The 6th US Ersatz Math Olympiad

Solutions and Results

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1 December 2024

Contents

1 Summary

§1.1 News and commentary

The sixth USEMO was held on October 26 – 27, 2024. A total of 59 students submitted at least one paper. This year's USEMO was sponsored by the CoRe Lab, Institute of Artificial Intelligence, Peking University. We are grateful for their support.

This USEMO came at a time for me when I was personally rather busy, because I was trying to complete all the calculations and writing for my PhD thesis before my deadline in December. I am grateful to my team of volunteer graders who worked quickly and efficiently in preparing the rubrics and marking the papers, even without much oversight from me. Most of the delay between the grading and the publication of the results is due to my own unresponsiveness. I am thankful to the volunteers and contestants for their patience!

§1.1a Expansion of USEMO to non-US students

There has been a lot of requests to consider expanding the USEMO to not be USA-only, but rather open to all high school students. Because of the sponsorship from CoRe Lab, I'd like to start doing this from 2025 onwards, but I need to think a bit about exactly how that would affect the eligibility for volunteers for grading and problem proposals. (Any suggestions to this end are welcome!) In any case, look for an announcement sometime in early 2025.

Note that at the moment I still plan to host the competition at a fixed time (12:30pm-5:00pm in US Eastern time), so *de facto* I expect most of the contestants to be in the Americas. But the proposed change is that *de jure* we won't have a US requirement anymore. Again, this is all just a proposal. Expect more details announced later.

§1.1b We finally got burned by a config issue

Finally, it is with a bit of chagrin that I need to remark that problem 3 of the USEMO as written was not correct in some cases due to a configuration issue that I did not notice while editing the problem statement from its original form. Thankfully (and maybe to nobody's surprise), nobody noticed the issue during the exam either and as far as I can tell it had no effect on the scores of the problem. In the final published version of the problems, we added the condition that triangle ABC is acute to avoid the angle equality issue.

§1.2 Thanks

I am once again grateful to many individuals who helped make this competition possible.

§1.2a Sponsors

We are grateful to be sponsored this year by the CoRe Lab, Institute of Artificial Intelligence, Peking University.

§1.2b Proposers of problems

I thank everyone who submitted problems for the USEMO, of which there are many. The list of authors who had at least one problem in the shortlist were Andrei Chirita, Archit Manas, Galin Totev, Jinyu Xie, Kornpholkrit Weraarchakul, Kristiyan Vasilev, Matsvei Zorka, Nikolai Beluhov, Siraphop Khawplad, and TongGeometry.

§1.2c Reviewers

I thank Andrew Gu, Maximus Lu, Nikolai Beluhov, Noah Walsh, and Oleg Kryzhanovsky for reviewing the proposed problems.

§1.2d Graders

Thanks to everyone who graded at least one paper: Abdullahil Kafi, Adit Aggarwal, Alec Sun, Alex Chui, Alexandru Bordei, Andrew Shishko, Anmol Tiwari, Arifa, Aryan Das, Atharv Harlalka, Basil Sousounis, Dinh Quoc Tri, George Nikolov, Hans Yu, Honjar Xing, Jeeho Byun, Kanav Talwar, Kevin Liu, Kevin Zhao, Kornpholkrit Weraarchakul (Numton), Krishiv Khandelwal, Lasitha Vishwajith Jayasinghe, Lavish Khariwal, Lee Yiu Sing, Luca Seiki Pereira Fujii, Luis André, Mandar Kasulkar, Mihir Singhal, Mixtilinear Graph, Ricky Chen, Sanjana Chacko, Seongjin Shim, Shreeansh Hota, Shreya Mundhada, Siripurapu Bhuvan, Smochina Vladislav, Tiger Li, Victor Kostadinov, Wietze Koops, William Liu, Yasser Merabet, Yousif Wameedh, Ziyad Elamrani.

Special thanks to those who served as problem captains (who this year not only had to design rubrics but also were asked to sign off on every paper for their assigned problems): Alec Sun, Hans Yu, Kevin Zhao, and Mihir Singhal.

§1.2e Other supporters

I would like to thank the Art of Problem Solving for offering the software and platform for us to run the competition. Special thanks to Jo Welsh for dealing with all my support requests.

2 Results

If you won one of the seven awards, please reach out to `usemo@evanchen.cc` to claim your prize!

§2.1 Top Scores

Congratulations to the top three scorers, who win the right to propose problems to future instances of USEMO.

1st place Alexander Wang (32 points)

2nd place Oron Wang (29 points)

3rd place Feodor Yevtushenko (28 points)

§2.2 Special awards

See the Rules for a description of how these are awarded. (Note in particular that students already in the top three above aren't considered for special awards.)

Youth prize Channing Yang

Top female Ekam Kaur

Top day 1 Andrew Brahms, Angela Liu, Evan Fan, Haofang Zhu, Jiahe Liu, Liam Reddy[1], Liran Zhou, Mingyue Yang, Rohan Bodke, Royce Yao

Top day 2 Daniel Ge

§2.3 Honorable mentions

This year we award Honorable Mention to anyone scoring at least 20 points (who is not in the top three already). The HMs are listed below in alphabetical order.

Channing Yang

Daniel Ge

Ekam Kaur

Evan Fan

Jiahe Liu

Liam Reddy

Rohan Bodke

Royce Yao

[1]From the output of `sort -R`, we randomly selected Liam Reddy for the monetary prize.

§2.4 Distinction

The Distinction award is awarded for either scoring at least 14 points or in the top 25 of scores, whichever is more inclusive. This year, the 25th place student scored 9 points, so Distinction awards recognize any student with at least this score. The Distinction awards are listed below in alphabetical order.

Andrew Brahms

Angela Liu

Atticus Stewart

Benjamin Fu

Benny Wang

Evil Chin

Grant Blitz

Haofang Zhu

Joey Zheng

Liran Zhou

Luka Stopar

Mingyue Yang

Ruilin Wang

Shruti Arun

3 Solutions and marking schemes

§3.1 USEMO 1 — proposed by Galin Totev

Problem statement

There are 1001 stacks of coins S_1, S_2, $\dots$, S_{1001}. Initially, stack S_k has k coins for each $k = 1, 2, \dots, 1001$. In an operation, one selects an ordered pair (i, j) of indices i and j satisfying $1 \le i < j \le 1001$ subject to two conditions:

- The stacks S_i and S_j must each have at least one coin.
- The ordered pair (i, j) must *not* have been selected in any previous operation.

Then, if S_i and S_j have a coins and b coins, respectively, one removes $\gcd(a, b)$ coins from each stack.

What is the maximum number of times this operation could be performed?

§3.1a Solution

The answer is $500 \cdot 501 = 250500$. Our solution is split into two parts.

¶ Construction. Firstly, we will give a valid construction. We start by performing operations $(1001, 1000)$, $(1001, 999)$, $\dots$, $(1001, 1)$, in order. By induction, at each step $(1001, j)$, S_{1001} will have $j + 1$ coins and thus, since $\gcd(j + 1, j) = 1$, one coin will be removed from each stack. At the end of this process, 1000 operations will have been performed. Then stack S_{1001} will have one coin; we discard it. The remaining (nonempty) stacks will have $1, 2, \dots, 999$ coins, and no operation will have been performed between any of them. Thus we can repeat this process, performing operations with the 999-coin stack and the rest of the stacks in descending order.

Repeating this process until all the stacks have been discarded, we perform

$$1000 + 998 + \cdots + 2 = 500 \cdot 501$$

operations, as desired.

¶ Proof of bound. To prove this is the maximum number of operations we can perform, we bound the total number of operations. The stacks S_1, $\dots$, S_{500} can only participate in at most

$$1 + \cdots + 500 = \frac{500 \cdot 501}{2}$$

operations (since each operation removes at least one coin from them). The remaining 501 stacks can only perform $\binom{501}{2} = \frac{500 \cdot 501}{2}$ operations between themselves, since each pair can only perform the operation once. Thus, in total, we can perform at most $500 \cdot 501$ operations.

§3.1b Marking scheme

The solution is split into two parts: the lower bound (construction), worth **4 points**, and the upper bound, worth **3 points**. These parts are completely additive.

In general, minor errors will be worth a deduction, but please message the channel when you find any that are not included in the rubric so that we can add them to the rubric for a deduction. Errors purely in arithmetic (even in the final answer), including incorrect summation of an arithmetic series, will **not** merit any deduction.

§3.1c Lower bound

The following items are available (here $n = 1001$):

- **1 point** for any construction that achieves $\Omega(n^2)$ moves.
- **2 points** for any construction that achieves $n^2/4 - O(n)$ moves.
- **4 points** for a correct lower bound construction (achieving $500 \cdot 501 = (n^2-1)/4$ moves).

Up to 1 point may be deducted for constructions that are correct as stated but for which insufficient justification is provided that the construction works. (For example, in the case of the construction of the official solution, no justification would be required, since the fact that it works is obvious enough not to require justification.)

§3.1d Upper bound

The following items are available:

- **1 point** for any correct upper bound that is **strictly** less than $\lfloor 1001 \cdot 1002/4 \rfloor = 250750$.
- **3 points** for a correct upper bound (of $500 \cdot 501 = (n^2-1)/4$).

§3.2 USEMO 2 — proposed by Andrei Chirita

Problem statement

Let k be a fixed positive integer. For each integer $1 \le i \le 4$, let x_i and y_i be positive integers such that their least common multiple is k. Suppose that the four points (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4) are the vertices of a non-degenerate rectangle in the Cartesian plane. Prove that $x_1x_2x_3x_4$ is a perfect square.

§3.2a Solution

It suffices to prove that $v_p(x_1x_2x_3x_4)$ is even for each prime $p \mid k$. Since the four points form a rectangle, we have

$$x_1 + x_3 = x_2 + x_4 \tag{3.1}$$
$$y_1 + y_3 = y_2 + y_4 \tag{3.2}$$
$$(x_1 - x_3)^2 + (y_1 - y_3)^2 = (x_2 - x_4)^2 + (y_2 - y_4)^2 \tag{3.3}$$
$$x_2x_4 - x_1x_3 = y_1y_3 - y_2y_4 \tag{3.4}$$

Let $v_p(k) = m$. For each $1 \le i \le 4$, we have

$$\max\left(v_p(x_i), v_p(y_i)\right) = m.$$

We split into cases.

- **Case 1.** If $v_p(x_i) = m$ for three i then $v_p(x_i) = m$ for the fourth by (3.1), so $2 \mid v_p(x_1x_2x_3x_4) = 4m$.
- **Case 2.** If $v_p(y_i) = m$ for three i then $v_p(y_i) = m$ for the fourth by (3.2). By (3.4) we have $p^{2m} \mid x_2x_4 - x_1x_3$. We now use the fact that $v_p(x \pm y) = \min(v_p(x), v_p(y))$ whenever $v_p(x) \ne v_p(y)$. Using the contrapositive, $v_p(x_1x_3), v_p(x_2x_4) \le 2m$ implies $v_p(x_1x_3) = v_p(x_2x_4)$ and hence $2 \mid v_p(x_1x_2x_3x_4)$.
- **Case 3.** Otherwise, $v_p(x_i) = m$ for exactly two i. If these i are consecutive (cyclically), for example $i = 1, 2$ without loss of generality, then from (3.1) we have $p^m \mid x_3 - x_4$. Since $v_p(x_3), v_p(x_4) \le m$, we have $v_p(x_3) = v_p(x_4)$ and hence $2 \mid v_p(x_1x_2x_3x_4)$. If these i are not consecutive, for example $i = 1, 3$ without loss of generality, then from (3.1) we have $p^m \mid x_2 + x_4$, and we can finish using the same argument as in the consecutive case.

> **Remark.** There are rectangles which satisfy the hypothesis, for instance (a, b), (a, ab), (ab, ab), (ab, b) where $\gcd(a, b) = 1$.

§3.2b Marking scheme

Recall the four equations

$$x_1 + x_3 = x_2 + x_4$$
$$y_1 + y_3 = y_2 + y_4$$

$$(x_1 - x_3)^2 + (y_1 - y_3)^2 = (x_2 - x_4)^2 + (y_2 - y_4)^2$$
$$x_2x_4 - x_1x_3 = y_1y_3 - y_2y_4$$

which were numbered (3.1), (3.2), (3.3) and (3.4) in the provided solution.

The following are awarded marks, with items being *non-additive*:

- **2 points** for writing down (3.1), (3.2), and (3.4). The condition $(y_2 - y_1)(y_4 - y_1) = -(x_2 - x_1)(x_4 - x_1)$ or similar in lieu of (3.4) is accepted but (3.3) or similar is not.
- **2 points** for solving the problem in at least one of Case 2 and Case 3, irrespective of whether the conditions (3.1), (3.2), and (3.4) are written down.
- **5 points** for both of the above items.
- **7 points** for a complete solution.

§3.3 USEMO 3 — proposed by Matsvei Zorka

Problem statement

Let ABC be an acute triangle with incenter I. Two distinct points P and Q are chosen on the circumcircle of ABC such that

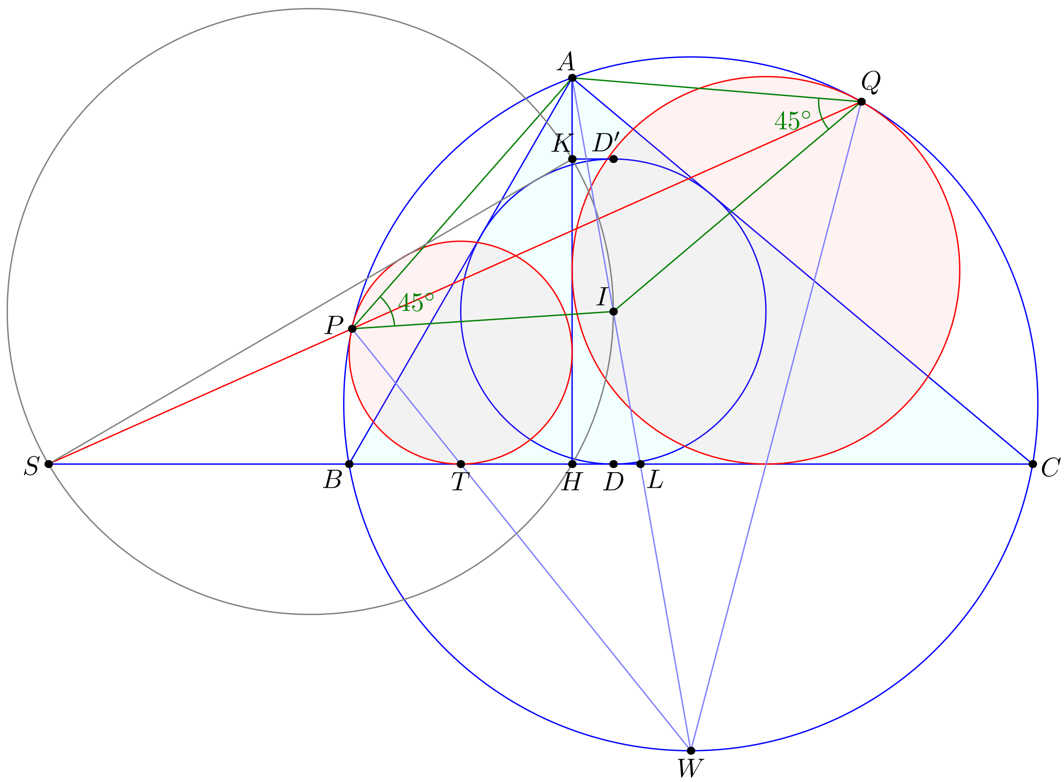
$$\angle API = \angle AQI = 45^\circ.$$

Lines PQ and BC meet at S. Let H denote the foot of the altitude from A to BC. Prove that $\angle AHI = \angle ISH$.

§3.3a Solution

We give three solutions.

¶ Solution via Tebault circles from the author. Construct the Tebault circles ω_1 and ω_2 which are tangent to (ABC), side BC, and cevian AH.



The key claim is that P and Q coincide with the tangency points of the circles we just drew:

Claim — Points P and Q are the tangency points of ω_i and (ABC).

Proof. Let ω_1 touch BC at T and (ABC) at P'. We will show that $\angle AP'I = 45^\circ$.

Let W be the arc midpoint of BC not containing A and let $L = \overline{AW} \cap \overline{BC}$. It is well known that W, T, P are collinear and (by "shooting lemma")

$$WT \cdot WP = WL \cdot WA = WI^2 = WB^2.$$

Hence we get

$$\measuredangle ITH = \measuredangle ITW - \measuredangle LTW = -\measuredangle P'IW - \measuredangle LTW.$$

From $AP'TL$ concyclic we also have

$$-\measuredangle P'IW - \measuredangle LTW = -\measuredangle P'IW - \measuredangle IAP' = \measuredangle AP'I.$$

It is left to remember that $\measuredangle ITH = \pm 45^\circ$ because IT must be parallel to an angle bisector of $\measuredangle AHB$ by the properties of Tebault circles.

Similarly, $\angle AQ'I = 45^\circ$. So P and Q coincide with P' and Q' in some order. □

Claim — Point S is the center of the positive homothety which maps ω_1 to ω_2.

Proof. This follows by Monge theorem on ω_1, ω_2 and (ABC). □

Let D be the tangency point of incircle ω of triangle ABC with BC with antipode D' and let ℓ be another tangent form S to ω_1, ω_2 (which is a tangent to ω as well because of the Tebault circles properties). Let $K = \ell \cap AH$.

Claim — $SHIK$ is cyclic.

Proof. Because of the Tebault circles property, the intersection of cevian and the second tangent lies on the tangent to ω at D'. In our case, it follows KD' is tangent to ω and thus parallel to the line BC. As a consequence, SI is parallel to another angle bisector of $\measuredangle SKD'$. Hence, $\measuredangle KIS = 90^\circ = \measuredangle KHS$ as desired. □

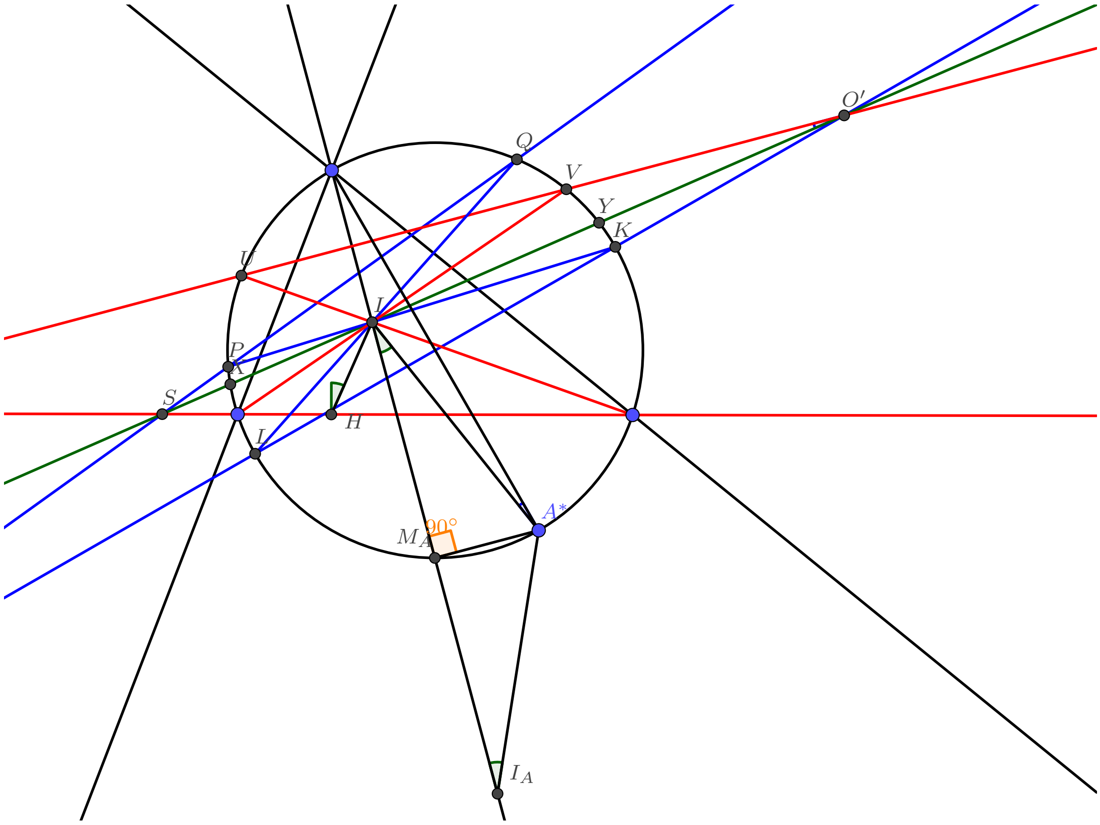
From the third lemma we may conclude that

$$\measuredangle KHI = \measuredangle KSI = \measuredangle ISH,$$

as desired.

Remark. For basic properties about Tebault circles contains the relevant facts (regrettably, it is in Russian only): `https://geometry.ru/articles/protasovtebo.pdf`.

¶ Solution by Nikolai Beluhov.



Let A^* be the antipodal point of A on (ABC).

Lemma 3.3.1

$\measuredangle AHI = \measuredangle A^*IA$.

Proof. Let I_A be the excenter opposite A. Since $ABH \sim AA^*C$ and $ACH \sim AA^*B$, we get that $AA^* \cdot AH = AB \cdot AC$. Since $ABI \sim AI_AC$ and $ACI \sim AI_AB$, similarly $AI \cdot AI_A = AB \cdot AC$. But also $\measuredangle A^*AI = \measuredangle HAI$, and we conclude that $AIH \sim AA^*I_A$. So $\measuredangle AHI = \measuredangle AI_AA^*$ Let M_A be the midpoint of II_A. Then as M_A is on (ABC), we know that $AM_A \perp M_AA^*$, and so A^* is on the perpendicular bisector of II_A. Thus $\measuredangle AI_AA^* = \measuredangle A^*IA$,as desired.

Note that $AIH \sim AA^*I_A$ can be seen simply by taking the $\sqrt{AB \cdot AC}$-inversion at A as well. $\square$

From now on, we will be proving $\measuredangle A^*IA = \measuredangle(SI, BC)$.

Let K, L be the second intersections of PI, QI and (ABC), respectively. Then the angle condition is equivalent to that KL is the perpendicular bisector of AA^*.

Lemma 3.3.2

SI passes through the circumcenter O' of triangle AIA^*.

Proof. Let U and V be the midpoints of the arcs AB and AC of (ABC). Then as U, V are both on the perpendicular bisector of AI, we know that $O' = UV \cap KL$. It thus suffices to show that UV, KL, SI are concurrent.

Let IS intersect (ABC) at X, Y. Then

$$(S, I; X, Y) \overset{C}{=} (B, U; X, Y) \overset{V}{=} (I, UV \cap SI; X, Y)$$

and

$$(S, I; X, Y) \overset{P}{=} (Q, K; X, Y) \overset{L}{=} (I, KL \cap SI; X, Y).$$

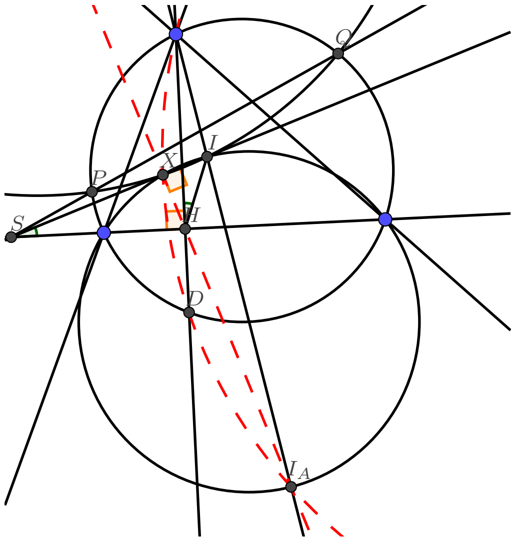
Thus $UV \cap SI = KL \cap SI$, as desired. $\square$

The rest is a straightforward angle chase. We know that

$$\measuredangle(SI, BC) = \measuredangle SO'K + \measuredangle(KL, BC) = \measuredangle IA^*A + \measuredangle A^*AI = \measuredangle A^*IA,$$

as desired.

¶ Solution by Hans Yu. Let I_A be the A-excenter. Now let X be the second intersection of (PIQ) and (BIC). Since S is the radical center of the circles $(PIQ), (BIC)$ and (ABC), we see that SXI are collinear.



Claim 3.3.3 — It suffices to show that I_AHX are collinear.

Proof of the claim. Suppose that I_AHX are collinear. It is well-known that BC bisects I_AHI (say, by harmonicity of $(I, I_A; A, BC \cap AI)$). Therefore $\measuredangle IHA = \measuredangle AHX = 90^\circ - \measuredangle(XI, AH) = 90^\circ - \measuredangle(SI, AH) = \measuredangle HSI$, as desired. Here we used that $\measuredangle HXI = \measuredangle I_AXI = 90^\circ$. $\square$

Now let AH intersects (ABC) again at D.

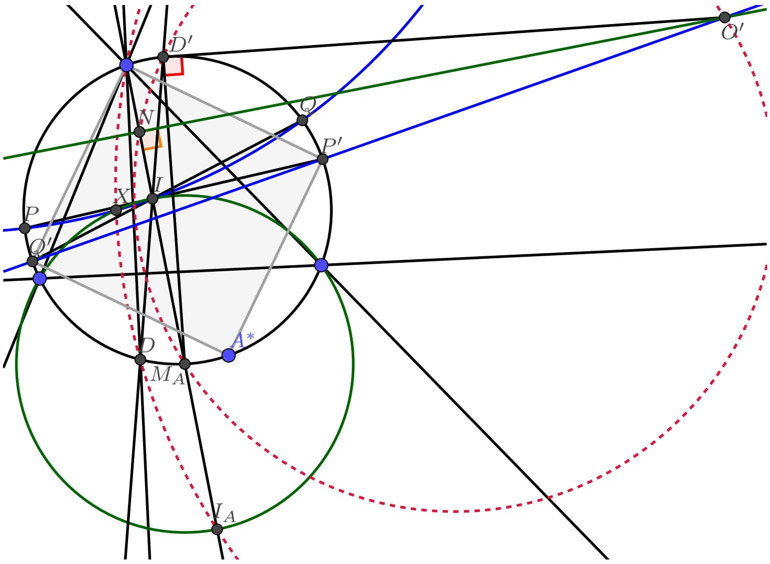
Claim 3.3.4 — It suffices to show that AI_AXD are concyclic.

Proof of the claim. Suppose that AI_AXD are concyclic, then $H = BC \cap AD$ is the radical center of $(ABCD), (AI_AXD)$ and $(BICI_A)$. Hence H is also on I_AX, and we are done by Claim 3.3.4. $\square$

Let PI, QI intersect the circumcircle of ABC again at P', Q', and let A^* be the antipodal point of A on the circumcircle of ABC. Then since $\angle APP' = \angle API = 45°$ and so is $\angle AQQ'$, we see that $AP'A^*Q'$ is a square.

Let M_A be the midpoint of II_A. Let O be the circumcenter of ABC. Then we can see that M_A is the midpoint of arc A^*D as well: to see this, note that $\measuredangle DAA^* = \measuredangle OM_AA = \measuredangle M_AAO = \measuredangle M_AAA^*$.

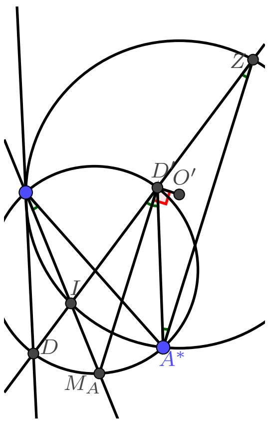
Let O' be the circumcenter of AIA^*, and let DI intersects (ABC) again at D'.



Claim 3.3.5 — It suffices to show that $M_AD' \perp O'D'$.

Proof of the claim. Suppose that $M_AD' \perp O'D'$. Let N be the midpoint of AI. Then $M_AND'O'$ are concyclic.

Consider the inversion at I sending A to M_A. This inversion fixes the circumcircle of ABC. It sends D' to D and N to I_A as $I_AI = 2M_AI$. Now to see where X is sent to, note that (PQI) is sent to the line $P'Q'$, which is the perpendicular bisector of AA^*. Moreover, (BI_AC) is sent to the line through N perpendicular to AI, which is just the perpendicular bisector of AI. Thus X is sent to the circumcenter of AIA^*, which is O'. As a consequence, O' is sent to X, and so the circle $M_AND'O'$ is sent to the circle AI_ADX, and we are done by Claim 2. $\square$



To finish off, we will show that $M_A D' \perp O'D'$. Let Z be on ID' such that $A^*Z \parallel D'M_A$. Then $\measuredangle IZA^* = \measuredangle ID'M_A = \measuredangle DD'M_A = \measuredangle M_A D'A^* = \measuredangle M_A AA^* = \measuredangle IAA^*$, showing that Z is on the circumcircle of AIA^*. As a consequence, O' is on the perpendicular bisector of A^*Z. However, since $\measuredangle D'ZA^* = \measuredangle DZA^* = \measuredangle DD'M_A = \measuredangle M_A D'A^* = \measuredangle ZA^*D'$, we have that D' is on the perpendicular bisector of A^*Z as well. This shows that $O'D' \perp A^*Z \parallel M_A D'$, as desired.

§3.3b Marking scheme

For all solutions, the following are *not awarded marks*:

- Rephrasing the angle condition in terms of $PI \cap (ABC)$ and $QI \cap (ABC)$.
- Swapping $\measuredangle AHI$ with some other angles, even if they are used in the official solutions.

For solutions not using Tebault circles, the following items are **not additive**:

- **2 points** Showing that SI passes through the circumcenter of AIA^*. Alternatively, show that SI, the perpendicular bisector of AI and the perpendicular bisector of AA^* are concurrent.

  Note: Points are still awarded if SI is replaced by some other two points that clearly lie on S, I, the perpendicular bisector of AI is replaced by the line connecting two points that are clearly on the perpendicular bisector, or the perpendicular bisector of AA^* is similarly replaced.
- **2 points** Show that if SI passes through the circumcenter of AIA^*, then the statement holds true.
- **7 points** Complete solution.

For solutions using Tebault circles, the following items are **additive**.

- **+1 point** Show that P, Q are tangency points of the Tebault circles to the circumcircle.
- **+1 point** Show that S is the center of homothety of the two Tebault circles.
- **+3 points** Construct K and show that $SHIK$ are concyclic.
- **+2 points** Finishing the solution.

§3.4 USEMO 4 — proposed by Kornpholkrit Weraarchakul

Problem statement

Find all sequences $a_1, a_2, \dots$ of nonnegative integers such that for all positive integers n, the polynomial

$$1 + x^{a_1} + x^{a_2} + \dots + x^{a_n}$$

has at least one integer root. (Here $x^0 = 1$.)

§3.4a Solution

The only answer is $a_1 = 1$ and $a_2 = a_3 = \dots = 0$.

It's clear that this works because for each n, the requested integer root is $x = -n$. We now prove this is the only solution.

In general, let

$$F_n(x) := 1 + x^{a_1} + \dots + x^{a_n}.$$

Claim — Let p be any prime. Then

$$F_{p-1}\left(-(p-1)\right) = 0.$$

Proof. Let $-r$ be the integer root of F_{p-1}, for $r > 0$. From

$$F_{p-1}(1) = p \text{ and } F_{p-1}(-r) = 0 \implies 1 + r \mid p$$

we conclude that $r = p - 1$ (since p is prime), as needed. □

We continue to focus on $F_{p-1}(-(p-1)) = 0$ for any prime p, that is,

$$1 + \sum_{i=1}^{p-1} (-1)^{a_i} (p-1)^{a_i} = 0.$$

The idea is that the big terms are *way* too big. Indeed, set $M := \max(a_1, \dots, a_{p-1})$ and assume that M occurs for $k \geq 1$ indices among $\{a_1, \dots, a_{p-1}\}$. Hence in the displayed sum, there are k terms equal to $(-1)^M (p-1)^M$. Hence

$$\begin{aligned} k \cdot (p-1)^M &= \left| 1 + \sum_{i : a_i < M} (-1)^{a_i} (p-1)^{a_i} \right| \\ &\leq 1 + (p-1-k) \cdot (p-1)^{M-1} \end{aligned}$$

which gives

$$1 \geq (p-1)^{M-1} \left[k \cdot (p-1) - (p-1-k) \right] = (p-1)^{M-1} \left[(k-1)(p-1) + k \right].$$

This could only happen if $k = 1$ and $M = 1$. In other words, for any prime p, the terms $(a_1, \dots, a_{p-1})$ consist of a single 1 and all other 0's.

In particular, for $p = 2$ we have $a_1 = 1$. Hence $a_2 = a_3 = \dots = 0$ as desired.

§3.4b Marking scheme

For all solutions, the following are *not awarded marks*:

- Getting the correct answer with no explanation.
- Showing that all roots are negative.

For correct solutions:

- **7 points** for a complete solution that shows $a_1 = 1$ and $a_{i>1} = 0$ is the only possible solution.

Solutions that are not complete will get the **maximum** points any item for partial credit may award them, and these points are *not additive*:

- **+2 points** for noting that if n is one less than a prime, $1 - p$ must be the integer root.
- **+4 points** for using Extremal Principle and taking the largest value of a_i into account.
- **+4 points** for showing that $P_{p-1}(0) = rQ_{p-1}(0)$ so $p - 1 \mid P_{p-1}(0)$.
- **+5 points** for setting the absolute value equation up in either of the two previous cases.
- **+5 points** for showing that $\sum_{i=0}^{p-1} x^{a_i} = x + (p - 1)$.

For all solutions which are incomplete with errors, the following deductions apply and are all additive. An incomplete solution can only get a deduction if it applies for the *complete* portion:

- **-1 point** for not checking that $a_1 = 1$ and $a_{i>1} = 0$ is indeed valid.
- **-1 point** for not indicating in any way that we check all primes p.

§3.5 USEMO 5 — proposed by Kornpholkrit Weraarchakul

Problem statement

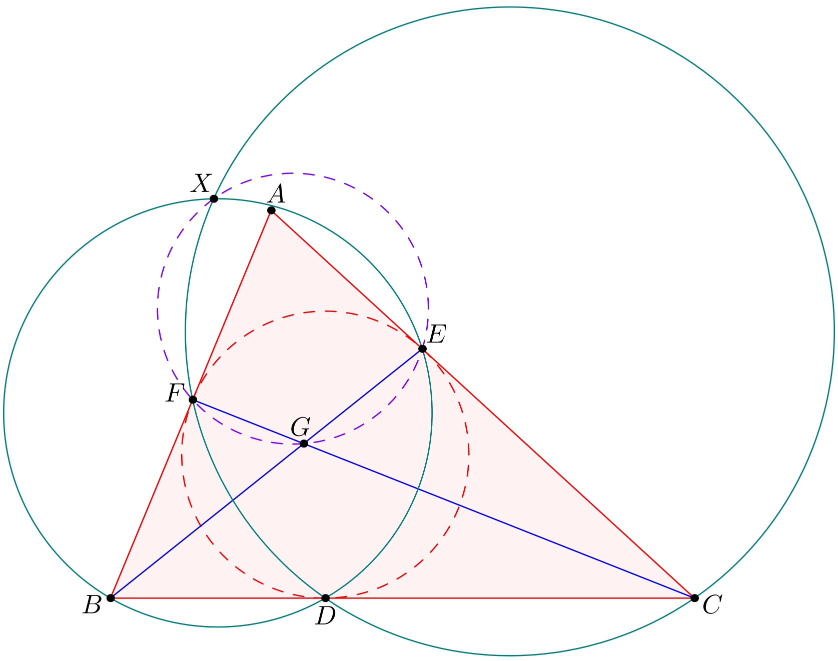
Let ABC be a scalene triangle whose incircle is tangent to BC, CA, AB at D, E, F respectively. Lines BE and CF meet at G. Prove that there exists a point X on the circumcircle of triangle EFG such that the circumcircles of triangles BCX and EFG are tangent, and

$$\angle BGC = \angle BXC + \angle EDF.$$

§3.5a Solution

We give two solutions.

¶ First solution. Let (BDE) and (CDF) intersect at X. We will show that this is the desired point.



Claim — X is on (EFG).

Proof. This is evident by Miquel's theorem on the triangle BGC with the points D, E, F on sides BC, BG and CG. Alternatively, angle chasing suffices:

$$\measuredangle EXF = \measuredangle EXD + \measuredangle DXF = \measuredangle EBD + \measuredangle DCF = \measuredangle EGF. \qquad \square$$

Claim — $\measuredangle BGC = \measuredangle BXC + \measuredangle EDF$.

Proof. Compute

$$\measuredangle BXC = \measuredangle BXD + \measuredangle DXC$$

$$\begin{aligned} &= \measuredangle BED + \measuredangle DFC \\ &= \measuredangle(BE, DE) + \measuredangle(DF, FC) \\ &= \measuredangle(BE, FC) - \measuredangle(DE, DF) \\ &= \measuredangle BGC - \measuredangle EDF. \end{aligned}$$

□

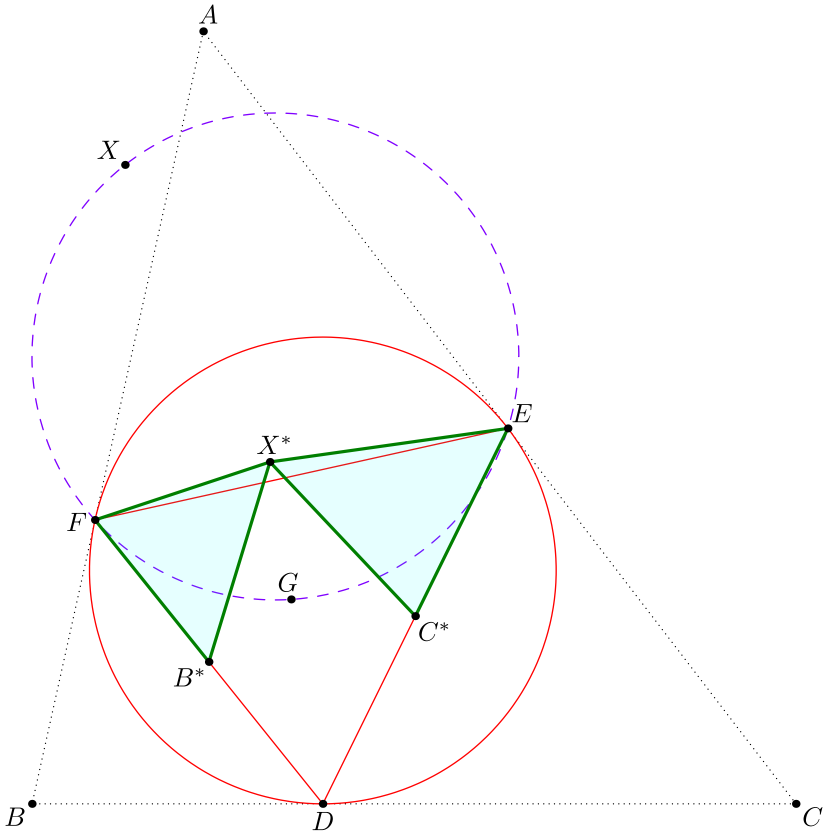
Claim — (BXC) is tangent to (EFX) at X.

Proof. It suffices to prove $\measuredangle BCX - \measuredangle FEX = \measuredangle BXF$. We do this by chasing the angles as follows.

$$\begin{aligned} \measuredangle BCX &= \measuredangle DCX \\ &= \measuredangle DFX \\ &= \measuredangle DFE + \measuredangle EFX \\ &= \measuredangle BDE + \measuredangle EFX \\ &= \measuredangle EGF - (\measuredangle EGF - \measuredangle BDE) + \measuredangle EFX \\ &= \measuredangle EGF - (\measuredangle EXF - \measuredangle BDE) + \measuredangle EFX \\ &= \measuredangle EGF + (\measuredangle BXE - \measuredangle EXF) + \measuredangle EFX \\ &= \measuredangle EGF + \measuredangle BXE + \measuredangle FEX \\ &= \measuredangle EXF + \measuredangle BXE + \measuredangle FEX \\ &= \measuredangle FEX + \measuredangle BXF. \end{aligned}$$

□

¶ **Second solution.** Here we give an alternative solution after showing the first two claims. Take the inversion with respect to the incircle. We denote the inverse of a point by $-^*$. Then $D = D^*$, $E = E^*$ and $F = F^*$. Moreover, B^* is the midpoint of DF and C^* is the midpoint of DE.



Since $X = (BDE) \cap (CDF)$, we have $X^* = (B^*DE) \cap (C^*DF)$. Note that since (B^*DE) and (C^*DF) intersect at D, X^* and also B^*F, C^*E pass through D, we have

$$\triangle X^*C^*E \sim \triangle X^*FB^*.$$

Thus, as $B^*C^* \parallel EF$, we have

$$\measuredangle B^*C^*X^* = \measuredangle FEX^* + \measuredangle EX^*C^* = \measuredangle FEX^* + \measuredangle B^*X^*F.$$

showing that $(B^*C^*X^*)$ and (EFX^*) are tangent at X^*. Inverting back, we get (BCX) and (EFX) are tangent at X, as desired.

§3.5b Marking scheme

For all solutions, the following items are additive:

- **+1 point** for a correct description of X **that allows a ruler-compass construction determining a unique** X. Most common examples are (1) $X = (BDE) \cap (CDF)$, and (2) $Y = DF \cap (EFG), Z = DE \cap (EFG)$ and $X = BY \cap CZ$.
- **+1 point** for showing that $(BDE), (CDF), (EFG)$ are concurrent either by angle chasing or stating Miquel's theorem.
- **+1 points** for proving that X satisfies the angle condition.
- **+2 points** for proving that X satisfies that (BCX) is tangent to (EFX).
- **+2 points** for getting all of the above items.

No partial points are awarded to non-synthetic solutions unless a synthetic statement clearly equivalent to one of the items above is stated and proved explicitly in the solution.

No deductions are made for configuration issues.

§3.6 USEMO 6 — proposed by Nikolai Beluhov

Problem statement

Let n be an odd positive integer and consider an $n \times n$ chessboard of n^2 unit squares. In some of the cells of the chessboard, we place a knight. A knight in a cell c is said to *attack* a cell c' if the distance between the centers of c and c' is exactly $\sqrt{5}$ (in particular, a knight does not attack the cell which it occupies).

Suppose each cell of the board is attacked by an even number of knights (possibly zero). Show that the configuration of knights is symmetric with respect to all four axes of symmetry of the board (i.e. the configuration of knights is both horizontally and vertically symmetric, and also unchanged by reflection along either diagonal of the chessboard).

§3.6a Solution

Let $n = 2k + 1$, and coordinatise the cells of the board by (x, y) with $0 \le x, y \le 2k$.

Consider the width-two outer frame F of the board formed by all cells (x, y) which satisfy at least one of the four conditions $x \le 1$, $x \ge 2k - 1$, $y \le 1$, and $y \ge 2k - 1$.

Observe that, if two valid configurations agree on F, then they agree everywhere. Indeed, suppose not, and consider the earliest cell (x, y) where they disagree, going from left to right and from top to bottom. The number of knights which attack cell $(x - 1, y - 2)$ will then differ by one between the two configurations, and we arrive at a contradiction.

Thus it suffices to show that F is fully symmetric.

Let $f(x, y) = 0$ when cell (x, y) is empty and $f(x, y) = 1$ when it is occupied by a knight. We treat the values of f as remainders modulo two, with $1 + 1 = 0$. We also set $f(x, y) = 0$ for all x and y which are not the coordinates of a cell.

Consider any set of cells S (these must all be valid cells), and let T be the set of all cells with an odd number of knight neighbours in S. Then, in a valid configuration, the sum of f over T will always be zero. (**A**)

For convenience, given a cell (x, y), let $f_{p,q}(x, y)$ be the sum of all eight expressions of the form either $f(x \pm p, y \pm q)$ or $f(x \pm q, y \pm p)$. (Notice that some of these expressions might coincide, and the coinciding ones will cancel out.)

For some $m \ge 1$, consider any subboard M of our board of size $(2m + 1) \times (2m + 1)$ centered at (x, y) (such that $m \le x, y \le 2k - m$). By (**A**) applied to the set of all cells in M with the same checkerboard colour as (x, y), we get that $f_{m-1,m}(x, y) + f_{m+1,m+2}(x, y) = 0$ (see Figure 3.1(i)).

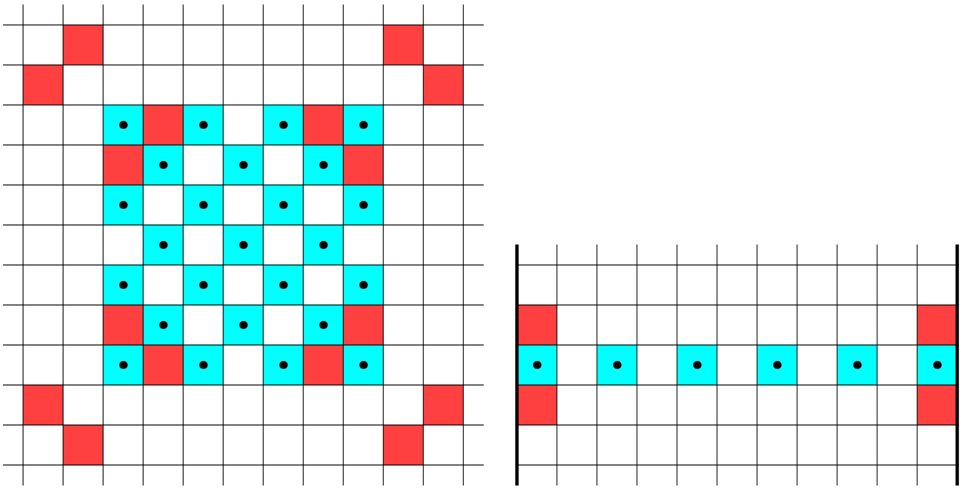
Let us add together these identities over m, $m - 2$, $m - 4$, $\dots$, all the way down to the remainder of m modulo two. Then all corresponding expressions of the form $f_{\ell,\ell+1}(x, y)$ with $\ell < m$ will cancel out, and in the end we will arrive at $f_{m+1,m+2}(x, y) = 0$, for all $m \le x, y \le 2k - m$. (**B**)

By (**B**) with $(x, y) = (k, k + 1)$ and $m = k - 1$, we get that $f(0, 0) = f(2k, 0)$. (**C**)

By (**A**) with $S = \{(0, y), (2, y), (4, y), \dots, (2k, y)\}$, we get that $f(0, y-1) + f(2k, y-1) = f(0, y + 1) + f(2k, y + 1)$ for all y with $0 \le y \le 2k$ (see Figure 3.1(ii)). (**D**)

By induction on y, with (**C**) for the base case and (**D**) for the induction step, we conclude that $f(0, y) = f(2k, y)$ for all y. (**E**)

By (**B**) with $(x, y) = (k + 1, k + 1)$ and $m = k - 1$, we get that $f(0, 1) = f(1, 0)$. (**F**)

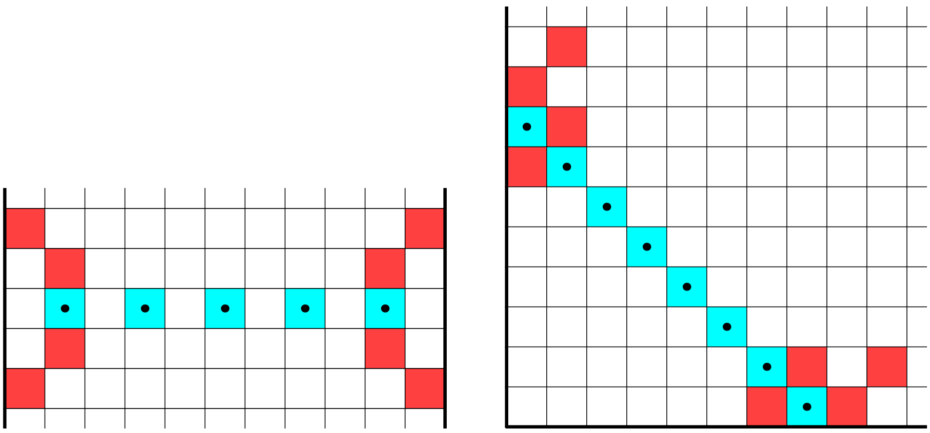


(i) (ii)

Figure 3.1: Visual demonstration of (**B**) and (**D**)

Similarly to (**F**), also $f(2k-1,0) = f(2k,1)$. By (**E**) with $y=1$, (**F**), and the previous identity, we arrive at $f(1,0) = f(2k-1,0)$. (**G**)

By (**A**) with $S = \{(1,y),(3,y),(5,y),\dots,(2k-1,y)\}$, we get that $f(1,y-1)+f(2k-1,y-1)+f(1,y+1)+f(2k-1,y+1)+f(0,y-2)+f(2k,y-2)+f(0,y+2)+f(2k,y+2)=0$. Using (**E**) where y is substituted with $y\pm 2$, the latter simplifies to $f(1,y-1)+f(2k-1,y-1) = f(1,y+1)+f(2k-1,y+1)$ for all y with $0 \le y \le 2k$. (**H**)



By induction on y, with (**G**) for the base case and (**H**) for the induction step, we conclude that $f(1,y) = f(2k-1,y)$ for all y. (**I**)

Similarly to (**E**) and (**I**), also $f(x,0) = f(x,2k)$ and $f(x,1) = f(x,2k-1)$ for all x. (**J**)

By (**A**) with $S = \{(x,0),(x-1,1),(x-2,2),\dots,(0,x)\}$, we get that $f(x-1,0)+f(x,1)+f(x+1,0)+f(x+2,1) = f(0,x-1)+f(1,x)+f(0,x+1)+f(1,x+2)$ for all x with $0 \le x \le 2k$. (**K**)

Similarly to (**K**), also $f(x-2,2k-1)+f(x-1,2k)+f(x,2k-1)+f(x+1,2k) = f(2k-1,x-2)+f(2k,x-1)+f(2k-1,x)+f(2k,x+1)$. By (**E**), (**I**), and (**J**), it follows that $f(x-2,1)+f(x-1,0)+f(x,1)+f(x+1,0) = f(1,x-2)+f(0,x-1)+f(1,x)+f(0,x+1)$ for all x with $0 \le x \le 2k$. (**L**)

By induction on x, with (**F**) for the base case and (**K**) and (**L**) for the induction step, we conclude that $f(x, 0) = f(0, x)$ and $f(x, 1) = f(1, x)$ for all x. (**M**)

Similarly to (**M**), also $f(x, 2k) = f(2k, x)$ and $f(x, 2k-1) = f(2k-1, x)$ for all x. (**N**)

By (**M**) and (**N**), we get that F is symmetric with respect to the unit-slope main diagonal of the board. By the same reasoning, F is symmetric with respect to the other main diagonal of the board as well.

Consider any cell (x, y) of F. When either $x \le 1$ or $x \ge 2k-1$, we get that $f(x, y) = f(2k-x, y)$ by (**E**) and (**I**). Otherwise, when either $y \le 1$ or $y \ge 2k-1$, we get that $f(x, y) = f(y, x)$ by diagonal symmetry, $f(y, x) = f(2k-y, x)$ by (**E**) and (**I**), and $f(2k-y, x) = f(2k-x, y)$ by diagonal symmetry once again.

Therefore, F is symmetric with respect to the vertical midline of the board. By the same reasoning, F is symmetric with respect to the horizontal midline of the board as well. The solution is complete.

> **Remark.** The number of knight configurations which satisfy the conditions of the problem is 2^n.
>
> This can be verified as follows. Let F_ℓ be the set of all cells (x, y) with $\min\{x, 2k-x, y, 2k-y\} = \ell$. Thus F_0, F_1, …, F_k form a partitioning of the board into pairwise disjoint concentric frames, with $F = F_0 \cup F_1$.
>
> First we place some knights on the cells of F so that F is fully symmetric. There are 2^n ways to do that.
>
> It is straightforward to see that we can now fill in F_2 uniquely so that every cell of F_0 is attacked by an even number of knights. Thus F_2 will be fully symmetric as well. After that, we can similarly fill in F_3 so that every cell of F_1 is attacked by an even number of knights, and so on and so forth. Therefore, every fully symmetric arrangement of knights within F can be extended to a valid configuration over the complete board in a unique manner.

> **Remark.** When n is even, the number of knight configurations which satisfy the conditions of the problem is 2^{2n}. Here follows a quick sketch of the proof.
>
> First we place some knights on rows 0 and 1 in an arbitrary manner. There are 2^{2n} ways to do that.
>
> It is straightforward to see that we can now fill in row 2 uniquely so that every cell of row 0 is attacked by an even number of knights. After that, we can similarly fill in row 3 uniquely so that every cell of row 1 is attacked by an even number of knights, and so on and so forth.
>
> It turns out that, at the end of this process, every cell of rows $n-2$ and $n-1$ will be attacked by an even number of knights as well. One proof relies on a series of applications of (**A**) to certain sets of cells S within rows 0, 1, …, $n-3$. Therefore, every arrangement of knights within the lowermost couple of rows can be extended to a valid configuration over the complete board in a unique manner.

§3.6b Marking scheme

We will give partial credit on an ad hoc basis, since we expect that the number of solutions worth partial credit will be very small. We give points for the following, though any other significant progress may be awarded partial credit, as we will decide when it comes up.

- **2 points** for proving symmetry by 180 degree rotation about the cell (k, k) (using the notation of the solution). This may be done on the entire board, only on F, or on any other set that determines the rest of the knight positions (without proof, unless it is not reasonably easy to see).

- **3 points** for proving any other type of symmetry, i.e., horizontal or vertical symmetry, symmetry across a main diagonal, or symmetry under 90 degree rotation.
- **7 points** for a complete solution.

Point deductions may be given for minor flaws, but we will be especially lenient with omitted details, since it's somewhat unreasonable to expect every step to be justified in detail.

4 Statistics

§4.1 Summary of scores for USEMO 2024

N	59	1st Q	4	Max	32
μ	9.95	Median	7	Top 3	28
σ	8.25	3rd Q	16	Top 12	17

§4.2 Problem statistics for USEMO 2024

	P1	P2	P3	P4	P5	P6
0	12	36	55	44	44	58
1	1	0	0	1	6	0
2	0	5	1	0	0	1
3	1	0	0	0	1	0
4	15	0	0	0	1	0
5	1	1	0	1	0	0
6	0	0	0	3	0	0
7	29	17	3	10	7	0
Avg	4.61	2.27	0.39	1.59	1.05	0.03
QM	5.36	3.86	1.60	3.25	2.52	0.26
#5+	**30**	**18**	**3**	**14**	**7**	**0**
%5+	%50.8	%30.5	%5.1	%23.7	%11.9	%0.0

§4.3 Rankings for USEMO 2024

Sc	Num	Cu	Per
42	0	0	0.00%
41	0	0	0.00%
40	0	0	0.00%
39	0	0	0.00%
38	0	0	0.00%
37	0	0	0.00%
36	0	0	0.00%
35	0	0	0.00%
34	0	0	0.00%
33	0	0	0.00%
32	1	1	1.69%
31	0	1	1.69%
30	0	1	1.69%
29	1	2	3.39%

Sc	Num	Cu	Per
28	1	3	5.08%
27	0	3	5.08%
26	0	3	5.08%
25	0	3	5.08%
24	0	3	5.08%
23	1	4	6.78%
22	1	5	8.47%
21	5	10	16.95%
20	1	11	18.64%
19	0	11	18.64%
18	0	11	18.64%
17	1	12	20.34%
16	3	15	25.42%
15	2	17	28.81%

Sc	Num	Cu	Per
14	5	22	37.29%
13	0	22	37.29%
12	1	23	38.98%
11	1	24	40.68%
10	0	24	40.68%
9	1	25	42.37%
8	1	26	44.07%
7	9	35	59.32%
6	1	36	61.02%
5	0	36	61.02%
4	13	49	83.05%
3	0	49	83.05%
2	0	49	83.05%
1	2	51	86.44%
0	8	59	100.00%

§4.4 Histogram for USEMO 2024