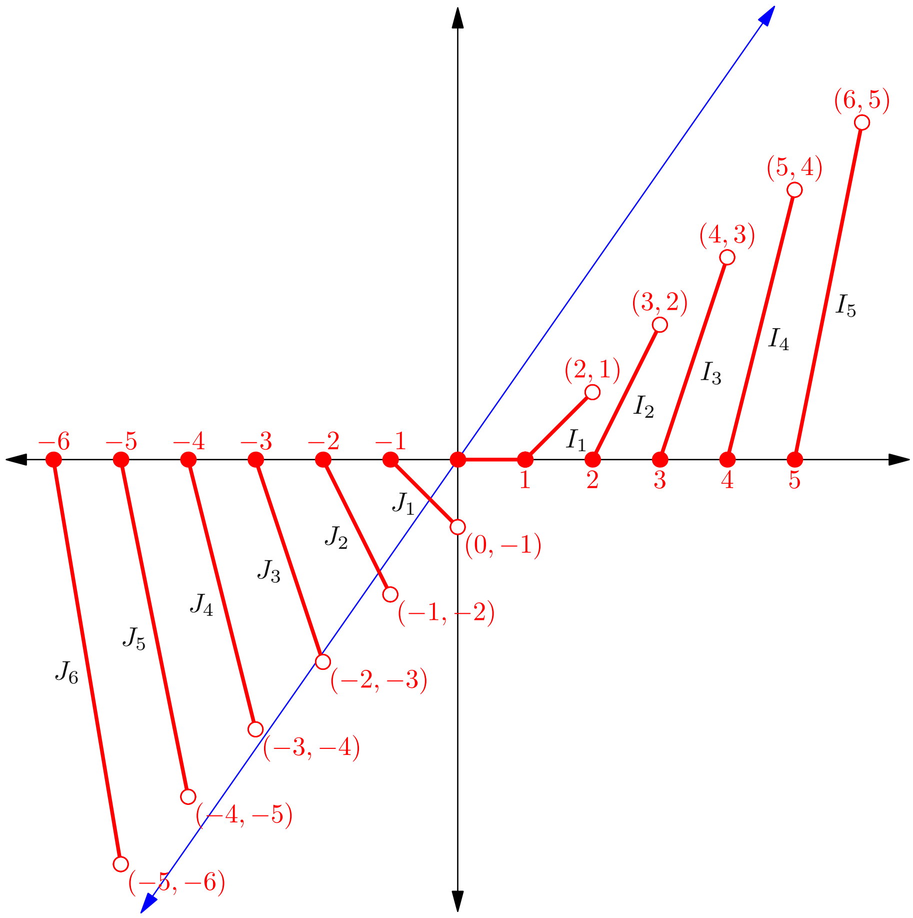
¶ **Problem author(s).** Joshua Liu and Ashvin Sinha

We present a (rigorous!) approach using the graph of the function, and one using only algebra. (This is a rare problem where graphical intuition is unusually useful.)

¶ **Graph-based solution.** Below in red, we have illustrated the graph of

$$y = \lfloor x \rfloor \cdot (x - \lfloor x \rfloor).$$

We are trying to choose $k > 0$ such that $y = kx$, (a line through the origin with slope k drawn in blue) meets the red graph in 24 points.



As shown above, we've labeled the line segments making up the red graph as follows:

- for $n \ge 1$, I_n is the segment joining $(n, 0)$ to $(n+1, n)$;
- for $n \ge 1$, J_n is the segment joining $(-n, 0)$ to $(-(n-1), -n)$.

We didn't label the line segment joining $(0, 0)$ to $(1, 0)$; the blue line always passes through the origin (and no other point on the x-axis).

We analyze where the blue line meets these segments:

- In Quadrant I: For $n \ge 1$, the blue line will meet I_n if and only if $k < \frac{n+1}{n}$. In particular, if the blue line meets I_n, it meets I_{n+1}, I_{n+2}, ...; thus $S(k)$ is infinite whenever $k < 1$.

  This case is not relevant for the problem, so we will henceforth assume always that $k \ge 1$ and hence no intersections occur in Quadrant I.

- In Quadrant III: for $n \ge 1$, the blue line will meet J_n if and only if $k < \frac{n+1}{n}$. Hence this means that if the blue line meets J_n, it will also meet J_{n-1}, J_{n-2}, ..., J_1.

Hence $|S(k)| = 24$ exactly when the blue line intersects J_1, J_2, ..., J_{23} but not J_{24}. (This gives 23 intersection points; the final one is the origin.) In other words, we want

$$\frac{24}{23} \le k < \frac{23}{22}.$$

The length of this interval is $\frac{23}{22} - \frac{24}{23} = \frac{1}{506}$, and hence the final answer should be $\boxed{506}$.

¶ Algebra-only solution. Suppose x was a solution, and write $n = \lfloor x \rfloor$, so $n \le x < n+1$. Then the equation reads $n(x-n) = kx$. We assume $n \ne k$, as $n = k$ implies $n^2 = 0 \implies k = 0$, contradiction. Then solving for x gives

$$x = \frac{n^2}{n-k} = n + \frac{kn}{n-k}.$$

In particular, for each value of n there is *at most* one candidate for x. However, this candidate only works if

$$0 \le \frac{kn}{n-k} < 1. \tag{2.1}$$

Conversely, we see that if (k, n) satisfy (2.1), then $x = n + \frac{kn}{n-k}$ does indeed satisfy the original equation. In other words, the cardinality of $S(k)$ is exactly equal to the number of integers n obeying such that (2.1). We solve this equation by analyzing three subcases:

- Note that if $k \le 1$, every negative integer n will be a solution to (2.1), so $S(k)$ will be infinite in that case. Therefore, we focus attention only on $k > 1$.
- We contend for $k > 1$, we cannot have $n > k$ in any solution to (2.1). If we did, then we would get $kn < n - k \implies (k-1)(n+1) < -1$, which is obviously absurd since $(k-1)(n+1) > 0$.
- Thus, we only need to examine the situation where $k > 1$ and $n - k < 0$. In that case, we in fact need $n \le 0$ and $kn > n - k$, which becomes

  $$-\frac{k}{k-1} < n \le 0.$$

  Hence, the number of valid solutions to (2.1) is exactly $\left\lceil \frac{k}{k-1} \right\rceil$.

Only the third case is relevant for us. Setting this ceiling equal to 24, we see that the half-open interval we seek is

$$23 < \frac{k}{k-1} \le 24 \iff \frac{24}{23} \le k < \frac{23}{22}$$

which is what we wanted.

Problem 7. Compute the number of 9-tuples $(a_0, a_1, \dots, a_8)$ of integers such that $a_i \in \{-1, 0, 1\}$ for $i = 0, 1, \dots, 8$ and the polynomial

$$a_8x^8 + a_7x^7 + \cdots + a_1x + a_0$$

is divisible by $x^2 + x + 1$.

¶ **Answer.** 831

¶ **Problem author(s).** Atticus Stewart

First, we give a necessary and sufficient condition for divisibility.

Claim — The divisibility occurs if and only if

$$a_6 + a_3 + a_0 = a_7 + a_4 + a_1 = a_8 + a_5 + a_2.$$

Proof. Notice that $x^3 \equiv 1 \pmod{x^2 + x + 1}$; in other words, we may take all exponents modulo 3. Consequently, the original polynomial satisfies the congruences

$$\begin{aligned} &a_8x^8 + a_7x^7 + \cdots + a_1x + a_0 \\ &\equiv a_8x^2 + a_7x^1 + a_6x^0 + a_5x^2 + \cdots + a_1x + a_0 \pmod{x^2 + x + 1} \\ &= (a_8 + a_5 + a_2)x^2 + (a_7 + a_4 + a_1)x + (a_6 + a_3 + a_0). \end{aligned}$$

This implies the result. □

To extract the count, we will do casework by the value of the common sum

$$S := a_6 + a_3 + a_0 = a_8 + a_5 + a_2 = a_7 + a_4 + a_1.$$

Owing to the symmetry, the contribution for a given S will be exactly the cube of the number of triplets achieving that sum S. This full list of such triplets is shown below.

Value of S	Number	Triplet(s), up to permutation
$S = 3$	1	$(1, 1, 1)$
$S = 2$	3	$(1, 1, 0)$
$S = 1$	6	$(1, 0, 0)$ and $(1, 1, -1)$
$S = 0$	7	$(1, -1, 0)$ and $(0, 0, 0)$
$S = -1$	6	$(-1, 0, 0)$ and $(-1, -1, 1)$
$S = -2$	3	$(-1, -1, 0)$
$S = -3$	1	$(-1, -1, -1)$

(Note that the triplets for S and $-S$ are in obvious bijection, which limits us to only four different cases. As a further check, the sum of the number of triplets is indeed $1 + 3 + 6 + 7 + 6 + 3 + 1 = 27 = 3^3$.)

Therefore, the final answer is

$$1^3 + 3^3 + 6^3 + 7^3 + 6^3 + 3^3 + 1^3 = \boxed{831}.$$

Problem 8. Let $n \ge k \ge 1$ be integers such that the binomial coefficient $\binom{n}{k}$ is a multiple of 1000. Compute the smallest possible value of $n + k$.

¶ Answer. 132

¶ Problem author(s). Amogh Akella

Write
$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n \cdot (n-1) \cdot \dots \cdot (n-k+1)}{1 \cdot 2 \cdot \dots \cdot k}.$$
We start with the following sufficient (but not necessary) condition:

Claim — Whenever $\binom{n}{k}$ is divisible by 1000, the set
$$S = \{n, n-1, \dots, n-k+1\}$$
must contain both a multiple of 125 and a multiple of 8.

Proof. The numerator of $\binom{n}{k}$ must contain 3 more factors of 5 than the denominator. Since there are k consecutive terms in both the numerator and denominator, the number of multiples of 5 in the numerator is at most 1 more than the denominator. The same holds with multiples of 25. Therefore, one of the numbers in the numerator must contain a multiple of 125.

An identical argument applies with powers of 2, meaning we must have a multiple of 8 in S. □

We now split our search into a few cases:

- Suppose $125 \in S$. Then if S contains multiples of 8, either $120 \in S$ or $128 \in S$. We consider each possibility in turn.
  - If $128 \in S$, then $n \ge 128$ and hence $125 \in S$ forces $k \ge 4$. Thus $n+k \ge 128+4$ in this case and the number $\binom{128}{4}$ does indeed work.
  - If $120 \in S$, then from $n \ge 125$ it follows $k \ge 6$. However, the binomial $\binom{125}{6}$ does not happen to be divisible by 8. So $n + k > 125 + 6$ in this case.
- If $125 \notin S$, then the multiple of 125 in S must be at least 250. So $n \ge 250$ and this case is much worse than the preceding one.

In all cases we examined, we found that $n+k \ge 132$ must hold. On the other hand, we also saw that $\binom{128}{4}$ does indeed work. Since we have a matching bound and construction, our answer is $\boxed{132}$.

Remark. The claim above also follows directly from *Kummer theorem*. However, Kummer's theorem is overkill in this case; the weaker (and easier to notice) claim is enough to get the desired bound.

Problem 9. Let ω be a circle with center O and radius 12. Points A, B, and C are chosen uniformly at random on the circumference of ω. Let H denote the orthocenter of $\triangle ABC$. Compute the expected value of OH^2.

¶ Answer. 432

¶ Problem author(s). Jiahe Liu

¶ First solution using complex numbers. We first consider the analogous problem where ω has radius 1 instead of 12 (and will multiply the answer by 144 to account for scaling at the end). Denote the points A, B, and C, by a, b, and c in the complex plane, lying on the unit circle. It is well known that $H = a + b + c$; therefore,

$$\begin{aligned} OH^2 &= |a+b+c|^2 = (a+b+c)(\overline{a+b+c}) = (a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \\ &= 3 + \frac{a}{b} + \frac{b}{a} + \frac{b}{c} + \frac{c}{b} + \frac{c}{a} + \frac{a}{c}. \end{aligned}$$

We now claim the six nonconstant terms each have expected value 0. To see why this is true, observe that if we fix b, $\frac{a}{b}$ is just a shifted counterclockwise by an angle of $\arg b$, so it still reflects a uniform point on the unit circle. Therefore, the expected value of this term is just 0. A similar argument applies to each of the five other terms. Thus, our expected value is just 3.

Finally, we multiply by 144 to account for the scaling at the start of the solution to obtain $144 \cdot 3 = \boxed{432}$ as our answer.

¶ Second solution using trig. As before, we will work on the unit circle, scaling at the end. In addition, it will be better to work with the centroid G instead, since $OG = \frac{1}{3}OH$ from the Euler line.

To that end, we let $A = (\cos\alpha, \sin\alpha)$, $B = (\cos\beta, \sin\beta)$, and $C = (\cos\gamma, \sin\gamma)$, where $\alpha, \beta, \gamma \in [0, 2\pi]$. Then

$$G = \left(\frac{\cos\alpha + \cos\beta + \cos\gamma}{3}, \frac{\sin\alpha + \sin\beta + \sin\gamma}{3}\right).$$

From this we are able to compute:

$$\begin{aligned} OH^2 &= 9OG^2 \\ &= (\cos\alpha + \cos\beta + \cos\gamma)^2 + (\sin\alpha + \sin\beta + \sin\gamma)^2 \\ &= \sum_{\text{cyc}} \left[\cos^2\alpha + 2\cos\alpha\cos\beta + 2\sin\alpha\sin\beta + \sin^2\alpha\right] \\ &= \sum_{\text{cyc}} \left[1 + 2\cos(\alpha - \beta)\right] \\ &= 3 + 2\cos(\alpha-\beta) + 2\cos(\beta-\gamma) + 2\cos(\gamma-\alpha). \\ \implies \mathbb{E}[OH^2] &= 3 + 2\mathbb{E}[\cos(\alpha-\beta)] + 2\mathbb{E}[\cos(\beta-\gamma)] + 2\mathbb{E}[\cos(\gamma-\alpha)] \end{aligned}$$

according to linearity of expectation. From the symmetry of the cosine function, each of the latter three expectations vanishes. This establishes $\mathbb{E}[OH^2] = 3$ as before.

Problem 10. Compute the number of integers $b \in \{1, 2, \dots, 1000\}$ for which there exists positive integers a and c satisfying

$$\gcd(a, b) + \operatorname{lcm}(b, c) = \operatorname{lcm}(c, a)^3.$$

¶ **Answer.** 026

¶ **Problem author(s).** Kenny Tran

¶ **Solution by Ritwin Narra.** Note that c divides both $\operatorname{lcm}(b, c)$ and $\operatorname{lcm}(c, a)$, so it follows that $c \mid \gcd(a, b)$ and thus c divides both a and b. Thus, we may eliminate c from the equation completely and obtain

$$\gcd(a, b) + b = a^3.$$

Claim — We can parameterize solutions as $(a, b, c) = (a, a^3 - d, c)$ where $c \mid d$ and $d \mid a$ for positive integers a.

Proof. We already see every solution must take this form from the discussion above. Conversely, all equations on the curve can be verified to work. □

Note that $a^3 - a \le b \le a^3 - 1$ in any valid (a, b, c). As the closed intervals $[a^3 - a, a^3 - 1]$ are disjoint across $a \in \mathbb{Z}_{>0}$, the cases for different values of a do not overlap. For a fixed a, the number of possible values of b is given by $\tau(a)$, the number of positive divisors of a. We need values of a with $2 \le a \le 10$, as shown below:

a	2	3	4	5	6	7	8	9	10
$\tau(a)$	2	2	3	2	4	2	4	3	4

Hence the final answer is

$$\sum_{a=2}^{10} \tau(a) = \boxed{26}.$$

Problem 11. Compute the number of ordered triples of positive integers (a, b, n) satisfying $\max(a, b) \le \min(\sqrt{n}, 60)$ and

$$\operatorname{Arcsin}\left(\frac{a}{\sqrt{n}}\right) + \operatorname{Arcsin}\left(\frac{b}{\sqrt{n}}\right) = \frac{2\pi}{3}.$$

¶ **Answer.** 640

¶ **Problem author(s).** Neil Kolekar

For $x \ge \max(a, b)$, define $L(x) := \operatorname{Arcsin}(a/x) + \operatorname{Arcsin}(b/x)$.

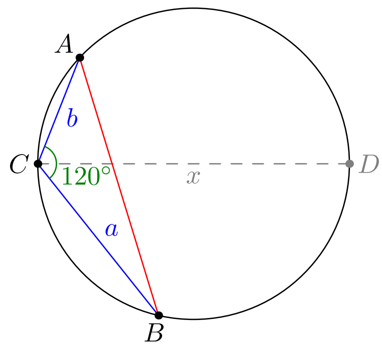
Claim — The equation $L(x) = \frac{2\pi}{3}$ has one solution if $a \le 2b$ and $b \le 2a$, else zero.

Proof. Now, the function L is strictly decreasing, so there is at most one such value of x; and there is one if and only if

$$\frac{2\pi}{3} \le \max_{x \ge \max(a,b)} L(x) = L(\max(a, b)) = \frac{\pi}{2} + \operatorname{Arcsin} \min(a/b, b/a). \qquad \square$$

Claim — If $L(x) = \frac{2\pi}{3}$, then $x = \frac{2}{\sqrt{3}} \cdot \sqrt{a^2 - ab + b^2}$.

Proof. Construct triangle ABC inscribed in a circle with diameter x with $CA = b$ and $CB = a$, as shown below. By construction, we have $\angle C = 120^\circ$.



By the (extended) law of sines and the law of cosines in tandem, we thus have

$$AB = \frac{\sqrt{3}}{2} x = \sqrt{a^2 - ab + b^2}. \qquad \square$$

We can thus put the two claims together to work out the extraction.

Claim — The answer is the number of pairs $(a, b) \in \{1, \dots, 60\}^2$ obeying $a \le 2b$, $b \le 2a$, and $a + b \equiv 0 \pmod 3$.

Proof. The first two conditions were already discussed. For the last condition, we are interested in when $n = x^2 = \frac{4}{3}(a^2 - ab + b^2)$ is an integer; that is, $3 \mid a^2 - ab + b^2$. From $a^2 - ab + b^2 = (a + b)^2 - 3ab$ the result follows. $\square$

To produce the final tally, we first count the number of pairs (a, b) with the added condition $a \le b$ by casework over the value of a. For each a, let $\ell(a)$ be the number of choices of b for that a.

- For $1 \le a \le 30$, we have $\ell(a) = \left\lceil \frac{a+1}{3} \right\rceil$, for $b \in \{2a, 2a-3, 2a-6, \dots\}$.
- For $31 \le a \le 60$, one can check $\ell(a) = 21 - \left\lfloor \frac{a}{3} \right\rfloor - (a \bmod 3)$.

Summing gives

$$\begin{aligned} \sum_{a=1}^{30} \ell(a) &= (1+1+2) + (2+2+3) + (3+3+4) + \dots + (10+10+11) \\ &= 4 + 7 + 10 + \dots + 31 = 175. \\ \sum_{a=31}^{60} \ell(a) &= (10+9+10) + (9+8+9) + (8+7+8) + \dots + (1+0+1) \\ &= 29 + 26 + 23 + \dots + 2 = 155. \end{aligned}$$

This gives a total count of $175 + 155 = 330$. Similarly, there are 330 pairs with $a \ge b$. Finally, there are 20 pairs with $a = b$ (namely $(3,3)$, $(6,6)$, …, $(60,60)$). So the final answer is $330 + 330 - 20 = \boxed{640}$.

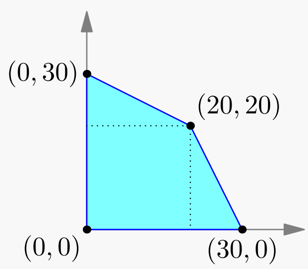
Remark (Krishna Pothapragada)**.** There is actually a way to compute the final answer extraction without casework, although finding it is perhaps more difficult than the original problem. The conditions $a \le 2b$, $b \le 2a$, and $a + b \equiv 0 \pmod 3$ can be reparametrized as saying there exist nonnegative integers (x, y) not both zero such that

$$a = 2x + y, \qquad b = x + 2y.$$

Indeed, $x = \frac{2a-b}{3}$ and $y = \frac{2b-a}{3}$. So the answer actually equals the number of lattice points besides the origin bounded by

$$x \ge 0, \quad y \ge 0, \quad 2x + y \le 60, \quad 2y + x \le 60.$$

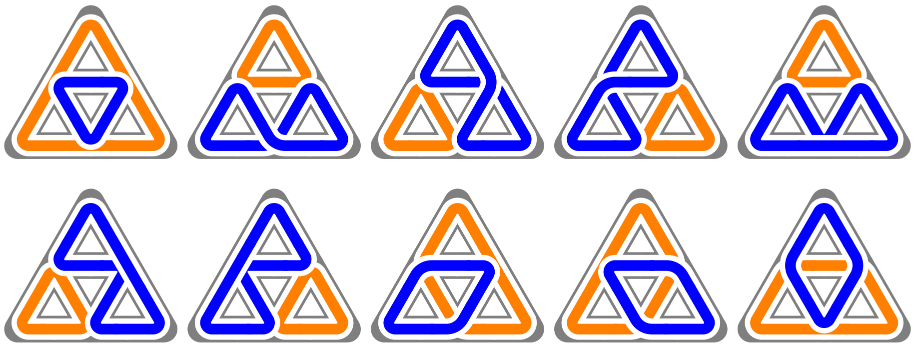
These four constraints bound a quadrilateral with vertices at $(0,0)$ $(0,30)$, $(20,20)$, and $(30,0)$, as shown below; it has area $20^2 + 100 + 100 = 600$.



To count the number of lattice points we use **Pick's theorem**; in the usual notation we have $I + \frac{B}{2} - 1 = 600$. It is easy to check that $B = 30 + 30 + 10 + 10 = 80$, so $I = 600 - 39 = 561$. Remembering to delete the origin, we get $I + B - 1 = 640$ as desired.

Problem 12. Let $\mathcal{G}_n$ denote a triangular grid of side length n consisting of $\frac{(n+1)(n+2)}{2}$ pegs. Charles the Otter wishes to place some rubber bands along the pegs of $\mathcal{G}_n$ such that every edge of the grid is covered by exactly one rubber band (and no rubber band traverses an edge twice). He considers two placements to be different if the sets of edges covered by the rubber bands are different or if any rubber band traverses its edges in a different order. The ordering of which bands are over and under does not matter.

For example, Charles finds there are exactly 10 different ways to cover $\mathcal{G}_2$ using exactly two rubber bands; the full list is shown below, with one rubber band in orange and the other in blue.



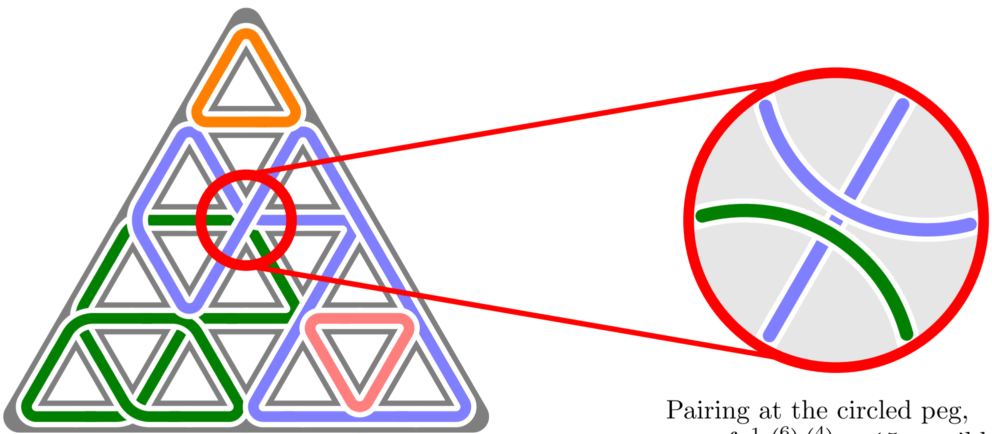
Let N denote the total number of ways to cover $\mathcal{G}_4$ with *any number* of rubber bands. Compute the remainder when N is divided by 1000.

¶ **Answer.** 125

¶ **Problem author(s).** Ethan Lee

The key idea is that one can solve the problem by looking only at individual pegs rather than the entire picture at a whole.

At each peg, define two edges using that peg to be a "pair" if there is a rubber band through both edges. Then we obtain a "pairing" of the edges at each peg. For example, the figure below shows an example placement for $\mathcal{G}_4$, with one peg circled in red, and the associated pairing of its six edges.



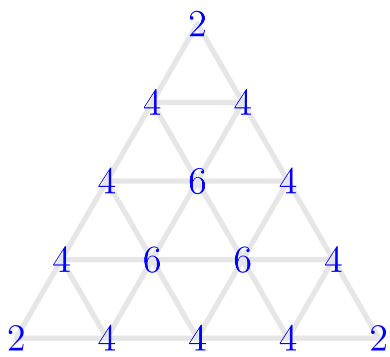
Example placement of $\mathcal{G}_4$

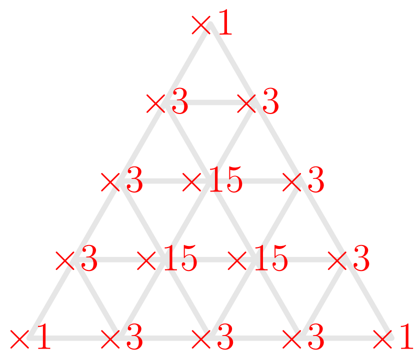
Pairing at the circled peg, one of $\frac{1}{3!}\binom{6}{2}\binom{4}{2} = 15$ possible.

Claim — Configurations of rubber bands are in bijection with choices of pairings at each each peg.

Proof. This is essentially by definition; if two configurations of rubber bands have the same pairs of edges, each rubber band in one configuration is also in the same position in the other configuration, so the configurations are identical. □

Consider the picture $\mathcal{G}_4$ below on the left; at each peg we've written the number of edges emanating it in blue. There is only one pairing at the three corners marked 2. At each of the nine pegs marked 4, there are $\frac{1}{2!}\binom{4}{2} = 3$ possible pairings (if we denote the edges by a, b, c, d, they are $\{ab, cd\}$, $\{ac, bd\}$ and $\{ad, bc\}$). Finally, at each of the three pegs marked 6, there are $\frac{1}{3!}\binom{6}{2}\binom{4}{2} = 15$ possible pairings. Thus, the number of ways is given by the multiplication in the right figure in red.





In summary, we obtain

$$N = 15^3 \cdot 3^9.$$

Observe that $N \equiv 0 \pmod{125}$ and $N \equiv (-1)^3 \cdot 3^9 \equiv -3 \pmod 8$. This gives us a unique answer modulo 1000, namely $N \equiv \boxed{125} \pmod{1000}$.

Problem 13. Let S denote the sum of all integers n such that $1 \le n \le 2024$ and exactly one of $n^{22}-1$ and $n^{40}-1$ is divisible by 2024. Compute the remainder when S is divided by 1000.

¶ Answer. 688

¶ Problem author(s). Raymond Zhu

Notice that $n = 2024$ does not work; and when $1 \le n < 2024$, whenever the number n works so does the number $2024 - n$. Therefore, we will have

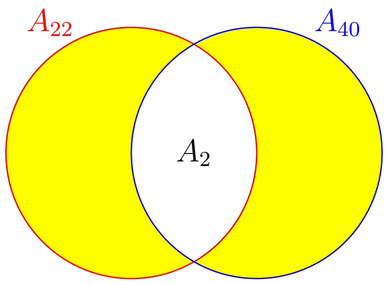
$$S = 1012M$$

where M is the *number* of values of n that work.

We know that $\gcd(n^{22}-1, n^{40}-1) = n^{\gcd(22,40)} - 1 = n^2 - 1$. So we are going to analyze three cases:

- Let A_{22} be the number of values of n with $n^{22} - 1 \equiv 0 \pmod{2024}$.
- Let A_{40} be the number of values of n with $n^{40} - 1 \equiv 0 \pmod{2024}$.
- Let A_2 be the number of values of n with $n^2 - 1 \equiv 0 \pmod{2024}$, which is the number of values counted in both A_{22} and A_{40}.

This gives us the following Venn diagram, with the values for M shaded in yellow.



Then

$$M = A_{22} + A_{40} - 2A_2.$$

and therefore it suffices to compute each A_e.

We now prime-factorize the year as

$$2024 = 2^3 \cdot 11 \cdot 23.$$

To proceed further, we need two key facts (which are standard in olympiad number theory):

- For each even integer e, $n^e \equiv 1 \pmod 8$ if and only if n is odd.
- If p is a prime and $e \ge 1$ is any exponent, then modulo p, there are exactly $\gcd(p-1, e)$ residues modulo p whose e^{th} power is $1 \pmod p$.

The first fact is obvious by taking $(2k+1)^2 \pmod 8$. The second one follows in general by taking a primitive root modulo p. However, for this problem we only need it for $e \in \{2, 22, 40\}$ and $p \in \{11, 23\}$, in which case one could also just describe the relevant residues manually (thus making the use of primitive roots unnecessary for the cases this problem uses):

$$\begin{aligned} n^{22} \equiv 1 \pmod{11} &\iff n \equiv \pm 1 \pmod{11} \\ n^{22} \equiv 1 \pmod{23} &\iff n \not\equiv 0 \pmod{23} \\ n^{40} \equiv 1 \pmod{11} &\iff n \not\equiv 0 \pmod{11} \\ n^{40} \equiv 1 \pmod{23} &\iff n \equiv \pm 1 \pmod{23} \\ n^{2} \equiv 1 \pmod{11} &\iff n \equiv \pm 1 \pmod{11} \\ n^{2} \equiv 1 \pmod{23} &\iff n \equiv \pm 1 \pmod{23}. \end{aligned}$$

Either way, together with the Chinese Remainder Theorem, this gives us an explicit formula for A_e for every even exponent e, by looking modulo each of 8, 11 and 23:

$$A_e = 4 \cdot \gcd(10, e) \cdot \gcd(22, e).$$

In particular,

$$\begin{aligned} A_{22} &= 4 \cdot 2 \cdot 22 = 176 \\ A_{40} &= 4 \cdot 10 \cdot 2 = 80 \\ A_2 &= 4 \cdot 2 \cdot 2 = 16. \end{aligned}$$

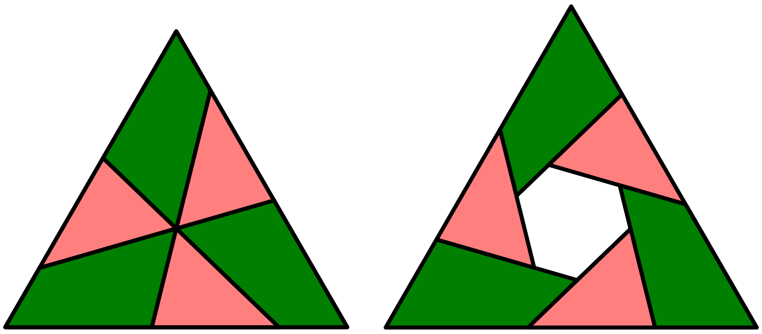
Therefore, we have

$$M = 176 + 80 - 2 \cdot 16 = 224.$$

To finish, we have

$$S = 1012 \cdot 224 \equiv 12 \cdot 224 \equiv \boxed{688} \pmod{1000}.$$

Problem 14. Ritwin the Otter has a cardboard equilateral triangle. He cuts the triangle with three congruent line segments of length x spaced at 120° angles through the center, obtaining six pieces: three congruent triangles and three congruent quadrilaterals. He then flips all three triangles over, then rearranges all six pieces to form another equilateral triangle with an equiangular hexagonal hole inside it, as shown below.

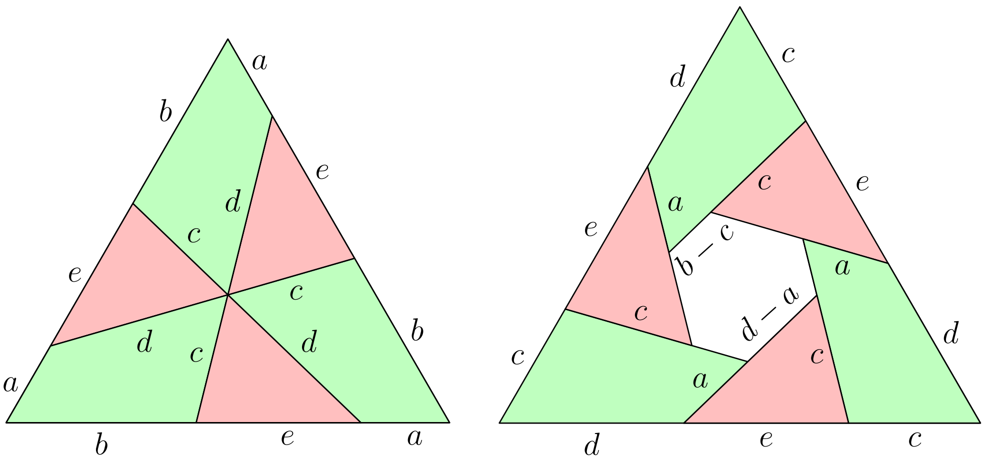


Given that the side lengths of the hole are 3, 2, 3, 2, 3, 2, in that order, the value of x can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute $m+n$.

¶ Answer. 759

¶ Problem author(s). Ethan Lee

Label the lengths in the picture as in the figure below, so that $x = c + d$.



Now we make several insights:

- First, we compare areas. Recall that an equilateral triangle with side length s has an area of $\frac{\sqrt{3}}{4}s^2$. For the left figure, the area is $\frac{\sqrt{3}}{4}(a+b+e)^2$, and for the right figure, it's $\frac{\sqrt{3}}{4}(c+d+e)^2$. The difference between these two areas equals the area of the hexagonal gap, which can be obtained by subtracting three equilateral triangles of side length 2 from a larger triangle of side length 7. Hence, the gap's area is

$$\frac{\sqrt{3}}{4}(7^2 - 3 \cdot 2^2) = \frac{\sqrt{3}}{4} \cdot 37.$$

Therefore, the difference in the areas of the two triangles is

$$\frac{\sqrt{3}}{4}(c+d+e)^2 - \frac{\sqrt{3}}{4}(a+b+e)^2 = \frac{\sqrt{3}}{4}\cdot 37,$$

which simplifies to

$$(c+d+e)^2 - (a+b+e)^2 = 37. \tag{2.2}$$

- The hexagonal gap in the right figure has side lengths of $b-c$ and $d-a$. We can deduce that

  $$b-c=2, \quad d-a=3 \tag{2.3}$$

  because the alternative ($b-c=3$, $d-a=2$) would result in a negative value for the left-hand side of (2.2).

- In the left figure, the red triangles with sides c, d, e are similar to the larger triangles with sides $a+e$, $b+e$, $c+d$. So, we have

  $$c:d:e = (a+e):(b+e):(c+d). \tag{2.4}$$

We have thus obtained five constraints (one from (2.2), two from (2.3), two from (2.4)) in five variables a, b, c, d, e; we show how to solve them.

From (2.3) we have $a=d-3$ and $b=c+2$. We substitute these into (2.2) to eliminate a and b completely; we find

$$(c+d+e)^2 - (c+d+e-1)^2 = 37.$$

Defining $S := c+d+e$ for brevity, we can solve $S^2-(S-1)^2=37$ to get $S=19$.

Doing the same substitution of (2.3) into (2.4) to eliminate the variables a and b gives

$$c:d:e = (d+e-3):(c+e+2):(c+d).$$

Now we use the following fact:

Fact. If $u:v:w = u':v':w'$, then both ratios are equal to $u+u' : v+v' : w+w'$.

Applying this fact to the above ratios, we get

$$\begin{aligned} c:d:e &= (c+d+e-3):(c+d+e+2):(c+d+e) \\ &= (S-3):(S+2):S \\ &= 16:21:19. \end{aligned}$$

Therefore,

$$c = \frac{16}{56}\cdot 19, \qquad d = \frac{21}{56}\cdot 19, \qquad e = \frac{19}{56}\cdot 19.$$

Now we compute

$$x = c+d = \frac{19}{56}\cdot(16+21) = \frac{703}{56},$$

which gives a final answer of $703+56 = \boxed{759}$.

> **Remark.** Amazingly, the obvious "law of cosines" on the triangle with side lengths c, d, e (which has a 60° angle opposite e) gives a relation which is *redundant* (it says $e^2 = c^2+d^2-cd$). In principle, one could have used this equation instead of (2.2).

Problem 15. A parabola in the Cartesian plane is tangent to the x-axis at $(1, 0)$ and to the y-axis at $(0, 3)$. The sum of the coordinates of the vertex of the parabola can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute $m + n$.

¶ Answer. 046

¶ Problem author(s). Wilbert Chu

¶ Solution by proposer. This solution splits cleanly into two halves. The first is computing the equation of the parabola, and the second is extracting the vertex from the equation.

We start by identifying the equation of the parabola.

Claim — The parabola is the zero locus of

$$P(x, y) = 9x^2 - 18x + 9 - 6y + y^2 - 6xy.$$

Proof. Since P is a conic, its equation should take the form

$$P(x, y) = Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

Our goal is determine all the constants A, B, ..., F; these constants will be unique up to scaling. Since $F \neq 0$ (as the parabola may not pass through the origin; it is already tangent to both axes at different points), we will assume by scaling $F = 9$.

Since the parabola is tangent to the x-axis at $x = 1$, we have that $P(x, 0)$ is of the form $k(x-1)^2$ for some k. Similarly, $P(0, y)$ is of the form $\ell(y-3)^2$ for some ℓ. Then these two facts forces

$$P(x, y) = 9x^2 - 18x + 9 - 6y + y^2 + Bxy.$$

This equation can only be a parabola if the second degree terms $9x^2 + y^2 + Bxy$ is a perfect square, so either $B = 6$ or $B = -6$. In the first case, we have

$$P(x, y) = (3x + y)^2 - 6(3x + y) + 9 = (3x + y - 3)^2$$

which is a single line, not a parabola. Therefore we need $B = -6$ and the proof is complete. □

In order to extract the vertex from this equation, we are going to reparametrize using a new coordinate system (X, Y) defined by

$$\begin{aligned} X &= 3x - y \\ Y &= x + 3y. \end{aligned}$$

This transformation corresponds to a rotation together with a scaling by a factor of $\sqrt{10}$. In this new coordinate, system, we can recover

$$P(x, y) = (3x - y)^2 - \frac{24}{5}(3x - y) - \frac{18}{5}(x + 3y) + 9$$

$$= X^2 - \frac{24}{5}X - \frac{18}{5}Y + 9.$$

In other words, in the new coordinate system, the parabola corresponds to

$$\frac{18}{5}Y = X^2 - \frac{24}{5}X + 9 = \left(X - \frac{12}{5}\right)^2 + \frac{81}{25}$$

or

$$Y = \frac{5}{18}\left(X - \frac{12}{5}\right)^2 + \frac{9}{10}$$

Written this way, it is clear that the vertex of the parabola is given exactly by

$$(X, Y) = \left(\frac{12}{5}, \frac{9}{10}\right).$$

Translating back to the original coordinate system, we solve $3x - y = \frac{12}{5}$ and $x + 3y = \frac{9}{10}$ to recover

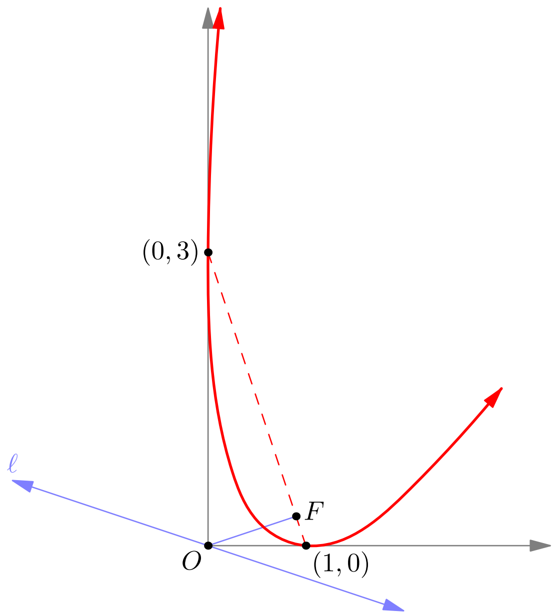
$$(x, y) = \left(\frac{81}{100}, \frac{3}{100}\right).$$

Finally, $\frac{81+3}{100} = \frac{21}{25}$, for an answer of $\boxed{46}$.

Remark. More generally, if the parabola had been tangent at $(p, 0)$ and $(0, q)$, the coordinates of the vertex would be

$$\left(\frac{pq^4}{(p^2+q^2)^2}, \frac{p^4 q}{(p^2+q^2)^2}\right).$$

¶ Second solution, by Karn Chutinan. Let ℓ and F be the directrix and focus of F of this parabola $\mathcal{P}$ respectively. Let O be the origin.



We use the following facts, which are proved at https://davidaltizio.web.illinois.edu/geom-conics.pdf.

Lemma

Line ℓ is the polar of F with respect to $\mathcal{P}$. Furthermore, the tangents from a point to $\mathcal{P}$ are perpendicular to each other if and only if that point lies on ℓ.

Lemma (Optical Property)

The tangent at a point P on the parabola bisects PF and the altitude from P to ℓ.

As such, it follows that F is the altitude from O to the line from $(0,3)$ to $(1,0)$, i.e. $3x + y = 3$, so $F = (\frac{9}{10}, \frac{3}{10})$.

By the above facts, O lies on the directrix. Moreover, by the optical property, the reflection of OF over either tangent is ℓ, so the directrix has equation $x = -3y$. From here we compute the midpoint of the altitude from F to ℓ, which yields $V = (\frac{81}{100}, \frac{3}{100})$ which gives the answer of $\frac{21}{25}$, and $21 + 25 = \boxed{46}$.

Remark. In generality, a set of points P and lines L such that $|P| + |L| = 5$ typically determine $2^{\min\{|P|,|L|\}}$ conics C through all points in P and tangent to all lines in L. Here, we take P to be $(0,3)$ and $(1,0)$, and L to be the axes and the line at infinity.

See https://www.youtube.com/watch?v=X83vac2uTUs for some exposition.

3 Statistics

§3.1 Total score statistics

Total score	Frequency
Total score = 0	2
Total score = 1	5
Total score = 2	3
Total score = 3	5
Total score = 4	9
Total score = 5	9
Total score = 6	15
Total score = 7	16
Total score = 8	11
Total score = 9	6
Total score = 10	3
Total score = 11	4
Total score = 12	1
Total score = 13	1
Total score = 14	1
Total score = 15	1

§3.2 Number of correct answers per problem

P#	Description	#Correct
1	Floors	61
2	$BOATIS$	81
3	Perry the panda	82
4	a_{a_i}	60
5	Pentagon $ABCDE$	72
6	Floor equation	35
7	Divisible by $x^2 + x + 1$	45
8	$\binom{n}{k}$	50
9	OH^2	33
10	GCD/LCM	19
11	arcsin	6
12	Rubber bands	7
13	$n^{22} - 1$ and $n^{40} - 1$	13
14	Cardboard triangle	8
15	Parabola	7

4 Behind the scenes

§4.1 Testsolving statistics

Read Section 1.2.1 for more about the Probase system we used for these testsolves. To summarize, testsolvers had a time limit and unlimited answer checks. We record

- the fastest solve time (in seconds) among testsolvers who got the answer in one try;
- the median solve time among testsolvers who got the answer in one try;
- the number of testsolvers that got the correct answer on the first try;
- the number that eventually got the correct answer before the timer expired (but may have required multiple tries). The timer was set as follows: the author was asked to estimate the difficulty of each problem on a scale of one light bulb to five light bulbs, where n light bulbs was a problem suitable for problem $3n$ on the AIME. Then solvers got $5(n+1)$ minutes for their testsolving session. (However, they could give up early if they didn't want to wait for the full time.)

Here's the data. Problems 11 was altered a lot during the editing process and the revised version never got a clean testsolve, so we don't have data for that problem.

P#	About	Fastest	Median	Correct	Finished
1	Floors	0:20	1:14	26	42
2	$BOATIS$	0:09	1:44	25	36
3	Perry the panda	0:10	1:32	48	60
4	a_{a_i}	1:39	3:31	11	16
5	Pentagon $ABCDE$	1:26	5:07	10	13
6	Floor equation	4:33	7:06	6	9
7	Divisible by x^2+x+1	1:38	4:18	13	16
8	$\binom{n}{k}$	0:34	3:57	10	17
9	OH^2	0:22	1:58	11	16
10	GCD/LCM	4:21	15:02	3	5
11	arcsin				
12	Rubber bands	3:00	19:19	2	3
13	$n^{22}-1$ and $n^{40}-1$	7:00	7:00	1	4
14	Cardboard triangle	19:12	21:29	4	4
15	Parabola	7:52	16:18	4	7

This data is mostly for comedic value than anything else (e.g., "holy crap someone got this in x seconds WTF??"). It shouldn't be taken too seriously because almost no effort was put into uniformizing the conditions in which testsolvers tried the problems, something which we hope to lock down much more carefully in future uses of Probase.

For example, here are issues with the data:

- We didn't take much effort to uniformize the testing conditions.
- Different versions of the problem came up over time, and we didn't separate the testsolve data for them. So some people worked on harder versions of the problem than others.

- Testsolvers were likely to be working on subjects they preferred.
- Often testsolvers would give up immediately if they looked at a problem and decided they weren't interested in the statement.

§4.2 Editing showcase

In this section I want to highlight some examples of problems that transformed significantly during the editing process and the ideas and thinking behind some of those edits.

§4.2.1 Problem 5 (pentagon geo)

The original statement read really differently:

Initial version of problem 5
Let $ABCD$ be a trapezoid, with $AB \parallel CD$ and $AB < CD$, that is inscribed in a circle with center O. Then P is the circumcenter of $\triangle COD$. Let E, F, and K be the intersections of $\overrightarrow{OP}$ with CD, AB, and the minor arc AB of $(ABCD)$ respectively. If $EF = 5$, and $FK = 1$, and $\frac{AB}{CD} = \frac{2}{3}$. If the circumradius of $(ABCD)$ is $\frac{m}{n}$, find $m + n$.

Compare that to the updated statement:

Final version of problem 5

Convex pentagon $ABCDE$ is inscribed in circle ω such that $\frac{AC}{DE} = \frac{2}{3}$, $AE = CD$, and $AB = BC$. Suppose the distance from B to line AC is 144 and the distance from B to line DE is 864. Compute the radius of ω.

We reworked the point names and definitions completely to avoid unnecessary definitions and get a statement that was much more succinct.

In addition, we went and scaled everything up so that the answer would be an integer. I suppose some students might think this is obnoxious, but I did it on purpose for two main reasons:

- It helps provide a confirmation that your answer is correct; if you find a factor of 5 in the denominator, for example, you know something went wrong.
- It avoids penalizing students who accidentally give the diameter of 1296 (which is larger than the maximum answer 1000) instead of the radius of 648, which I thought might be a common misstep.

§4.2.2 Problem 11 (arcsin's)

The original statement looked almost unrecognizable compared to the edited version:

Initial version of problem 11

For any pair of positive real numbers (a, b) denote $f(a, b)$ by the unique value of $x > \max(a, b)$ such that

$$\arcsin(a/x) + \arcsin(b/x) = 120^\circ$$

provided that such x exists and 0 if such x does not exist. Find the remainder when

$$\frac{3}{4}\left[\sum_{m=1}^{2023}\sum_{n=1}^{2023} f(m, n)^2\right]$$

is divided by 1000.

Overall, the idea behind this version was to determine that

$$f(a, b) = \begin{cases} \sqrt{\frac{4}{3}(a^2 - ab + b^2)} & \text{if } a < 2b \text{ and } b < 2a \\ \text{doesn't exist} & \text{otherwise.} \end{cases}$$

The sum above was the original way of trying to get an integer answer out of this closed form. However, it obviously looks a lot more complicated.

We floated through several other ideas for how to work with the "meat" of the problem, which was determining when x existed and getting the closed form when it does. It took us a while before we settled on the idea that we could compress both halves of the problem by asking *when* $f(a, b)$ was the square root of an integer: you needed to know the boundaries $a \le 2b$ and $b \le 2a$, whereas the closed form was used to extract $a + b \equiv 0 \pmod 3$ for the integer condition. This eliminated the need to have f-notation at all.

In this way, we were able to get the statement down to a single sentence:

Final version of problem 11

Compute the number of ordered triples of positive integers (a, b, n) satisfying $\max(a, b) \le \min(\sqrt{n}, 60)$ and

$$\operatorname{Arcsin}\left(\frac{a}{\sqrt{n}}\right) + \operatorname{Arcsin}\left(\frac{b}{\sqrt{n}}\right) = \frac{2\pi}{3}.$$

The choice of 60 was again an intentional guard-rail to help with error-checking: a student who didn't realize $a \le 2b$ or $b \le 2a$ was a necessary condition would find out when there answer ended up being over 1000. (Looking back, perhaps this was actually worse for some students; it could lead the student to spend a lot of time trying to find their mistake but not being able to figure out where it was.)

§4.2.3 Problem 12 (rubber bands)

This is a problem that didn't need any mathematical copy-editing, but needed a *ton* of English editing. It's one of those problems that, no matter what you write, somebody will manage to misinterpret it, and examples are necessary.

Initial version of problem 12

Charles decides to place one or more rubber bands on the triangular grid in the first figure such that every edge of the grid is covered by exactly one rubber band. An example is given in the second figure. Given that there are N ways to do this, find the last 3 digits of N.

(Two ways are considered different if the sets of edges covered by the rubber bands are different or if any rubber band traverses its edges in a different order. Going over/under does not change the configuration.)

The original proposal did have a figure showing an example of a rubber band tiling. The issue was that the definition of "different" also needed an example.

Eventually, we decided it'd be a good idea to introduce the notation $\mathcal{G}_n$, so that we could give examples on a smaller grid instead. However, another issue was that even for $\mathcal{G}_2$, there are already $3^3 = 27$ different valid ways (and there is only one way for $\mathcal{G}_1$). Not only was this too many examples to draw out, but the form of the number also would be an excessive hint on the solution method.

It took a few tries before we hit on the idea of showing all coverings that used *exactly two* rubber bands instead. This resolved both issues at once: it turned out there were 10 valid ways, which is both few enough to draw but also does not give away the solution idea.

§4.2.4 Problem 13 ($n^{22} - 1$ and $n^{40} - 1$)

The initial proposal for this problem was meant to be a standard exercise in principle of inclusion-exclusion:

Initial version of problem 13

Find the number of nonnegative integers n less than 1009 such that exactly one of $n^{84} - 1$ or $n^{72} - 1$ is divisible by 1009.

Note that 1009 is prime. However, we thought the problem could be made a lot more substantial if we used a better *composite* modulus, so that contestants would need to use the Chinese remainder theorem.

Final version of problem 13

Let S denote the sum of all integers n such that $1 \le n \le 2024$ and exactly one of $n^{22} - 1$ and $n^{40} - 1$ is divisible by 2024. Find the remainder when S is divided by 1000.

We also tried to make it less obvious it was a counting problem by asking for the sum rather than the number; because all the exponents involved are even, valid solutions pair as x and $2024 - x$, so they are equivalent, but adds one more step to the problem.

§4.3 "Raw" versions of all fifteen problems

If you're curious what the problems looked like before the magic editing touch, here they are. (Problems 12 and 14 had figures, but we didn't include them in this appendix.)

1. Compute the number of real values of x such that $0 < x \le 100$ which satisfy the equation
$$x^2 = \lfloor x \rfloor \lceil x \rceil.$$

2. Evan is throwing darts at a regular hexagon $BOATIS$. The probability that Evan's dart lands on the interior of exactly one of the quadrilaterals $BOAT$ and $OTIS$ can be expressed as $\frac{m}{n}$. Find $m+n$.

3. Perry the panda eats in a very unique way. On Monday, he eats 14 pieces of bamboo, and on each following day, Perry eats one less than three times the previous day or one more than the previous day, with equal probability. At the end of the day on Friday, find the expected number of pieces of bamboo Perry has eaten throughout the week.

4. Let N be the number of sequences $a_1, a_2, \dots, a_7$ such that $a_i \in \{1, 2, \dots, 7\}$ for $i = 1, 2, \dots, 7$ and the sequences $a_1, a_2, \dots, a_7$ and $a_{a_1}, a_{a_2}, \dots, a_{a_7}$ are the same. Find the remainder when N is divided by 1000.

5. Let $ABCD$ be a trapezoid, with $AB \parallel CD$ and $AB < CD$, that is inscribed in a circle with center O. Then P is the circumcenter of $\triangle COD$. Let E, F, and K be the intersections of $\overrightarrow{OP}$ with CD, AB, and the minor arc AB of $(ABCD)$ respectively. If $EF = 5$, and $FK = 1$, and $\frac{AB}{CD} = \frac{2}{3}$. If the circumradius of $(ABCD)$ is $\frac{m}{n}$, find $m+n$.

6. The equation $\lfloor x \rfloor \cdot \{x\} = kx$ has exactly 24 solutions when $k \in [a, b)$ for real numbers a and b. Compute $\frac{1}{b-a}$.

7. Find the number of 9-tuples $(a_0, a_1, \dots, a_8)$ such that $a_i \in \{-1, 0, 1\}$ for $i = 0, 1, \dots, 8$ and the polynomial $a_8x^8 + a_7x^7 + \cdots + a_1x + a_0$ is divisible by $x^2 + x + 1$.

8. Let n and k be positive integers such that $\binom{n}{k}$ is a multiple of 1000 and the sum $n+k$ is minimized. What is $n+k$?

9. Let ω be a circle with center O and radius 1. Points A, B, and C are chosen uniformly at random on ω, and suppose that H is the orthocenter of $\triangle ABC$. Find the expected value of OH^2.

10. Over all solutions to the equation

$$\gcd(a, b) + \operatorname{lcm}(b, c) = \operatorname{lcm}(c, a)^3$$

in positive integers, how many three-digit values of b are possible?

11. For any pair of positive real numbers (a, b) denote $f(a, b)$ by the unique value of $x > \max(a, b)$ such that

$$\arcsin(a/x) + \arcsin(b/x) = 120^\circ$$

provided that such x exists and 0 if such x does not exist. Find the remainder when

$$\frac{3}{4}\left\lceil \sum_{m=1}^{2023} \sum_{n=1}^{2023} f(m, n)^2 \right\rceil$$

is divided by 1000.

12. Charles decides to place one or more rubber bands on the triangular grid in the first figure such that every edge of the grid is covered by exactly one rubber band. An example is given in the second figure. Given that there are N ways to do this, find the last 3 digits of N.

(Two ways are considered different if the sets of edges covered by the rubber bands are different or if any rubber band traverses its edges in a different order. Going over/under does not change the configuration.)

13. Find the number of nonnegative integers n less than 1009 such that exactly one of $n^{84} - 1$ or $n^{72} - 1$ is divisible by 1009.

14. An equilateral triangle is cut into six pieces with three line segments of length x through its center; these six pieces are rearranged to form another equilateral triangle with an equiangular hexagonal hole inside it. Given that the hole has side lengths 3 and 2, x can be written as $\frac{m}{n}$ where m and n are relatively prime positive integers. Find $m + n$.

15. A parabola in the Cartesian plane is tangent to the x-axis at $x = 1$ and to the y-axis at $y = 3$. The sum of the coordinates of the vertex of the parabola can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute $m + n$.

§4.4 AI attempts

It is apparently now a tradition to feed problem statements from competitions into large language models and see how many correct answers it obtains. ChatGPT 3.5 and Bard did not answer any questions correctly, but ChatGPT 4 was able to generate Python code correctly answering the following four problems. A full transcript of ChatGPT 4's attempt at the problems is available at https://chat.openai.com/share/fae13bbd-38a9-4034-865d-49e24eb49840.

1. Floors

```
from sympy import *

# Define the range
x = symbols('x')
integer_solutions = [i for i in range(1, 101) if i**2 == i*i]

non_integer_solutions = []
for n in range(1, 100):
    x = Symbol('x')
    equation = Eq(x**2, n*(n + 1))
    solutions = solve(equation, x)
    non_integer_solutions.extend([sol.evalf() for sol in
        solutions if 0 < sol <= 100 and sol != n])

total_solutions = len(integer_solutions) +
    len(non_integer_solutions)
```

3. Perry the panda

```
bamboo_monday = 14
expected_bamboo_tuesday = (3*bamboo_monday - 1 + bamboo_monday
    + 1) / 2

```

```
def expected_bamboo_next_day(bamboo_previous_day):
    return (3*bamboo_previous_day - 1 + bamboo_previous_day +
        1) / 2

total_expected_bamboo = bamboo_monday
for day in range(4):  # For Tuesday to Friday
    total_expected_bamboo +=
        expected_bamboo_next_day(total_expected_bamboo)
```

7. Divisible by $x^2 + x + 1$

```
from sympy import symbols, solve
from itertools import product

# Define the variables
a0, a1, a2, a3, a4, a5, a6, a7, a8 = symbols('a0:9')

# Generate all possible combinations of a_i values
combinations = list(product([-1, 0, 1], repeat=9))

# Count the number of valid combinations
valid_combinations = 0
for combo in combinations:
    # Assign values to a_i
    a0, a1, a2, a3, a4, a5, a6, a7, a8 = combo

    # Check if the sums of coefficients for x^2, x, and
        constant terms are equal
    if (a8 + a5 + a2) == (a7 + a4 + a1) == (a6 + a3 + a0):
        valid_combinations += 1
```

13. $n^{22} - 1$ and $n^{40} - 1$

```
def is_divisible_by_2024(number):
    return number % 2**3 == 0 and number % 11 == 0 and number %
        23 == 0

total_sum = 0

for n in range(1, 2025):
    divisible_22 = is_divisible_by_2024(n**22 - 1)
    divisible_40 = is_divisible_by_2024(n**40 - 1)

    if divisible_22 != divisible_40:  # Only one of them is
        divisible
        total_sum += n

remainder = total_sum % 1000
```

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