The 7$^{\text{th}}$ US Ersatz Math Olympiad

Solutions and Results

Evan Chen《陳誼廷》

30 December 2025

Contents

1 Summary

§1.1 News and commentary

The seventh USEMO took place from October 25 – 26, 2025. A total of 139 students submitted at least one paper. This year's USEMO was again sponsored by the CoRe Lab, Institute of Artificial Intelligence, Peking University and we are grateful for their support.

Like last year, most of the delay between the grading and the publication of the results is due to my own unresponsiveness. I am thankful to the volunteers and contestants for their patience!

§1.1a Anant Mudgal will be the future director for the USEMO

I am happy to announce that starting from USEMO 2026, I am passing my role of director to **Anant Mudgal**.

Anant is one of my earliest students, and a longtime member of the math olympiad community — a competitor on the Indian IMO team from 2015 to 2018, and one of the Indian team coaches since 2019. He was also on the Problem Selection Committee for IMO 2025. I'm delighted that he has agreed to help organize the USEMO for me.

§1.2 Thanks

I am once again grateful to many individuals who helped make this competition possible.

§1.2a Sponsors

We are grateful to be sponsored this year by the CoRe Lab, Institute of Artificial Intelligence, Peking University.

§1.2b Proposers of problems

I thank everyone who submitted problems for the USEMO, of which there are many. The list of authors who had at least one problem in the shortlist were Alan Zaripov, Andrey Kandrashkin, Holden Mui, Iman Maghsoudi, Imhan Maghsoudi, Jaewon Son, Kaixin Wang, Kornpholkrit Weraarchakul, Krishna Pothapragada, Mikhail Raikhman, Miroslav Marinov, Mjtaba Zareh Bidaki, Mojtaba Zareh Bidaki, Oleg Kryzhanovsky, Petko Lazarov, Rutthee Youyongwatanakul, Sathyaram Baskar, Sathyaram Basker, Siraphop Khawplad, Tran Quang Hung, Xu Zi Jie, Yi Wang.

§1.2c Reviewers

I thank the reviewers of the shortlisted problems:

- Alansha Jiang
- Aprameya Tripathy
- Carlos Rodriguez
- Krishna Pothapragada
- Maximus Lu
- Milan Haiman
- Noah Walsh

§1.2d Graders

Thanks to everyone who graded at least one paper: Aaron Fanxi Su, Aatmik Krishna, Abdullahil Kafi, Ahem Garg, Alansha Jiang, Alec Sun, Alex Chui, Alex Gautam, Alex Yan, Alexander Wang, Anant Mudgal, Andrew Shishko, Anmol Tiwari, Arham Gada, Atharv Harlalka, Cyrus Nemati, Daniel Ji, Debarchan Neogi, Evan Chen, Gasser Elatfy, Hannah Fox, Harini KS, Ishan Prabhu, Jake Tan, Kaixin Wang, Kamalesh Sarkar, Kevin Liu, Kornpholkrit Weraarchakul, Lasitha Vishwajith Jayasinghe, Leon Lau, Liam Celinski, Marius Cerlat, Mauricio Flores Claros, Miroslav Marinov, Monamy Zaman, Namish Durgapal, Niranjana Pottekkat, Oron Wang, Paras Kumar, Paul Dao, Pedro Henrique de Almeida Ursino, Petko Lazarov, Reyaansh Agrawal, Ryan Li, Sathyaram Basker, Shreya Mundhada, Smochina Vladislav, Taes Padhihary, and Zhaopeng Wu.

Special thanks to those who served as problem captains (who this year not only had to design rubrics but also were asked to sign off on every paper for their assigned problems):

- Alec Sun
- Anant Mudgal
- Hannah Fox
- Kaixin Wang.

§1.2e Other supporters

I thank the Art of Problem Solving for offering the software and platform for us to run the competition. Special thanks to Jo Welsh for dealing with all my support requests.

2 Results

If you won one of the seven awards, please reach out to `usemo@evanchen.cc` to claim your prize!

§2.1 Top Scores

Congratulations to the top three scorers.

1st place Jiahe Liu (35 points)

1st place Royce Yao (35 points)

3rd place Darsh Patel (30 points)

§2.2 Special awards

See the Rules for a description of how these are awarded. (Note in particular that students already in the top three above aren't considered for special awards.)

Two of the special awards this year are tied. For the monetary award, we randomly (using `random.org`, which outputted 1 twice when asked for a number from 1 to 2) selected the first student to receive the cash award, but otherwise recognize both students equally.

Youth prize Arjun Suresh, Joey Zheng

Top female Sanjana Philo Chacko, Xinyi Li

Top day 1 Benjamin Fu

Top day 2 Yoll (Gurt) Feng

§2.3 Honorable mentions

This year we award Honorable Mention to anyone scoring at least 26 points (who is not in the top three already). The HMs appear below in alphabetical order.

Arjun Suresh

Ciobotea Alexandru

Grant Blitz

Joey Zheng

Nguyen Duc Gia Bach

Rafał Żebruń

Ruoxue Lin

Shihan Kanungo

Yoll (Gurt) Feng

§2.4 Distinction

The Distinction award is awarded for either scoring at least 14 points or in the top 25 of scores, whichever is more inclusive. This year, the 25th place student scored 19 points, so Distinction awards recognize any scores of 14 or higher. The Distinction awards appear below in alphabetical order.

Anik Sardar

Alexander Svoronos

Atticus Stewart

Ben Jump

Benjamin Fu

Benny Wang

Channing Yang

David Kurniady

Eden He

Elena Beckman

Enzo Holanda Sampaio

George Zhao

Hirbod Hemmatian

Hongming Allan Zhao

Huanqi Zhang

Hyun-Jin Kim

Hyunjun Jang

Hà Mạnh Hùng

Ivar Lee Fevang

Justin Jia

Keshav Karumbunathan

Krithik Manoharan

Le Yi Tan

Leo Wu

Mai Thanh Lam

Maria Radu

Pablo Freire Fernández

Radin Nikeghbali

Ryan Shao

Ryan Zhang

Sami El-Hajjar

Sanjana Philo Chacko

Seongjin Shim

Tarun Rapaka

Victoria Lund Søraas

Vincent Wang

Xinyi Li

Youran Gu

3 Solutions and marking schemes

§3.1 USEMO 1 — proposed by Alan Zaripov

Problem statement

Find all real numbers λ for which there exists an integer $n \geq 2$ and an arithmetic progression $(a_0, a_1, \dots, a_n)$ of real numbers (in that order) such that the identity

$$(X-\lambda)(X-\lambda^2)\dots(X-\lambda^n) = a_0X^n + a_1X^{n-1} + a_2X^{n-2} + \dots + a_n$$

is true for every real number X.

§3.1a Solution

The answer is $\lambda = -1$ only. This works because

$$(X-(-1))(X-(-1)^2) = 1X^2 + 0X + (-1).$$

We turn to showing that $\lambda = -1$ is the only possible answer, for which we present several approaches.

¶ Vieta solution. The idea behind this solution is to use Vieta formulas on the polynomial P defined by

$$\begin{aligned} P(X) &\coloneqq (X-1)\cdot(X-\lambda)(X-\lambda^2)\dots(X-\lambda^n) \\ &= (X-1)\cdot\sum_{r=0}^{n} a_{n-r}X^r \\ &= X^{n+1} + d(X^n + \dots + X^1) - a_n \end{aligned}$$

where d is the common difference of the arithmetic progression. So, the coefficients of X, X^2, …, X^n in $P(X)$ are equal.

Claim — We must have $|\lambda| = 1$.

Proof. It is enough to consider the X and X^n coefficients of P. By Vieta's formulas on P, we also know the coefficient of X^n in $P(X)$ is

$$-(1+\lambda+\dots+\lambda^n)$$

while the coefficient of X is

$$(-1)^n\sum_{r=0}^{n}\lambda^{\frac{n(n+1)}{2}-r} = (-1)^n\lambda^{\frac{n(n-1)}{2}}(1+\lambda+\dots+\lambda^n).$$

Therefore,

$$-(1+\lambda+\dots+\lambda^n) = (-1)^n\lambda^{\frac{n(n-1)}{2}}(1+\lambda+\dots+\lambda^n)$$

$$\implies 0 = \left(1 + (-1)^n \lambda^{\frac{n(n-1)}{2}}\right)(1 + \lambda + \cdots + \lambda^n).$$

But if $|\lambda| \neq 1$, both of the factors above are nonzero (the latter factor equals $\frac{\lambda^{n+1}-1}{\lambda-1}$). This completes the proof. □

Finally, it is easy to see that $\lambda = 0$ and $\lambda = 1$ do not work (for $\lambda = 1$ we get $a_0 = 1$, $a_1 < 0$, $a_2 > 0$; for $\lambda = 0$ we get $a_0 = 1$ and $a_1 = a_2 = 0$). The solution is complete.

¶ Descartes rule of signs solution. From the remark at the end of the last solution, it is enough to consider the case

$$\lambda \neq -1, 0, 1 \tag{3.1}$$

We will show that (3.1) produces a contradiction.

The main observation for this approach is:

Claim — If $a_0, a_1, \ldots, a_n$ is any arithmetic progression of real numbers, then the polynomial $a_0X^n + a_1X^{n-1} + \cdots + a_n$ has at most one positive root.

Proof. This follows from Descartes rules of signs, because the sequence $(a_0, \ldots, a_n)$ has at most one sign change. □

However, if we had $n \geq 4$ and (3.1), then λ^2 and λ^4 are distinct positive roots. Hence, we only need to consider the cases $n = 2$ and $n = 3$.

- Suppose $n = 2$. Then,

$$\begin{aligned}(X-\lambda)(X-\lambda^2) &= a_0X^2 + a_1X + a_2\\ X^2 + (-\lambda-\lambda^2)X + \lambda^3 &= a_0X^2 + a_1X + a_2.\end{aligned}$$

  So,

$$\begin{aligned}a_2 - 2a_1 + a_0 &= 0\\ \lambda^3 - 2(-\lambda^2 - \lambda) + 1 &= 0\\ (\lambda+1)(\lambda^2+\lambda+1) &= 0.\end{aligned}$$

  This produces the desired contradiction to (3.1).

- Suppose $n = 3$. Then,

$$\begin{aligned}(X-\lambda)(X-\lambda^2)(X-\lambda^3) &= a_0X^3 + a_1X^2 + a_2X + a_3\\ X^3 + (-\lambda-\lambda^2-\lambda^3)X + (\lambda^3+\lambda^4+\lambda^5)X + (-\lambda^6) &= a_0X^3 + a_1X^2 + a_2X + a_3.\end{aligned}$$

  Therefore,

$$\begin{aligned}-a_3 + a_2 + a_1 - a_0 &= 0\\ \lambda^6 + \lambda^5 + \lambda^4 - \lambda^2 - \lambda - 1 &= 0\\ (\lambda^4 - 1)(\lambda^2 + \lambda + 1) &= 0.\end{aligned}$$

  This produces the desired contradiction to (3.1).

¶ Bounding solution Following the calculation at the end of the last solution, we assume $n \geq 4$.

Claim — We have $\lambda < 0$.

Proof. If $\lambda \geq 0$, then $a_0 = 1$, $a_1 \leq 0$, and $a_2 \geq 0$, a contradiction. □

Claim — We have $\lambda \leq -1$.

Proof. Assume towards a contradiction that $-1 < \lambda < 0$. We have

$$\begin{aligned} a_2 &= \sum_{1 \leq i < j \leq n} \lambda^{i+j} \\ &= \sum_{i=1}^{n} \lambda^i \sum_{j=i+1}^{n} \lambda^j \\ &= \sum_{i=1}^{n} \lambda^i \frac{\lambda^{n+1} - \lambda^{i+1}}{\lambda - 1} \\ &= \frac{1}{\lambda - 1} \sum_{j=1}^{n} (\lambda^{n+i+1} - \lambda^{2i+1}). \end{aligned}$$

Since $\frac{1}{\lambda-1}$ is negative and $(\lambda^{n+i+1} - \lambda^{2i+1})$ is positive, $a_2 < 0$. Thus, $a_n < -1$. However,

$$|a_n| = \left| (-1)^n \lambda^{\frac{n(n+1)}{2}} \right| < 1.$$ □

Claim — If λ is a solution, then $\frac{1}{\lambda}$ is a solution.

Proof. By the previous claim, $\lambda \neq 0$. We have

$$\begin{aligned} \left(X - \frac{1}{\lambda}\right)\left(X - \frac{1}{\lambda^2}\right) \cdots \left(X - \frac{1}{\lambda^n}\right) &= \frac{1}{(-\lambda)^n}(1 - \lambda X)(1 - \lambda^2 X) \cdots (1 - \lambda^n X) \\ &= \frac{1}{(-\lambda)^n}(a_0 + a_1 X + \cdots + a_n X^n). \end{aligned}$$ □

Since λ and $\frac{1}{\lambda}$ are solutions, we have $\lambda, \frac{1}{\lambda} \leq -1$. This is only possible when $\lambda = -1$.

Remark (Author comments)**.** At one of the stages of the St. Petersburg 2008 School Olympiad, such a problem was proposed, which, however, is not the motivation for creating the one given earlier: *a polynomial of degree n with rational coefficients has n different roots forming a geometric progression. What values can the number n take?*

§3.1b Marking scheme

For incomplete solutions, the following partials apply:

- **0 points** for claiming that $\lambda = -1$ is the only solution.
- **0 points** for proving that if $\lambda \neq 0$ is a solution, then $\frac{1}{\lambda}$ is a solution.
- **0 points** for showing $\lambda < 0$.
  - Similarly, showing $\lambda \neq 0$, $\lambda \neq 1$, or other special cases is not worth any points.

- **0 points** for using Vieta's formulas to calculate specific coefficients.
- **1 point** for proving that $\lambda = -1$ is a solution.
  - Saying "$\lambda = -1$ and $n = 2$ works" suffices.
  - Saying that $n = 2$ implies $\lambda = -1$ is not sufficient.
- **2 points** for proving $|\lambda| \geq 1$ or $|\lambda| \leq 1$.
- **2 points** for calculating $a_0 - a_1$ and $a_{n-1} - a_n$ in terms of λ and n.
  - Note that solutions must calculate both of these to get the points.
- **1 point** for multiplying the given polynomial by $(X-1)$ and showing that the coefficients of $x, x^2, \dots, x^n$ are equal.
- **1 point** for multiplying the given polynomial by $(X-1)^2$ and showing that the coefficients of $x^2, x^3, \dots, x^n$ are 0.

The point for the construction is additive with other points, but other than that, these partials are non-additive.

For complete solutions, the following additive deductions apply:

- **-0 points** for sign errors that don't significantly affect the solution.
  - For example, writing
  $$a_n - a_{n-1} = a_1 - a_0 \implies (-\lambda)^{\frac{n(n-1)}{2}} \left(\sum_{i=0}^{n} \lambda^i \right) = -\sum_{i=0}^{n} \lambda^i$$
  and concluding that $(-\lambda)^{\frac{n(n-1)}{2}} = 1$ or $\sum_{i=0}^{n} \lambda^i = 0$ is not a dock.
- **-1 point** for not showing $\lambda = -1$ works.
  - Saying "$\lambda = -1$ and $n = 2$ works" suffices, but only saying "$\lambda = -1$ works" does not.
  - Solutions that miss the condition that $n \geq 2$ but still show that $\lambda = -1$ is the only solution where $n \geq 2$ will not get docked.
- **-1 point** for a solution that shows $\lambda \in \{-1, 0, 1\}$ but does not show $\lambda = -1$, or a solution that has division by 0 issues.
  - Solutions that handle the $\lambda = 0$ but not the $\lambda = 1$ case will also receive this dock.
  - The division-by-zero dock only applies when the solution cancels a term from both sides of an equation without justifying that it is nonzero. In particular, writing fractions without justifying that the denominator is nonzero is OK, as long as this does not lead to actual missed cases.
  - If a term is clearly nonzero, then saying that it is clearly nonzero before canceling it is enough to not get this dock.
- **-2 points** for a solution that misses at least one of the following two cases:
  - $n = 2$
  - $n = 3$

  This item is not additive with the previous item. A solution that misses larger values of n will not be considered complete.

§3.2 USEMO 2 — proposed by Iman Maghsoudi, Ayeen Izady

Problem statement

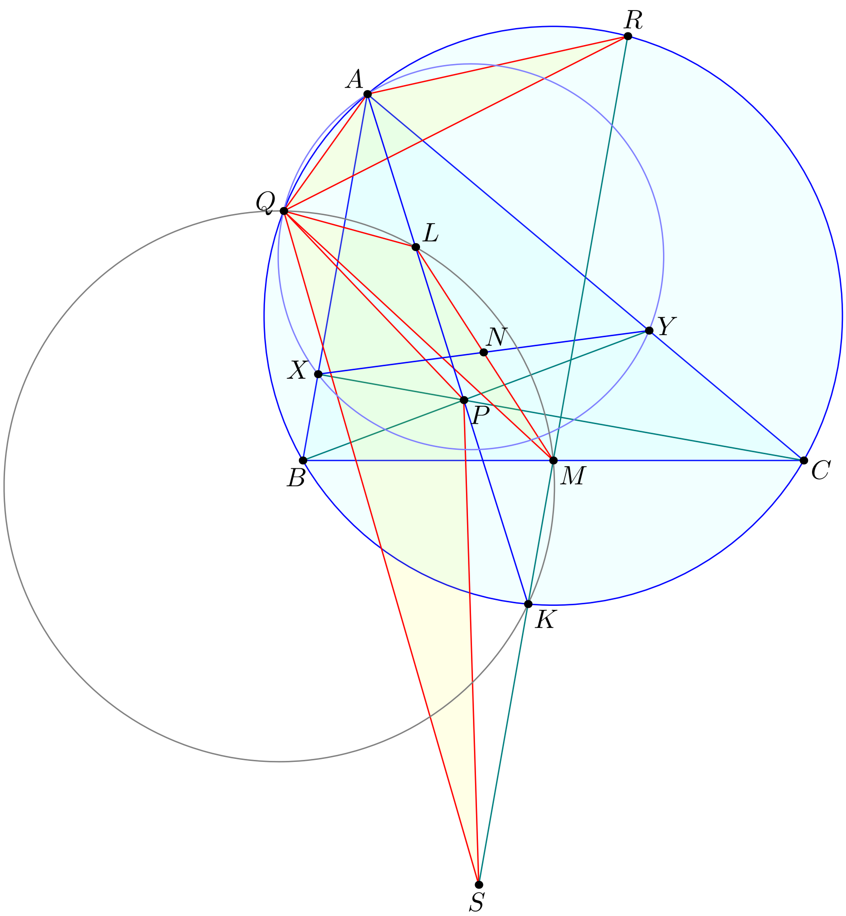
Let ABC be a fixed triangle with circumcircle ω. Consider P a variable point inside ABC. Ray BP meets side AC at Y while ray CP meets side AB at X. Let Q be the second intersection of ω and the circumcircle of triangle AXY. Let K be the second intersection of ray AP and ω.

Prove that as P varies, the circumcircles of triangle QPK all have a common radical center.

§3.2a Solution

Let M be the midpoint of $\overline{BC}$. We show that M is the desired fixed point.

¶ Classical approach using spiral similarity (from author). Let KM meet ω again at R, and let S be the reflection of R in M. Also N and L be the midpoints of $\overline{XY}$ and $\overline{AP}$.



Claim — Points L, M, N are collinear.

Proof. They lie on the Newton-Gauss line of $AXPY$. □

Claim — Quadrilateral $QLMK$ is concyclic.

Proof. Since $Q = (AXY) \cap (ABC)$, we know $\triangle QXB \overset{+}{\sim} \triangle QYC$. Since N and M are midpoints of $\overline{XY}$ and $\overline{BC}$, we in fact have

$$\triangle QXB \overset{+}{\sim} \triangle QNM \overset{+}{\sim} \triangle QYC.$$

Together with LMN collinear, we can get the angle chase

$$\measuredangle LMQ = \measuredangle NMQ = \measuredangle XBQ = \measuredangle ABQ = \measuredangle LKQ$$

to deduce that $QLMK$ is concyclic. □

Claim — Quadrilateral $QPKS$ is cyclic.

Proof. Since $Q = (KAR) \cap (KLM)$ was just shown, we know $\triangle QAR \overset{+}{\sim} \triangle QLM$. Since L and M are midpoints of $\overline{AP}$ and $\overline{RS}$, we in fact have

$$\triangle QAR \overset{+}{\sim} \triangle QLM \overset{+}{\sim} \triangle QPS.$$

Hence this gives $QPKS$ is cyclic. □

To finish, we can now show M has fixed power $BC^2/4$ by noting:

$$\operatorname{Pow}(M, (QPK)) = MK \cdot MS = MK \cdot MR = MB \cdot MC = \frac{BC^2}{4}.$$

¶ Second solution via moving points (Anant Mudgal) Let B_1, C_1 be points on lines AB, AC respectively such that the circumcircle of $\triangle AQB_1$ is tangent to line AC and the circumcircle of $\triangle AQC_1$ is tangent to line AB.

Claim 3.2.1 (Moving Points) — Point P lies on line $\overline{B_1C_1}$.

Proof. Fix Q and animate X with constant velocity on $\overline{AB}$. Since $AQXY$ are concyclic and $\triangle QXY \sim \triangle QBC$ has fixed shape, $X \mapsto Y$ is linear. Thus, lines $\overline{BY}$ and $\overline{CX}$ are moving lines with degrees equal to 1. Further, the two lines $\overline{BY}$ and $\overline{CX}$ coincide if and only if $X = B$. Thus, by Zack's lemma,

$$\deg P \leqslant \deg \overline{BY} + \deg \overline{CX} - 1 = (1+1) - 1 = 1$$

so P moves on a line. Now $P = B_1$ when $Y = A$ and $P = C_1$ when $X = A$, hence $\overline{B_1C_1}$ is the locus of P, as desired. We use the fact that $Q \neq A$ and $AQXY$ cyclic to imply that $X \neq Y$ and so B_1, C_1 are distinct points. □

Claim 3.2.2 — The circumcircles of BB_1Q and CC_1Q are tangent to line BC.

Proof. Since $\overline{AC}$ is tangent to the circumcircle of AQB_1, we conclude

$$\measuredangle(BQ, QC) = \measuredangle(BA, AC) = \measuredangle(B_1A, AC) = \measuredangle(B_1Q, QA).$$

Further, $\measuredangle(CQ, QA) = \measuredangle(CB, BA) = \measuredangle(CB, BB_1)$. Hence

$$\begin{aligned}\measuredangle(BQ, QB_1) &= \measuredangle(BQ, QA) - \measuredangle(B_1Q, QA) \\ &= \measuredangle(BQ, QC) + \measuredangle(CQ, QA) - \measuredangle(B_1A, AC) \\ &= \measuredangle(CQ, QA).\end{aligned}$$

Thus, $\measuredangle(BQ, QB_1) = \measuredangle(CB, BB_1)$, proving that $\overline{BC}$ is tangent to circumcircle of BB_1Q. Similarly, $\overline{BC}$ is tangent to circumcircle of CC_1Q, proving the claim. □

Suppose the circumcircles of BB_1Q and CC_1Q meet at point Z.

Claim — Point Z lies on line $\overline{B_1C_1}$ and on circle Γ.

Proof. Note that

$$\measuredangle(B_1Z, ZQ) = \measuredangle(B_1B, BQ) = \measuredangle(AB, BQ)$$

and likewise $\measuredangle(C_1Z, ZQ) = \measuredangle(AC, CQ)$ so Z lies on $\overline{B_1C_1}$. Now

$$\measuredangle(PZ, ZQ) = \measuredangle(B_1Z, ZQ) = \measuredangle(B_1B, BQ) = \measuredangle(AB, AQ).$$

Further, $\measuredangle(PK, KQ) = \measuredangle(AK, KQ) = \measuredangle(AB, BQ)$, so $\measuredangle(PZ, ZQ) = \measuredangle(PK, KQ)$, proving the claim. □

Finally, let M be the midpoint of $\overline{BC}$. Since

$$\operatorname{Pow}(M, \omega_B) = MB^2 = MC^2 = \operatorname{Pow}(M, \omega_C)$$

we conclude that M lies on the radical axis $\overline{QZ}$ of ω_B, ω_C. By the above claims, we conclude that

$$\operatorname{Pow}(M, \Gamma) = MQ \cdot MZ = MB^2$$

hence $MQ \cdot MZ = MB^2$ is a constant, hence M is our desired point.

§3.2b Marking scheme

This rubric can be split into two schemes:

- **0+ Scheme.** All partial solution attempts will be assessed accordingly to this scheme. The maximum possible score for a solution in this scheme will be 3 points.
- **7- Scheme.** All correct or essentially correct solutions will be awarded according to this scheme. The minimum possible score in this scheme will be 5 points.

Note on contingencies. If a solution is correct contingent on a claim being true, but the claim has **not** been proven, the solution will be considered in the 0+ **scheme** *unless* the claim is precisely the statement of a well-known theorem (for example, the existence of the Newton-Gauss line or spiral-similarity occuring in pairs) or a minor omission.

Note on minor errors related to configurations and typos. No deductions should be applied for typographical issues or configuration issues that can be resolved by directed angles.

- **0 Points.**

- All incomplete algebraic solutions [complex numbers, trigonometry, barycentric, Cartesian or other coordinate systems, degree-counting or moving points] without significant synthetic progress should be awarded 0 points by default unless they exhibit significant progress in the form of an independent synthetic geometry claim and its proof. The grade for such scripts should be awarded according to the value of the synthetic claim.
- **In this marking scheme, claims and conjectures without proof are awarded 0 points**, besides the 1 point award for the fixed point M below.
- Stating known theorems or facts such as moving points, gliding principle, $\triangle QXY \sim \triangle QBC$, Newton-Gauss line, Zack's lemma, etc. without any significant progress should be marked as 0 points as well.

- **1 Point.**

  An additive 1 point is awarded for correctly claiming that the fixed point is the midpoint of BC.

- **2 Points.** Proving any of the below claims should be worth 2 points
  - Proving Q, L, M, K are concyclic but no progress towards finishing the solution.
  - Proving P lies on $\overline{B_1C_1}$.
  - Proving Z lies on $\overline{B_1C_1}$ and on Γ.
  - Any other synthetic claim found along an alternate solution path with comparable value to the above will also be considered for 2 points. These should be highlighted accordingly to maintain consistency.

  The 2 points in this scheme are non-additive to each other and to any other progress, except the additive 1 point for claiming M is the fixed point.

- **3 Points.** Claiming M is fixed point and proving any item on the 2 Points scheme. This is the maximum possible score for all incomplete solutions.

- **5 Points.** If a solution is essentially correct but incomplete due to a minor fixable detail, it should be marked with 5 points. For example, a singular missing moving points case in a list of many cases that is otherwise degenerate and everything else is correct or a missing equality in an angle chase that is obvious from context within the script may fall under this category. *This item should only be considered if no other rubric item is a better descriptor of the work presented in the script.*

- **6 Points.** For solutions that are essentially correct and assume (without proof) a well-known fact such as the spiral similarity lemmas, Newton-Gauss line or more egregiously, Zack's lemma, without proper citation of the statement of said well-known fact.

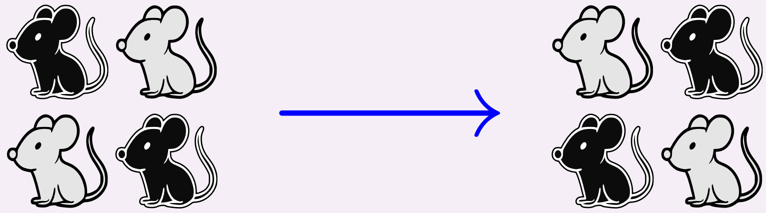
  This should only be considered if it is sufficiently evident that no further mathematical work needs to be done to interpret the assumption as an instance of the well-known fact. Otherwise, the 0+ scheme should apply.

- **7 Points.** Complete solution with no errors.

§3.3 USEMO 3 — proposed by Holden Mui

Problem statement

Suppose 2025 black and white mice are arranged in a 45×45 grid. A set of four mice is *special* if the four mice form a contiguous 2×2 square, the top-left mouse and the bottom-right mouse are both black, and the bottom-left mouse and top-right mouse are both white. A *move* swaps the positions of the black and white mice in a special set, as shown below.



Across all possible initial configurations of mice, what is the maximum number of moves that one could make?

§3.3a Solution

The maximum possible number of moves is $(22 \cdot 23)^2 = \boxed{256036}$.

¶ Proof of upper bound. To prove that at most $(22 \cdot 23)^2$ moves can be made, label the 45 columns from left to right with the integers -22 to 22, and label the 45 rows from bottom to top with the integers -22 to 22. Define the *weight* of a position to be the product of its row label and its column label, and define the *energy* of a configuration to be the sum of the weights of the black stones in the configuration.

Claim 3.3.1 — Making a move always increases the energy by exactly one.

Proof. Applying a move on a special set of four stones with row labels $\{a, a+1\}$ and column labels $\{b, b+1\}$ increases the sum of the weights of the black stones by

$$ab + (a+1)(b+1) - a(b+1) - b(a+1) = 1. \qquad \square$$

To finish, note that the minimum possible initial energy is

$$\sum_{a=-22}^{-1} \sum_{b=1}^{22} ab + \sum_{a=1}^{22} \sum_{b=-22}^{-1} ab = -2(1+2+\ldots+22)^2 = -\tfrac{1}{2}(22 \cdot 23)^2,$$

and the maximum possible final energy is

$$\sum_{a=-22}^{-1} \sum_{b=-22}^{-1} ab + \sum_{a=1}^{22} \sum_{b=1}^{22} ab = 2(1+2+\ldots+22)^2 = \tfrac{1}{2}(22 \cdot 23)^2.$$

Therefore, Claim 3.3.1 implies that at most $(22 \cdot 23)^2$ moves can be made.

¶ Alternate proof of upper bound. Label the rows and columns using the integers $1, \dots, 45$.

Claim 3.3.2 — Let $\mathcal{M}(a,b)$ denote the number of times that the 2×2 square spanning row labels $\{a, a+1\}$ and column labels $\{b, b+1\}$ can be moved, and define $\mathcal{M}(a,b) = 0$ if $a = 0$ or $b = 0$. Then

$$\mathcal{M}(a,b) + \mathcal{M}(a-1,b-1) - \mathcal{M}(a-1,b) - \mathcal{M}(a,b-1) \le 1$$

for every pair of positive integers (a,b).

Proof. The number of times that the stone in with coordinates (a,b) is flipped from white to black is $\mathcal{M}(a-1,b-1) + \mathcal{M}(a,b)$, and the number of times that the stone is flipped from black to white is $\mathcal{M}(a-1,b) + \mathcal{M}(a,b-1)$. The bound follows after observing that the difference of these quantities must be at most 1. □

Now,

$$\begin{aligned}
&\sum_{a=1}^{22}\sum_{b=1}^{22}\mathcal{M}(a,b) \\
&= \sum_{a=1}^{22}\sum_{b=1}^{22}((23-a)(23-b)+(23-a)(22-b)-(23-a)(22-b)-(22-a)(23-b))\mathcal{M}(a,b) \\
&= \sum_{a=1}^{22}\sum_{b=1}^{22}(23-a)(23-b)\left(\mathcal{M}(a,b)+\mathcal{M}(a-1,b-1)-\mathcal{M}(a-1,b)-\mathcal{M}(a,b-1)\right) \\
&\le \sum_{a=1}^{22}\sum_{b=1}^{22}(23-a)(23-b) = (1+\ldots+22)^2 = \tfrac{1}{4}(22\cdot 23)^2,
\end{aligned}$$

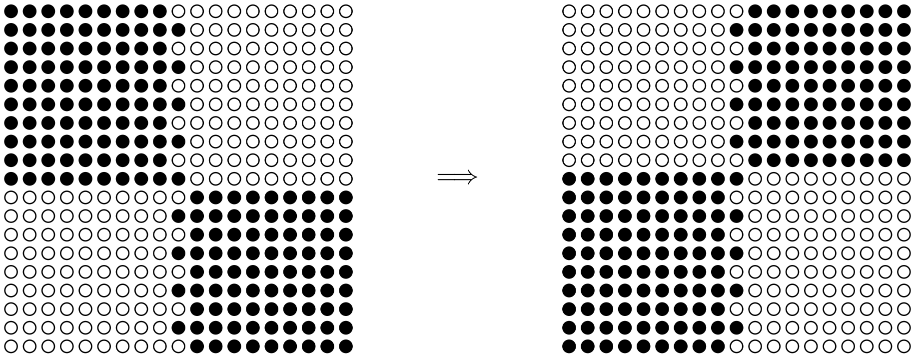
where the last inequality follows from Claim 3.3.2. Therefore,

$$\begin{aligned}
&\sum_{a=1}^{44}\sum_{b=1}^{44}\mathcal{M}(a,b) \\
&= \sum_{a=1}^{22}\sum_{b=1}^{22}\mathcal{M}(a,b) + \sum_{a=1}^{22}\sum_{b=23}^{44}\mathcal{M}(a,b) + \sum_{a=23}^{44}\sum_{b=1}^{22}\mathcal{M}(a,b) + \sum_{a=23}^{44}\sum_{b=23}^{44}\mathcal{M}(a,b) \\
&\le \tfrac{1}{4}(22\cdot 23)^2 + \tfrac{1}{4}(22\cdot 23)^2 + \tfrac{1}{4}(22\cdot 23)^2 + \tfrac{1}{4}(22\cdot 23)^2 \\
&= (22\cdot 23)^2
\end{aligned}$$

since the last inequality follows by symmetry from rotating the grid.

Remark. If this solution seems contrived to you, you should try to find an explicit upper bound for $\mathcal{M}(a,b)$.

¶ Construction achieving 256036 moves. To prove attainability, consider the following initial and final configurations, which have energies of $-\frac{1}{2}(22\cdot 23)^2$ and $\frac{1}{2}(22\cdot 23)^2$ respectively. It suffices to show that the left configuration can be turned into the right configuration, by Claim 3.3.1.



Claim 3.3.3 — If two rows are adjacent and are color complements (i.e. every black stone in the top row corresponds to a white stone in the bottom row, and vice versa), then a sequence of moves can be made to result in a configuration where the top row has its black stones on the right and the bottom row has its black stones on the left.

Proof. Repeatedly make moves until no more can be made, and note that the rows remain color complements throughout the entire process. By construction, no white stone in the top row can lie to the left of a black stone in the top row; this forces the desired final configuration. □

Call a sequence of moves achieving the goal in Claim 3.3.3 a *row flip*. Since the number of black stones contained in any two adjacent rows is always 45, repeatedly applying row flips until no more can be made will result in a configuration where every black stone in the rightmost column is above every white stone in the rightmost column. Since row flipping forces each row to be color-segregated (i.e. a block of white stones followed by a block of black stones, or vice versa), the final configuration must be the one given in the figure above, as desired.

Remark. The problem has previously appeared in the case m and n are both even, in the Iran-Taiwan competition 2022, proposed by CSJL, see `https://aops.com/community/p25490083`. Before the contest, there is no completely correct solution in the thread.

§3.3b Marking scheme

We will award 1 points for the correct answer, 1 point for the construction and 4 points for the bound, with $1+1=2, 1+4=5, 1+1+4=7$. For example, if the submission only correctly proves the correct bound, or only gives the correct construction, and claims a wrong answer, it is awarded 1 or 4 points.

An incomplete/incorrect proof for the bound earns the maximum number of points in a single solution path.

Answer (1 point)

Award 1 point for the answer.

Proof of bound (4 points)

¶ Solution path 1.

- **0 point** for observing that the number of black stones in each row and each column does not change.
- **0 point** for considering the monovariant as $\sum_{(i,j) \text{ black}} ij$ without saying $-22 \le i, j \le 22$, even if the solution notes that each operation increases this monovariant by 1.
- **1 point** for considering the monovariant as $\sum_{(i,j) \text{ black}} ij$ with $-22 \le i, j \le 22$ and the solution notes that each operation increases this monovariant by 1.
- **3 points** for finishing **using** this monovariant (so this item can be awarded only if the previous one is awarded)

¶ Solution path 2.

- **0 points** Define $\mathcal{M}(a, b)$ as the number of times the square with row labels $\{a, a+1\}$ and column labels $\{b, b+1\}$ is moved.
- **0 points** For observing that a 2×2 square on the boundary can be toggled at most once.
- **1 point** Observe that
$$\mathcal{M}(a, b) + \mathcal{M}(a-1, b-1) - \mathcal{M}(a-1, b) - \mathcal{M}(a, b-1) \le 1$$
- **1 point** Consider $S_j := \sum_{(i,j) \text{ black}} i$, and observe that the difference between maximum and minimum is at most $22 \cdot 23$, and explain that each move increments S_{j+1} by 1 and decrements S_j by 1 for some j.
- **2 points** Using or not using the above item, prove that for all $1 \le a, b \le 44$ (i.e. all a, b that correspond to a genuine 2×2 square)
$$\mathcal{M}(a, b) \le \min\{a, 45 - a\} \cdot \min\{b, 45 - b\}$$
(There is another way to do this: a 2×2 square partitions the 45×45 grid into four grids of dimensions $a \times b$, $(45 - a) \times b$, $a \times (45 - b)$, $(45 - a) \times (45 - b)$, respectively. Then consider the number of white/black squares in each of the four grids)
- **4 points** Finish (using or not using the above item).

Note that at most one item can be awarded in this solution path. If the submission is eligible for both 1 point items, it earns 1 point.

For a correct solution that does not follow one of the solution paths, we will give full credit if it's fully correct, and no credit otherwise.

¶ Deductions. For solutions that are essentially correct, the following deductions may apply

- **-1 point** Some algebra/calculation mistake that affects the solution.

Construction (1 point)

- **0 points** for a wrong construction. This includes the bogus construction where I color (x, y) white iff both or none of $x \le 22$ and $y \le 22$ holds. (Or something of the same idea)
- **1 point** for a correct construction. Justification is not necessary; for example, just giving the correct construction for 5×5 that easily generalizes gets full credit.

§3.4 USEMO 4 — proposed by Xu Zi Jie

Problem statement

Determine all odd integers $n \ge 3$ with the following property: Let S denote the set of all positive integers less than n which are relatively prime to n, and let $k = \frac{1}{2}|S|$. Then one can label the integers in S by $a_1, \dots, a_{2k}$ in some order such that

$$\sum_{i=1}^{k} a_i^2 = \frac{1}{2}\sum_{i=1}^{k} a_i a_{i+k}.$$

§3.4a Solution

The answer is all odd positive integers greater than 1.

> **Remark.** This is not always true for even $n > 2$. A necessary condition is $\varphi(n)$ to be a multiple of 4, which for example shows that all numbers in the form $2p$ do not work, where $p \equiv 3 \pmod 4$ is a prime. However, $8, 12$ also does not work.

If we let the $2k$ numbers be $b_1 < b_2 < \dots < b_{2k}$, then we claim the following construction works:

$$(a_1, \dots, a_{2k)} = (b_1, b_2, \dots, b_k, b_{2k}, b_{2k-1}, \dots, b_{k+1}).$$

In other words, we seek to prove

$$2\sum_{i=1}^{k} b_i^2 = \sum_{i=1}^{k} b_i b_{2k+1-i}.$$

We give three different methods for proving that this construction is valid.

¶ First proof (author). Firstly, since $\gcd(x, n) = 1$ iff $\gcd(n - x, n) = 1$ for all positive integers $x < n$,

$$b_1 + b_{2k} = b_2 + b_{2k-1} = \dots = b_k + b_{k+1} = n.$$

Furthermore, since n is odd,

$$b_1 < b_2 < \dots < b_k < \frac{n}{2} < b_{k+1} < \dots < b_{2k}.$$

Thus, each of $2b_1, \dots, 2b_k$ is a positive integer less than n and coprime to n. Hence together with $n - 2b_1, \dots, n - 2b_k$, these are $2k$ pairwise distinct positive integers less than n and coprime to n, meaning that $2b_1, \dots, 2b_k, n - 2b_1, \dots, n - 2b_k$ must be a permutation of $b_1, b_2, \dots, b_{2k}$, so their sum of squares are equal. Combining this with $n - 2b_i = b_{2k+1-i} - b_i$ for all $1 \le i \le k$, we see that

$$\begin{aligned}\sum_{i=1}^{2k} b_i^2 &= \sum_{i=1}^{k} (2b_i)^2 + \sum_{i=1}^{k} (b_{2k+1-i} - b_i)^2 \\ &= \sum_{i=1}^{k} 4b_i^2 + \sum_{i=1}^{2k} b_i^2 - 2\sum_{i=1}^{k} b_i b_{2k+1-i}.\end{aligned}$$

Therefore

$$2\sum_{i=1}^{k} b_i^2 = \sum_{i=1}^{k} b_i b_{2k+1-i}.$$

¶ Second proof (Helio Ng). Write $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}$, where $p_1 < p_2 < \dots < p_r$ are primes. For any $s \in S = \{1, 2, \dots, n\}$, define

$$f(s) = \begin{cases} 3s^2 - ns & \text{if } s < n/2 \\ 0 & \text{if } s > n/2 \end{cases}$$

then we have

$$\begin{aligned} 2\sum_{i=1}^{k} a_i^2 - \sum_{i=1}^{k} a_i a_{i+k} &= \sum_{i=1}^{k} (2b_i^2 - b_i b_{2k+1-i}) \\ &= \sum_{i=1}^{k} (2b_i^2 - b_i(n - b_i)) \\ &= \sum_{i=1}^{k} (3b_i^2 - nb_i) \\ &= \sum_{\substack{s \in S \\ (s,n)=1}} f(s). \end{aligned}$$

Using the inclusion–exclusion principle, we further rewrite this as

$$\begin{aligned} \sum_{\substack{s \in S \\ (s,n)=1}} f(s) &= \sum_{s \in S} f(s) - \sum_{\substack{s \in S \\ (s,n)>1}} f(s) \\ &= \sum_{s \in S} f(s) - \sum_{i} \sum_{\substack{s \in S \\ p_i \mid s}} f(s) + \sum_{i<j} \sum_{\substack{s \in S \\ p_i p_j \mid s}} f(s) - \dots + (-1)^r \sum_{\substack{s \in S \\ p_1 p_2 \dots p_r \mid s}} f(s). \end{aligned}$$

We will prove that each individual summation equals zero. In fact, for any positive divisor k of n, we write $n = (2m + 1)k$ and note that

$$\begin{aligned} \sum_{i=1}^{2m+1} f(ik) &= \sum_{i=1}^{m} (3(ik)^2 - n(ik)) = \sum_{i=1}^{m} (3(ik)^2 - (2m+1)k(ik)) \\ &= \frac{3k^2(m)(m+1)(2m+1)}{6} - \frac{k^2(2m+1)(m)(m+1)}{2} = 0 \end{aligned}$$

so

$$\begin{aligned} &\sum_{s \in S} f(s) - \sum_{i} \sum_{\substack{s \in S \\ p_i \mid s}} f(s) + \sum_{i<j} \sum_{\substack{s \in S \\ p_i p_j \mid s}} f(s) - \dots + (-1)^r \sum_{\substack{s \in S \\ p_1 p_2 \dots p_r \mid s}} f(s) \\ &= 0 - \sum_{i} 0 + \sum_{i<j} 0 - \dots + (-1)^r (0) = 0 \end{aligned}$$

as desired.

§3.4b Marking scheme

¶ For incomplete solutions At most **1 point** can be earned in this section.

- **1 point** Claiming that if I order the $2k$ numbers in S in increasing order $b_1 < \dots < b_{2k}$ then $(b_1, \dots, b_k, b_{2k}, b_{2k-1}, \dots, b_{k+1})$ works.
- **1 point** Proving it works for prime p, even if the recipe for the general construction is not explained.
- **0 point** Mention pairing between x and $n - x$.

¶ Solution 1 (author) The following are additive.

- **2 points** Stating that $2b_1, \dots, 2b_k$ and $n-2b_1, \dots, n-2b_k$ comprise of all elements of S.
- **4 points** for finishing.

¶ Solution 2 (Helio Ng) The following are additive.

- **0 points** for reducing the problem to

$$3\sum_{j=1}^{k} b_j^2 = n\sum_{j=1}^{k} b_j.$$

- **0 points** for stating Mobius inversion or PIE.
- **1 point** for applying Mobius inversion or PIE, and explain what we do for each term in the sum. For example, state that we are proving a version of $3\sum_{j=1}^{k} b_j^2 = n\sum_{j=1}^{k} b_j$ where b_j range over all multiples of m instead of b_j required to be coprime to n.
- **5 points** for finishing.

¶ Deductions For essentially complete solutions, the following deductions apply, and are additive.

- **-1 points** for wrong answer. (For example, saying a small number doesn't work).
- **-1 points** For another mistake

§3.5 USEMO 5 — proposed by Kornpholkrit Weraarchakul

Problem statement

Azza and Bob are playing the *squeakuences game*, a game whose rules depend on two positive integers n and g known to both players. A *squeakuence* is an ordered sequence of 100 integers (not necessarily positive). At the start of the game, Azza gives Bob a list of n different squeakuences, and Bob secretly picks one squeakuence and copies it into a notebook which Azza cannot see.

On each turn of the game, Azza makes up to g guesses for the squeakuence currently in the notebook. Bob hears all of Azza's guesses. If any of Azza's guesses are correct, the game ends and Azza wins. Otherwise, Bob privately chooses an index $1 \le i \le 100$ and an integer $\delta \in \{-1, 0, 1\}$, and secretly modifies the squeakuence written in the notebook by adding δ to its ith entry. (Azza is not told either i or δ. The values of i and δ can change from turn to turn.)

Find the smallest real number α for which there exists $C > 0$ making the following statement true: Azza can always guarantee winning the squeakuences game provided that $g > Cn^\alpha$.

§3.5a Solution

The answer is $\alpha = 0.99$.

¶ Azza's strategy when $\alpha = 0.99$. For $\alpha = \frac{99}{100}$, let $m := \left\lceil n^{\frac{1}{100}} \right\rceil$. Azza will choose $n \le m^{100}$ different sequences with terms in $\{1, 2, \dots, m\}$.

Consider all the sequences that have at least one term not in $\{2, 3, \dots, m-2\}$, of which there are

$$m^{100} - (m-3)^{100} = O\left(n^{\frac{99}{100}}\right).$$

Azza guesses all of them, and in doing so can ensure that the next sequence after Bob's modification has terms in $\{1, 2, \dots, m-1\}$. Continuing inductively, we see that after $m - 3$ rounds Azza ensures that the sequence has terms in $\{1, 2, 3\}$ at which point Azza can guess all such sequences.

¶ Proof that Azza cannot win when $\alpha < 0.99$ We show that if $\alpha < \frac{99}{100}$ then Azza cannot guarantee winning.

Lemma

Let X be any set of lattice points in $\mathbb{R}^{100}$ and define X_j to be the projection of X onto the $x_j^\perp$ hyperplane. Then there exists j for which

$$|X_j| \ge \frac{1}{100} \cdot |X|^{\frac{99}{100}}.$$

Proof. Define a *column* to be the set of points obtained by taking all points along a line in the x_j-direction for some j.

First we claim that if $\sum_j |X_j| \le B$ then there exists a column of points of size at most $B^{\frac{1}{99}}$. Assume not, so that every column has more than $B^{\frac{1}{99}}$ points, and in particular take such a column in a x_1-line. Taking the x_2-lines through these points, there are more than $B^{\frac{2}{99}}$ points in a x_1x_2-plane. Repeating this yields more than B points in a $x_{100}^{\perp}$-plane, contradiction.

Next we show that $\sum_j |X_j| \le B$ implies $|X| \le B^{\frac{100}{99}}$ by induction with base case $B = 1$. By the claim, there exists a column C with at most $B^{\frac{1}{99}}$ points, and let $X' = X \setminus C$. Then $\sum_j |X_j| \le B - 1$, and by the induction hypothesis $|X'| \le (B-1)^{\frac{100}{99}}$, hence $|X| \le |X'| + B^{\frac{1}{99}} \le B^{\frac{100}{99}}$ as desired. The lemma follows from the claim: if for all j we have $|X_j| < \frac{1}{100} \cdot |X|^{\frac{99}{100}}$ then $\sum_j |X_j| < |X|^{\frac{99}{100}}$, so by the claim we have $|X| < \left(|X|^{\frac{99}{100}}\right)^{\frac{100}{99}}$, contradiction. □

Define two sequences P and Q to be *adjacent*, denoted by $P \sim Q$, if one can be modified to the other by Bob.

Corollary

Let X be any set of sequences and define $S = \{(P, Q) \mid P \in X, Q \notin X, P \sim Q\}$. Then $|S| \ge \frac{1}{100} \cdot |X|^{\frac{99}{100}}$.

Proof. By the lemma, there exists a projection X_j of X onto the $x_j^{\perp}$ hyperplane with $|X_j| \ge \frac{1}{100} \cdot |X|^{\frac{99}{100}}$. Then S contains the points that are one unit in the x_j coordinate below the $|X_j|$ points in X with the minimum x_j coordinates that project to the same element of X_j. □

By the corollary, the size of the set of possible modified sequences will increase by at least $\frac{1}{100} \cdot n^{\frac{99}{100}}$ at every turn. Since $\alpha < \frac{99}{100}$, this increase is greater than the number g of Azza's guesses for sufficiently large n, so Azza cannot guarantee winning.

§3.5b Marking scheme

The crux is to realize that the problem is related to the surface area of subsets of integer lattices.

- **1 point** for claiming the answer of $\alpha = \frac{99}{100}$.
- **2 points** for *either* establishing Azza's strategy, or proving $\alpha < 0.99$ is losing for Azza conditioned on the corollary.
- **5 points** for proving either bound as well as the corollary.
- **5 points** for proving both bounds conditioned on the corollary.
- **7 points** for proving both bounds as well as the corollary.

A citation that includes a complete and accurate statement of a named theorem such as the Loomis-Whitney Inequality and a proof that this theorem implies the corollary constitutes a valid proof of the corollary. However, a reference to a "well-known result" without proper citation does not constitute a valid proof.

§3.6 USEMO 6 — proposed by Kornpholkrit Weraarchakul

Problem statement

Let k be a positive integer not divisible by 6. Suppose that there exists a prime p such that p divides both $2025^k - 1$ and $2026^k - 1$. Prove that $p < 3^k$.

§3.6a Solution

We show two approaches, an elementary one and an advanced one.

¶ First solution using polynomials. The basic idea of the solution is that (when $6 \nmid k$), we have $X^k - 1$ and $(X+1)^k - 1$ are coprime. Hence Bézout theorem promises the existence of $P_0, Q_0 \in \mathbb{Q}[x]$ such that

$$(X^k - 1) \cdot P_0 + \left((X+1)^k - 1\right) \cdot Q_0 = 1.$$

Our basic strategy is to give an upper bound on a positive integer N such that, when we multiply both sides by N, clears the denominators in the coefficients of P_0 and Q_0. Then by choosing $X = 2025$, we get a bound $p \mid N$. By writing N as the product of not-too-big integers, we can bound p.

We show the details now.

Claim — When $6 \nmid k$, the polynomials $X^k - 1$ and $(X+1)^k - 1$ do not have common roots in $\mathbb{C}$ (i.e. are relatively prime in $\mathbb{Q}[X]$).

Proof. If z was a common root, then we would have $|z| = |z+1| = 1$. This only occurs if $z = \frac{-1 \pm \sqrt{3}i}{2}$ and $z + 1 = \frac{1 \pm \sqrt{3}i}{2} = e^{\pm \frac{\pi i}{3}}$, which would imply $6 \mid k$. □

For the rest of the solution, set $\omega := e^{\frac{2i\pi}{k}}$. Also assume $k > 1$.

Claim — There exists polynomials P and Q with *integer* coefficients such that

$$(X^k - 1) \cdot P(X) + ((X+1)^k - 1) \cdot Q(X) = N$$

for

$$N := k \cdot \prod_{u=0}^{k-1} ((\omega^u + 1)^k - 1).$$

Proof. Let $P_0, Q_0 \in \mathbb{Q}[X]$ satisfy $(X^k - 1)P_0(X) + ((X+1)^k - 1)Q_0(X) = 1$ as described at the start by Bézout lemma; we may assume $\deg P_0 \le k - 1$ and $\deg Q_0 \le k - 1$.

We are going to make P_0 and Q_0 explicit via Lagrange interpolation. For all $u \in \{0, 1, \dots, k-1\}$, by substituting $X = \omega^u$, we get

$$Q_0(\omega^u) = \frac{1}{((\omega^u + 1)^k - 1)}.$$

Hence, Lagrange interpolation gives

$$Q_0(X) = \sum_{u=0}^{k-1} \frac{1}{(\omega^u + 1)^k - 1} \prod_{v \ne u} \frac{X - \omega^v}{\omega^u - \omega^v} = \sum_{r=0}^{k-1} \left(\sum_{u=0}^{k-1} \frac{w^{-u(r+1)}}{k((\omega^u + 1)^k - 1)} \right) X^r.$$

Then,

$$NQ_0(X) = \sum_{r=0}^{k-1} \left(\sum_{u=0}^{k-1} w^{-u(r+1)} \prod_{v \neq u} ((\omega^v + 1)^k - 1) \right) X^r.$$

Each coefficient is a symmetric polynomial in the kth roots of unity, which must be an integer. So, $P(X) = NP_0(X)$ and $Q(X) = NQ_0(X)$ satisfy the claim. $\square$

To continue, rewrite N as

$$N = k \cdot \prod_{u=0}^{k-1} ((\omega^u + 1)^k - 1) = k \cdot \prod_{u=0}^{k-1} \left(\prod_{v=0}^{k-1} (\omega^u - \omega^v + 1) \right).$$

So as a crude estimate, N certainly divides the number

$$N' := k \cdot \prod_{a=0}^{k-1} \prod_{b=0}^{k-1} \left(\prod_{u=0}^{k-1} (\omega^{au} - \omega^{bu} + 1) \right)$$

(it is possible to be more careful with avoiding double-counting pairs (a, b), but this N' is good enough for the bound we want, and this way we don't have to think too much about prime factors of k). For each fixed ordered pair (a, b), the number

$$T_{a,b} := \prod_{u=0}^{k-1} (\omega^{au} - \omega^{bu} + 1)$$

is an integer with absolute value at most $|T_{a,b}| \le 3^k$, since it can be written as a symmetric polynomial in the kth roots of unity.

Finally, note that $p \mid N'$, either p divides k or p divides some $T_{a,b}$. Either way, $p \le 3^k$.

> **Remark.** It is possible to rephrase a more refined version of this result in terms of the so-called "resultant" R of the polynomials $X^k - 1$ and $(X+1)^k - 1$, which has the property $p \mid R$ exactly when the polynomials have a common root modulo p. This is a more advanced way to phrase the solution above.

¶ Modification to the first solution Let r be a primitive kth root of unity mod p. Then, there exist $a, b < k$ such that $2025 = r^a$ and $2026 = r^b$. Therefore, $r^b = r^a + 1 \implies r^b - r^a - 1 = 0$. Also, $r^k - 1 = 0$. This solution is similar to the first solution, but uses r in place of 2025 and $X^b - X^a - 1$ in place of $(X+1)^k - 1$.

Because $6 \nmid k$, $r^k - 1$ and $r^b - r^a - 1$ do not have common roots. By Bézout theorem, there exists polynomials P_0 and Q_0 such that

$$(X^k - 1) \cdot P_0 + \left(X^b - X^a - 1 \right) \cdot Q_0 = 1.$$

Plugging in $X = \omega^u$ for $u = 0, 1, \dots, k-1$, we get

$$Q_0(\omega^u) = \frac{1}{\omega^{bu} - \omega^{au} - 1}.$$

By Lagrange interpolation,

$$Q_0(X) = \sum_{u=0}^{k-1} \frac{1}{\omega^{bu} - \omega^{au} - 1} \prod_{v \neq u} \frac{X - \omega^v}{\omega^u - \omega^v} = \sum_{u=0}^{k-1} \frac{\sum_{m=0}^{k-1} x^m \omega^{-um}}{k(\omega^{bu} - \omega^{au} - 1)}.$$

Let $N = k\prod_{u=0}^{k-1}(\omega^{bu} - \omega^{au} - 1)$. Let $P = NP_0$ and $Q = NQ_0$. Then, P and Q are integer polynomials such that

$$(X^k - 1) \cdot P + \left(X^b - X^a - 1\right) \cdot Q = N.$$

Plugging in $X = r$, we get $p \mid N$. Clearly $p \nmid k$, so $p \mid \prod_{u=0}^{k-1}(\omega^{bu} - \omega^{au} - 1)$. However,

$$\left|\prod_{u=0}^{k-1}(\omega^{bu} - \omega^{au} - 1)\right| = \prod_{u=0}^{k-1}|\omega^{bu} - \omega^{au} - 1| \le 3^k.$$

¶ Solution via algebraic number theory (Kaixin Wang). As before, set $\omega := e^{\frac{2i\pi}{k}}$. Let $K = \mathbb{Q}(\omega)$, $\mathcal{O}_K$ be the integral closure of $\mathbb{Z}$ in K, and choose any prime ideal $\mathfrak{p}$ of $\mathcal{O}_K$ lying over $p\mathbb{Z}$.

We know that

$$\prod_{i=0}^{k-1}(2025 - \omega^i) = 2025^k - 1 \in p \cdot \mathcal{O}_K \subseteq \mathfrak{p}$$

and since $\mathfrak{p}$ is prime ideal, we know $2025 - \omega^i \in \mathfrak{p}$ for some i. Similarly $2026 - \omega^j \in \mathfrak{p}$ for some j. Subtracting,

$$(2026 - \omega^j) - (2025 - \omega^i) = 1 + \omega^i - \omega^j \in \mathfrak{p} \implies p \mid \mathrm{Norm}_{K/\mathbb{Q}}(1 + \omega^i - \omega^j).$$

Since $6 \nmid k$, it follows that $1 + \omega^i - \omega^j \neq 0$, and so p is at most the norm above. However, straight from the definitions,

$$\left|\mathrm{Norm}_{\mathbb{Q}(\omega)/\mathbb{Q}}(1 + \omega^i - \omega^j)\right| = \prod_{\gcd(m,k)=1, 1\le m\le k} |1 + \omega^{mi} - \omega^{mj}| \le 3^k.$$

> **Remark.** Although this is not needed for the above solution, we mention that in fact $\mathcal{O}_K = \mathbb{Z}[\omega]$ exactly.

> **Remark** (Alexander Wang)**.** It is also possible to get a bound that "depends on 2025", in the following way. Notice that $2025^a 2026^b$ should be a kth power modulo p for all a and b. If we choose (a, b) among nonnegative integers with $a + b < 2\sqrt{k}$, then all the integers are distinct and there are more than k of them. So p must be smaller than the maximum of them, hence
> $$p < 2026^{2\sqrt{k}}.$$

§3.6b Marking scheme

Incomplete solutions can receive the following non-additive partial credit points.

- **0 Points** for proving the statement for sufficiently high k when the bound depends on the numbers 2025 and 2026.
- **0 Points** for showing that $x^k - 1$ and $(x+1)^k - 1$ have no common roots when $6 \nmid k$. (We consider this fact to be well-known, and that the difficulty of the problem is instead realizing how to use it, e.g. with Bézout.)

- **2 Points** for showing that there exist rational-coefficient polynomials $P(x)$ and $Q(x)$ such that
$$(x^k - 1)P(x) + ((x+1)^k - 1)Q(x) = 1.$$
Similar statements that will also receive credit (non-exhaustive list):
  - 1 is replaced by an unknown constant rational number
  - $P(x)$ and $Q(x)$ are integer polynomials, and 1 is replaced by an unknown constant integer
  - $((x+1)^k - 1)$ is replaced by $((x-1)^k - 1)$

  This item is only worth **1 point** if the solution does not at least claim that $x^k - 1$ and $(x+1)^k - 1$ do not have common roots.

- **5 Points** for proving the second claim from the first solution or a similar statement.
  - In particular, any solution that explicitly computes an integer N such that there exists integer polynomials P and Q where $(x^k-1)P(x)+((x+1)^k-1)Q(x) = N$ will get this item.

Complete solutions can receive the following deductions.

- **-1 Point** for applying Bézout's theorem without noting that $x^k - 1$ and $(x+1)^k - 1$ do not have common roots; analogously, if one uses the solution via algebraic NT and doesn't explain why $1 - \zeta_k^{j'} + \zeta_k^j$ is nonzero.

4 Statistics

§4.1 Summary of scores for USEMO 2025

N	139	1st Q	2	Max	35
μ	10.62	Median	8	Top 3	30
σ	8.46	3rd Q	15	Top 12	26

§4.2 Problem statistics for USEMO 2025

	P1	P2	P3	P4	P5	P6
0	31	86	123	30	102	131
1	36	31	2	29	3	1
2	7	0	3	2	11	5
3	1	0	0	0	0	1
4	2	0	1	0	0	0
5	1	2	2	0	17	1
6	19	1	1	5	0	0
7	42	19	7	73	6	0
Avg	3.41	1.29	0.55	4.13	1.09	0.14
QM	4.55	2.75	1.82	5.22	2.35	0.63
#5+	**62**	**22**	**10**	**78**	**23**	**1**
%5+	%44.6	%15.8	%7.2	%56.1	%16.5	%0.7

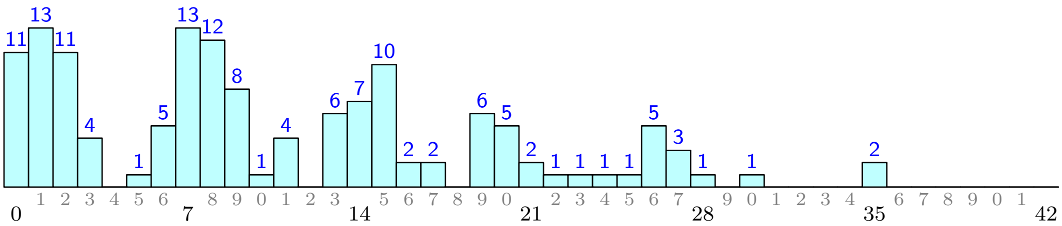
§4.3 Rankings for USEMO 2025

Sc	Num	Cu	Per
42	0	0	0.00%
41	0	0	0.00%
40	0	0	0.00%
39	0	0	0.00%
38	0	0	0.00%
37	0	0	0.00%
36	0	0	0.00%
35	2	2	1.44%
34	0	2	1.44%
33	0	2	1.44%
32	0	2	1.44%
31	0	2	1.44%
30	1	3	2.16%
29	0	3	2.16%

Sc	Num	Cu	Per
28	1	4	2.88%
27	3	7	5.04%
26	5	12	8.63%
25	1	13	9.35%
24	1	14	10.07%
23	1	15	10.79%
22	1	16	11.51%
21	2	18	12.95%
20	5	23	16.55%
19	6	29	20.86%
18	0	29	20.86%
17	2	31	22.30%
16	2	33	23.74%
15	10	43	30.94%

Sc	Num	Cu	Per
14	7	50	35.97%
13	6	56	40.29%
12	0	56	40.29%
11	4	60	43.17%
10	1	61	43.88%
9	8	69	49.64%
8	12	81	58.27%
7	13	94	67.63%
6	5	99	71.22%
5	1	100	71.94%
4	0	100	71.94%
3	4	104	74.82%
2	11	115	82.73%
1	13	128	92.09%
0	11	139	100.00%

§4.4 Histogram for USEMO 2025



§4.5 Full stats for USEMO 2025

Rank	P1	P2	P3	P4	P5	P6	Σ
1.	7	7	7	7	7	0	35
1.	7	7	7	7	7	0	35
3.	7	7	2	7	5	2	30
4.	7	7	0	7	7	0	28
5.	6	7	0	7	7	0	27
5.	6	7	0	7	5	2	27
5.	6	0	5	7	7	2	27
8.	7	7	0	7	5	0	26
8.	7	7	0	7	0	5	26
8.	7	1	6	7	5	0	26
8.	7	0	7	7	5	0	26
8.	7	0	7	7	5	0	26
13.	7	7	0	7	2	2	25
14.	1	7	2	7	7	0	24
15.	7	7	0	7	2	0	23
16.	7	1	7	2	5	0	22
17.	7	1	1	7	5	0	21
17.	7	0	7	7	0	0	21
19.	7	6	0	7	0	0	20
19.	7	1	0	7	5	0	20
19.	7	1	0	7	5	0	20
19.	2	0	4	7	5	2	20
19.	1	0	7	7	5	0	20
24.	7	5	0	7	0	0	19
24.	7	0	0	7	5	0	19
24.	7	0	0	7	5	0	19
24.	6	0	5	7	1	0	19
24.	6	0	0	7	5	1	19
24.	5	7	0	7	0	0	19
30.	7	1	0	7	2	0	17
30.	7	1	0	7	2	0	17
32.	7	0	0	7	2	0	16
32.	1	7	0	6	2	0	16
34.	7	1	0	7	0	0	15
34.	7	1	0	7	0	0	15
34.	7	1	0	7	0	0	15
34.	7	1	0	7	0	0	15

Rank	P1	P2	P3	P4	P5	P6	Σ
34.	7	1	0	7	0	0	15
34.	7	0	2	1	5	0	15
34.	7	0	1	7	0	0	15
34.	7	0	0	7	1	0	15
34.	3	0	0	7	5	0	15
34.	1	7	0	7	0	0	15
44.	7	7	0	0	0	0	14
44.	7	1	0	6	0	0	14
44.	7	0	0	7	0	0	14
44.	7	0	0	7	0	0	14
44.	7	0	0	7	0	0	14
44.	6	1	0	7	0	0	14
44.	6	0	0	7	1	0	14
51.	6	7	0	0	0	0	13
51.	6	1	0	6	0	0	13
51.	6	0	0	7	0	0	13
51.	6	0	0	7	0	0	13
51.	6	0	0	7	0	0	13
51.	6	0	0	7	0	0	13
57.	4	0	0	7	0	0	11
57.	2	0	0	7	2	0	11
57.	1	7	0	1	2	0	11
57.	1	1	0	7	2	0	11
61.	1	1	0	6	2	0	10
62.	7	1	0	1	0	0	9
62.	6	1	0	2	0	0	9
62.	2	0	0	7	0	0	9
62.	1	7	0	1	0	0	9
62.	1	1	0	7	0	0	9
62.	1	1	0	7	0	0	9
62.	1	1	0	7	0	0	9
62.	0	0	0	7	2	0	9
70.	7	0	0	1	0	0	8
70.	7	0	0	1	0	0	8
70.	7	0	0	1	0	0	8
70.	7	0	0	1	0	0	8
70.	7	0	0	1	0	0	8
70.	1	0	0	7	0	0	8
70.	1	0	0	7	0	0	8
70.	1	0	0	7	0	0	8
70.	1	0	0	7	0	0	8
70.	1	0	0	7	0	0	8
70.	1	0	0	7	0	0	8
70.	0	1	0	7	0	0	8
82.	6	0	0	1	0	0	7
82.	6	0	0	1	0	0	7
82.	1	5	0	1	0	0	7
82.	0	7	0	0	0	0	7
82.	0	0	0	7	0	0	7

Rank	P1	P2	P3	P4	P5	P6	Σ
82.	0	0	0	7	0	0	7
82.	0	0	0	7	0	0	7
82.	0	0	0	7	0	0	7
82.	0	0	0	7	0	0	7
82.	0	0	0	7	0	0	7
82.	0	0	0	7	0	0	7
82.	0	0	0	7	0	0	7
82.	0	0	0	7	0	0	7
95.	6	0	0	0	0	0	6
95.	6	0	0	0	0	0	6
95.	6	0	0	0	0	0	6
95.	4	1	0	1	0	0	6
95.	0	0	0	6	0	0	6
100.	0	1	0	1	0	3	5
101.	2	0	0	1	0	0	3
101.	1	1	0	1	0	0	3
101.	1	1	0	1	0	0	3
101.	1	1	0	1	0	0	3
105.	2	0	0	0	0	0	2
105.	2	0	0	0	0	0	2
105.	2	0	0	0	0	0	2
105.	1	1	0	0	0	0	2
105.	1	1	0	0	0	0	2
105.	1	0	0	1	0	0	2
105.	1	0	0	1	0	0	2
105.	1	0	0	1	0	0	2
105.	1	0	0	1	0	0	2
105.	1	0	0	1	0	0	2
105.	1	0	0	1	0	0	2
116.	1	0	0	0	0	0	1
116.	1	0	0	0	0	0	1
116.	1	0	0	0	0	0	1
116.	1	0	0	0	0	0	1
116.	1	0	0	0	0	0	1
116.	1	0	0	0	0	0	1
116.	1	0	0	0	0	0	1
116.	0	1	0	0	0	0	1
116.	0	0	0	1	0	0	1
116.	0	0	0	1	0	0	1
116.	0	0	0	1	0	0	1
116.	0	0	0	1	0	0	1
116.	0	0	0	1	0	0	1
129.	0	0	0	0	0	0	0
129.	0	0	0	0	0	0	0
129.	0	0	0	0	0	0	0
129.	0	0	0	0	0	0	0
129.	0	0	0	0	0	0	0
129.	0	0	0	0	0	0	0
129.	0	0	0	0	0	0	0

Rank	P1	P2	P3	P4	P5	P6	Σ
129.	0	0	0	0	0	0	0
129.	0	0	0	0	0	0	0
129.	0	0	0	0	0	0	0
129.	0	0	0	0	0	0	0