The 2nd US Ersatz Math Olympiad

Solutions and Results

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Contents

1 Summary

§1.1 Overview of the 2nd USEMO

The second USEMO was held on October 24 – 25, 2020. A total of 135 students took part.

Overall, I am happy with the quality of all six of the chosen problems. The main surprise on the paper was an unusually difficult problem 4; I think one can make the case that problem 5 was similar in difficulty, so the order of these two problems on the second day was rather arbitrary. So this is the usual lesson in why it is valuable to try all problems on a given day.

Given the difficulty of the competition, solving any single problem is a fine achievement; like last year I did not "water down" the exam despite the fact that it is a public event, and instead probably made it too difficult.

I continue to get requests to open the USEMO to a broader audience, either by having a multi-division contest or allowing international students to compete. As you all suspect, the concern is a grading bottleneck; even this year, we had difficulties completing the grading and resolving disagreements in the scores within a timely manner. Therefore, I think it is unlikely that the 2021 USEMO (which will likely take place in fall 2021, by which point the pandemic has hopefully ended) would be enlarged, but the possibility is still on the table for future contests as the volunteer base gradually grows larger over time. We will have to wait and see!

We hope everyone stays safe as this year draws to a close and look forward to the next event.

§1.2 Thanks

I am once again grateful to many individuals who helped make this competition possible.

I would like to thank the Art of Problem Solving for offering the software and platform for us to run the competition. Special thanks to Corinne who was my main contact this time.

§1.2.1 Proposers of problems

I gratefully acknowledge the receipt of 23 proposals from Anas Chentouf, Anant Mudgal, Ankan Bhattacharya, Arnav Pati, Borislav Kirilov, David Altizio, Galin Totev, Hu Man Keat, Jaedon Whyte, Jeffery Li, Konstantin Garov, Luke Robitaille, Nikolai Beluhov, Pitchayut Saengrungkongka, Pulkit Agarwal, Valentio Iverson.

§1.2.2 Reviewers

I am indebted to the reviewers of the packet, namely Anant Mudgal, Andrew Gu, Ankan Bhattacharya, Ashwin Sah, Krit Boonsiriseth, Mihir Singhal, Nikolai Beluhov, Sasha Rudenko, Tristan Shin, Vincent Huang.

§1.2.3 Graders

Thanks to everyone who signed up to help grade the competition (even if you ended up not being able to contribute during these difficult times): Aayam Mathur, Adam Kelly, Ahmed Shaaban, Anant Mudgal, Anas Chentouf, Ankan Bhattacharya, Anubhab Ghosal, Arifa Alam, Aritra Barua, Arman Raayatsanati, Aron Thomas, Bobby Shen, Brandon Wang, Brian Reinhart, Carl Schildkraut, Daniel Naylor, Daniel Sheremeta, David Schmitz, Dylan Dalida, Ejaife Ogheneobukome, Hadyn Tang, Hector Osuna, Hu Man Keat, Ivan Borsenco, Jeffery Li, Jeffrey Kwan, Jit Wu Yap, Justin Hua, Kai Wang, Kazi Aryan Amin, Krit Boonsiriseth, Le Duc Minh, Lim Jeck, Luke Robitaille, Matija Delic, Michael Greenberg, Mihir Singhal, Milica Vugdelić, Minjae Kwon, RedPig, Risto Atanasov, Rohan Goyal, Sasha Rudenko, Shashwat Kasliwal, Srijon Sarkar, Taes Padhihary, Tahmid Hameem Chowdhury, Tamim Iqbal, Thomas Luo, Ting-Wei Chao, Tristan Shin, Valentio Iverson, Xinke Guo-Xue, Yannick Yao, Zawadul Hoque.

2 Results

If you won one of the seven awards, please reach out to `usemo@evanchen.cc` to claim your prize!

§2.1 Top Scores

Congratulations to the top three scorers, who win the right to propose problems to future instances of USEMO.

1st place Noah Walsh (42 points)

2nd place Ankit Bisain (41 points)

3rd place Gopal Goel (33 points)

§2.2 Special awards

See the Rules for a description of how these are awarded. For the purposes of awarding monetary prizes, ties are broken more or less arbitrarily by considering the presentation of elegance of solutions (which is obviously subjective). When this occurs, the names of tied students are noted as well.

Top female Sanjana Das (28 points)

Youth prize Ram Goel (28 points); tied with...

- Kevin Wu

Top day 1 Brandon Chen (21 points on Day 1)

Top day 2 Edward Yu (14 points on Day 2); tied with...

- Kevin Wu
- Rishabh Das
- Samuel Wang
- Sogand Kamani
- William Yue

§2.3 Honorable mentions

This year we award Honorable Mention to anyone scoring at least 26 points. The HM's are listed below in alphabetical order.

Brandon Chen

Kevin Wu

Ram Goel

Rishabh Das

Sanjana Das

§2.4 Distinction

We award Distinction to anyone scoring at least 14 points (two fully solved problems). The Distinction awards are listed below in alphabetical order.

Alex Hu

Amol Rama

Andrew Yuan

Benjamin Jeter

David Dong

Dennis Chen

Derek Liu

Edward Xiong

Edward Yu

Ethan Liu

Ethan Zhou

Jason Cheah

Jeffrey Chen

Justin Lee

Kevin Min

Matthew Chen

Maximus Lu

Maxwell Sun

Paul Hamrick

Ramyro Corrêa Aquines

Raymond Feng

Rich Wang

Robert Yang

Rowechen

Ryan Li

Samuel Wang

Serena An

Sogand Kamani

Warren Bei

William Yue

3 Solutions to the Problems

§3.1 Solution to USEMO1, proposed by David Altizio

Which positive integers can be written in the form

$$\frac{\operatorname{lcm}(x,y)+\operatorname{lcm}(y,z)}{\operatorname{lcm}(x,z)}$$

for positive integers x, y, z?

Let k be the desired value, meaning

$$-k\operatorname{lcm}(x,z)+\operatorname{lcm}(x,y)+\operatorname{lcm}(y,z)=0.$$

Our claim is that the possible values are even integers.

Indeed, if k is even, it is enough to take $(x,y,z)=(1,k/2,1)$.

For the converse direction we present a few approaches.

First approach using ν_2 only We are going to use the following fact:

Lemma

If u, v, w are nonzero integers with $u+v+w=0$, then either

$$\begin{aligned}\nu_2(u)&>\nu_2(v)=\nu_2(w);\\ \nu_2(v)&>\nu_2(w)=\nu_2(u);\quad\text{or}\\ \nu_2(w)&>\nu_2(u)=\nu_2(v).\end{aligned}$$

Proof. Let's assume WLOG that $e=\nu_2(w)$ is minimal. If both $\nu_2(u)$ and $\nu_2(v)$ are strictly greater than e, then $\nu_2(u+v+w)=e$ which is impossible. So assume WLOG again that $\nu_2(v)=\nu_2(w)=e$. Then

$$u=-(2^e\cdot\text{odd}+2^e\cdot\text{odd})=-2^e\cdot\text{even}$$

so $\nu_2(u)\geq e+1$. □

However, if we assume for contradiction that k is odd, then

$$\begin{aligned}\nu_2(-k\operatorname{lcm}(x,z))&=\max(\nu_2(x),\nu_2(z))\\ \nu_2(\operatorname{lcm}(x,y))&=\max(\nu_2(x),\nu_2(y))\\ \nu_2(\operatorname{lcm}(y,z))&=\max(\nu_2(y),\nu_2(z)).\end{aligned}$$

In particular, the *largest* two numbers among the three right-hand sides must be equal. So by the lemma, there is no way the three numbers $(-k\operatorname{lcm}(x,z),\operatorname{lcm}(x,y),\operatorname{lcm}(y,z))$ could have sum zero.

Second approach using ν_p for general p We'll prove the following much stronger claim (which will obviously imply k is even).

Claim — We must have $\operatorname{lcm}(x, z) \mid \operatorname{lcm}(x, y) = \operatorname{lcm}(y, z)$.

Proof. Take any prime p and look at three numbers $\nu_p(x)$, $\nu_p(y)$, $\nu_p(z)$. We'll show that

$$\max(\nu_p(x), \nu_p(z)) \le \max(\nu_p(x), \nu_p(y)) = \max(\nu_p(y), \nu_p(z)).$$

If $\nu_p(y)$ is the (non-strict) maximum, then the claim is obviously true.

If not, by symmetry assume WLOG that $\nu_p(x)$ is largest, so that $\nu_p(x) > \nu_p(y)$ and $\nu_p(x) \ge \nu_p(z)$. However, from the given equation, we now have $\nu_p(\operatorname{lcm}(y, z)) \ge \nu_p(x)$. This can only occur if $\nu_p(z) = \nu_p(x)$. So the claim is true in this case too. □

Third approach without taking primes (by `circlethm`) By scaling, we may as well assume $\gcd(x, y, z) = 1$.

Let $d_{xy} = \gcd(x, y)$, etc. Now note that $\gcd(d_{xy}, d_{xz}) = 1$, and cyclically! This allows us to write the following decomposition:

$$\begin{aligned} x &= d_{xy} d_{xz} a \\ y &= d_{xy} d_{yz} b \\ z &= d_{xz} d_{yz} c. \end{aligned}$$

We also have $\gcd(a, b) = \gcd(b, c) = \gcd(c, a) = 1$ now.

Now, we have

$$\begin{aligned} \operatorname{lcm}(x, y) &= d_{xy} d_{xz} d_{yz} ab \\ \operatorname{lcm}(y, z) &= d_{xy} d_{xz} d_{yz} bc \\ \operatorname{lcm}(x, z) &= d_{xy} d_{xz} d_{yz} ac \end{aligned}$$

and so substituting this in to the equation gives

$$k = b \cdot \left(\frac{1}{a} + \frac{1}{c} \right).$$

For a, b, c coprime this can only be an integer if $a = c$, so $k = 2b$.

Remark. From $a = c = 1$, the third approach also gets the nice result that $\operatorname{lcm}(x, y) = \operatorname{lcm}(y, z)$ in the original equation.

§3.2 Solution to USEMO2, proposed by Pitchayut Saengrungkongka

Calvin and Hobbes play a game. First, Hobbes picks a family $\mathcal{F}$ of subsets of $\{1, 2, \dots, 2020\}$, known to both players. Then, Calvin and Hobbes take turns choosing a number from $\{1, 2, \dots, 2020\}$ which is not already chosen, with Calvin going first, until all numbers are taken (i.e., each player has 1010 numbers). Calvin wins if he has chosen all the elements of some member of $\mathcal{F}$, otherwise Hobbes wins. What is the largest possible size of a family $\mathcal{F}$ that Hobbes could pick while still having a winning strategy?

The answer is $4^{1010} - 3^{1010}$. In general, if 2020 is replaced by $2n$, the answer is $4^n - 3^n$.

Construction: The construction is obtained as follows: pair up the numbers as $\{1, 2\}$, $\{3, 4\}$, ..., $\{2019, 2020\}$. Whenever Calvin picks a numbers from one pair, Hobbes elects to pick the other number. Then Calvin can never obtain a subset which has both numbers from one pair. There are indeed $2^{2n} - 3^n$ subsets with this property, so this maximum is achieved.

Bound: The main claim is the following.

Claim — Fix a strategy for Hobbes and an integer $0 \le k \le n$. Then there are at least $\binom{n}{k}2^k$ sets with k numbers that Calvin can obtain after his kth turn.

Proof, due to Andrew Gu. The number of ways that Calvin can choose his first k moves is

$$2n \cdot (2n-2) \cdot (2n-4) \cdot \ldots (2n-2(k-1)).$$

But each k-element set can be obtained in this way in at most $k!$ ways (based on what order its numbers were taken). So we get a lower bound of

$$\frac{2n \cdot (2n-2) \cdot (2n-4) \cdot \ldots (2n-2(k-1))}{k!} = 2^k \binom{n}{k}. \qquad \square$$

Thus by summing $k = 0, \dots, n$ the family S is missing at least $\sum_{k=0}^{n} 2^k \binom{n}{k} = (1+2)^n = 3^n$ subsets, as desired.

Alternate proof of bound: Fix a strategy for Hobbes, as before. We proceed by induction on n to show there are at least 3^n missing sets (where a "missing set", like in the previous proof, is a set that Calvin can necessarily reach). Suppose that if Calvin picks 1 then Hobbes picks 2. Then the induction hypothesis on the remaining game gives that:

- there are 3^{n-1} missing sets that contain 1 but not 2;
- there are also 3^{n-1} missing sets that contain neither 1 nor 2.
- But imagining Calvin picking 2 first instead, applying the induction hypothesis again we find that there are 3^{n-1} missing sets which contain 2.

These categories are mutually exclusive, so we find there are at least 3^n missing sets, as needed.

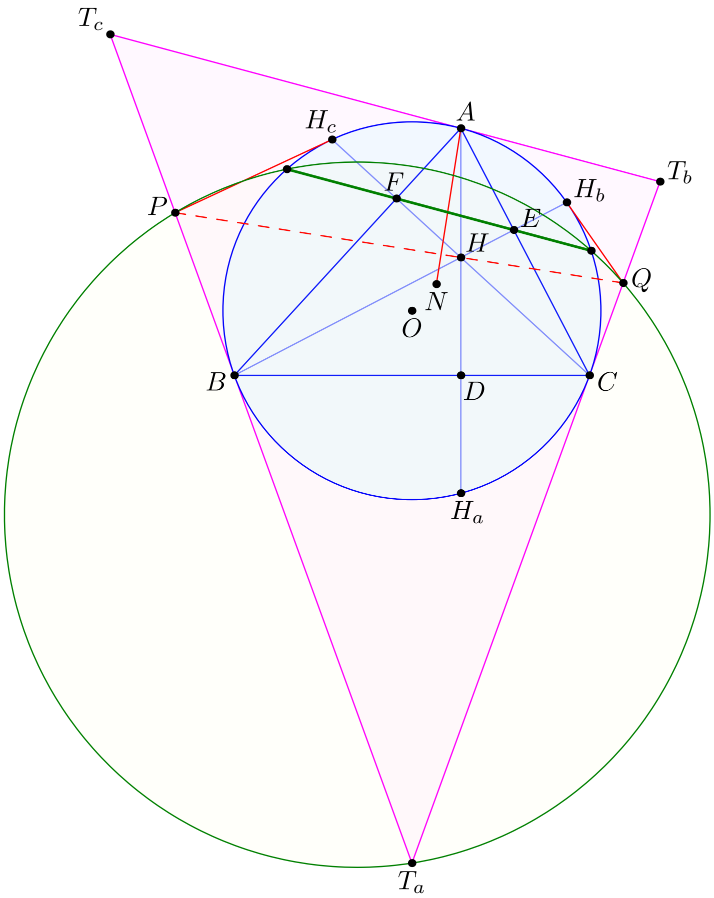
§3.3 Solution to USEMO3, proposed by Anant Mudgal

Let ABC be an acute triangle with circumcenter O and orthocenter H. Let Γ denote the circumcircle of triangle ABC, and N the midpoint of $\overline{OH}$. The tangents to Γ at B and C, and the line through H perpendicular to line AN, determine a triangle whose circumcircle we denote by ω_A. Define ω_B and ω_C similarly.

Prove that the common chords of ω_A, ω_B, and ω_C are concurrent on line OH.

We begin by introducing several notations. The orthic triangle is denoted DEF and the tangential triangle is denoted $T_aT_bT_c$. The reflections of H across the sides are denoted H_a, H_b, H_c. We also define the crucial points P and Q as the poles of $\overline{H_cB}$ and $\overline{H_bC}$ with respect to Γ.

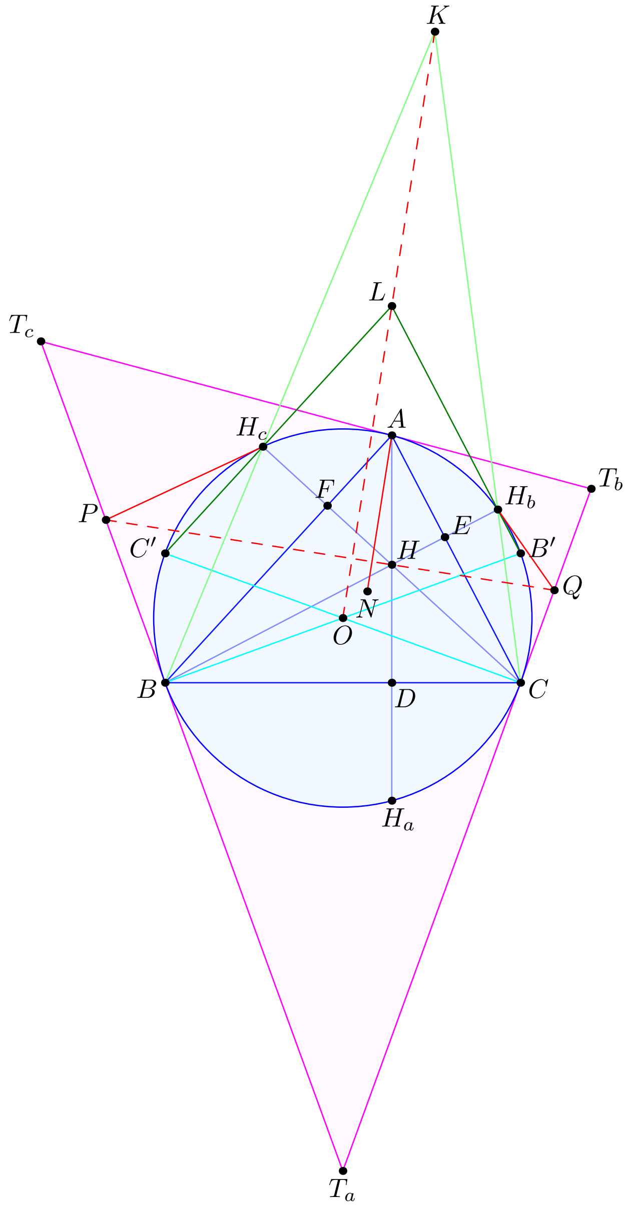
The solution, based on the independent solutions found by Anant Mudgal and Nikolai Beluhov, hinges on two central claims: that ω_A is the circumcircle of $\triangle T_aPQ$, and that $\overline{EF}$ is the radical axis of Γ and ω_A. We prove these two claims in turn.



Claim (Characterization of ω_A) — Line PQ passes through H and is perpendicular to $\overline{AN}$.

Proof. The fact that H lies on line PQ is immediate by Brokard's theorem.

Showing the perpendicularity is the main part. Denote by B' and C' the antipodes of B and C on Γ. Also, define $L = \overline{H_cC'} \cap \overline{H_bB'}$ and $K = \overline{BH_c} \cap \overline{CH_b}$, as shown.



We observe that:

- We have $\overline{OK} \perp \overline{PQ}$ since K is the pole of line $\overline{PQ}$ (again by Brokard).
- The points O, K, L are collinear by Pascal's theorem on $BH_cC'CH_bB'$.
- The point L is seen to be the reflection of H across A, so it follows $\overline{AN} \parallel \overline{OL}$ by a $\frac{1}{2}$-factor homothety at H.

Putting these three observations together completes the first claim. $\square$

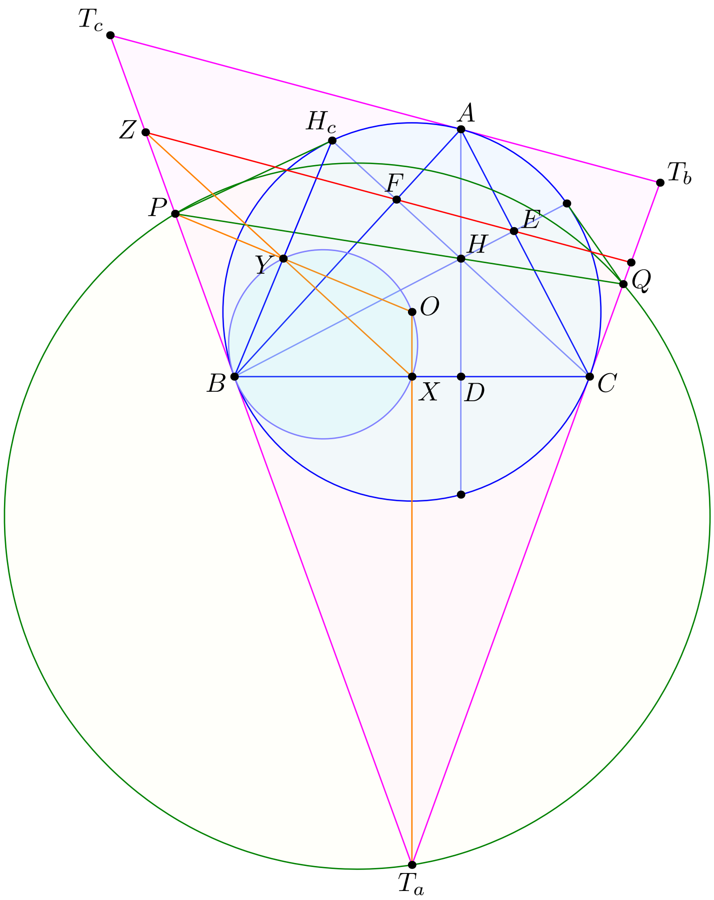
Remark (First claim is faster with complex numbers)**.** It is also straightforward to prove the first claim by using complex numbers. Indeed, in the usual setup, we have that the intersection of the tangents at B and H_c is given explicitly by

$$p = \frac{2b \cdot \left(-\frac{ab}{c}\right)}{b - \frac{ab}{c}} = \frac{2ab}{a-c}$$

and one explicitly checks $p - (a+b+c) \perp (b+c-a)$, as needed.

Claim (Radical axis of ω_A and Γ) — Line EF coincides with the radical axis of ω_A and Γ.

Proof. Let lines EF and T_aT_c meet at Z. It suffices to show Z lies on the radical axis, and then repeat the argument on the other side.



Since $\measuredangle FBZ = \measuredangle ABZ = \measuredangle BCA = \measuredangle EFA = \measuredangle ZFB$, it follows $ZB = ZF$. We introduce two other points X and Y on the perpendicular bisector of $\overline{BF}$: they are the midpoints of $\overline{BC}$ and $\overline{BH_c}$.

Since $OX \cdot OT_a = OB^2 = OY \cdot OP$, it follows that $XYPT_a$ is cyclic. Then

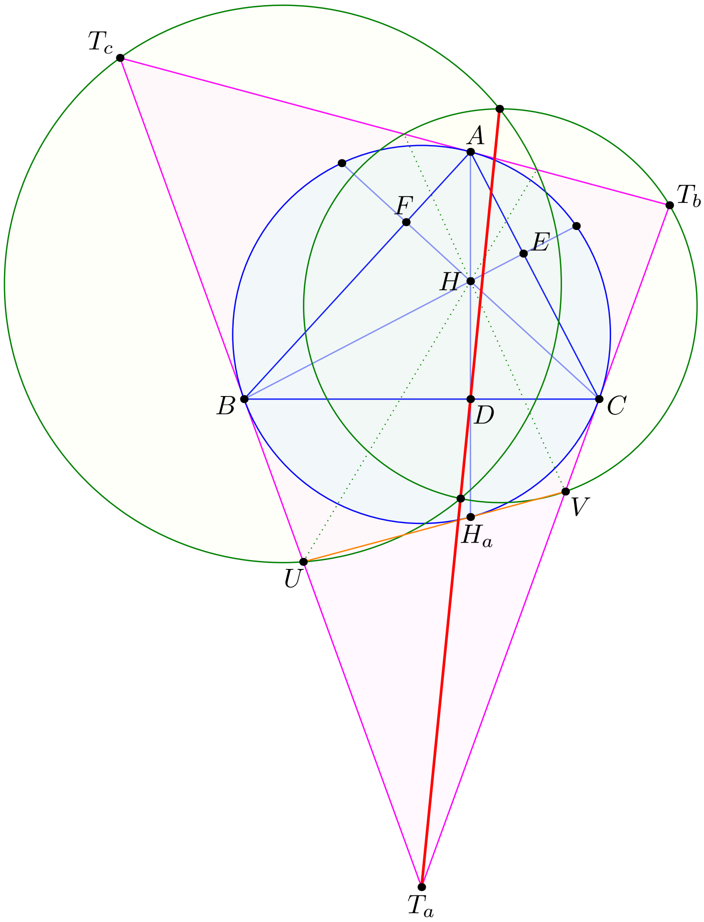
$$ZP \cdot ZT_a = ZX \cdot ZY = ZB^2$$

with the last equality since the circumcircle of $\triangle BXY$ is tangent to Γ (by a $\frac{1}{2}$-homothety at B). So the proof of the claim is complete. $\square$

Finally, we are ready to finish the problem.

Claim — Line DT_a coincides with the radical axis of ω_B and ω_C.

Proof. The point D already coincides with the radical axis because it is the radical center of Γ, ω_B and ω_C. As for the point T_a, we let the tangent to Γ at H_a meet $\overline{T_aT_c}$ at U and V; by the first claim, these lie on ω_C and ω_B respectively.



We need to show $T_aU \cdot T_aT_c = T_aV \cdot T_aT_b$.

But UVT_bT_c is apparently cyclic: the sides $\overline{T_bT_c}$ and $\overline{UV}$ are reflections across a line perpendicular to $\overline{AH_a}$, while the sides $\overline{UT_c}$ and $\overline{VT_b}$ are reflections across a line perpendicular to $\overline{BC}$. So this is true. $\square$

Now since $\triangle DEF$ and $\triangle T_aT_bT_c$ are homothetic (their opposite sides are parallel), and their incenters are respectively H and O, the problem is solved.

Remark (Barycentric approaches with respect to $\triangle T_aT_bT_c$)**.** Because the first claim is so explicit, it is possible to calculate the length of the segment PB. This opens the possibility of using barycentric coordinates with respect to the reference triangle $T_aT_bT_c$, and in fact some contestants were able to complete this approach. Writing $a = T_bT_c$, $b = T_cT_a$, $c = T_aT_b$ one can show that the radical center is the point

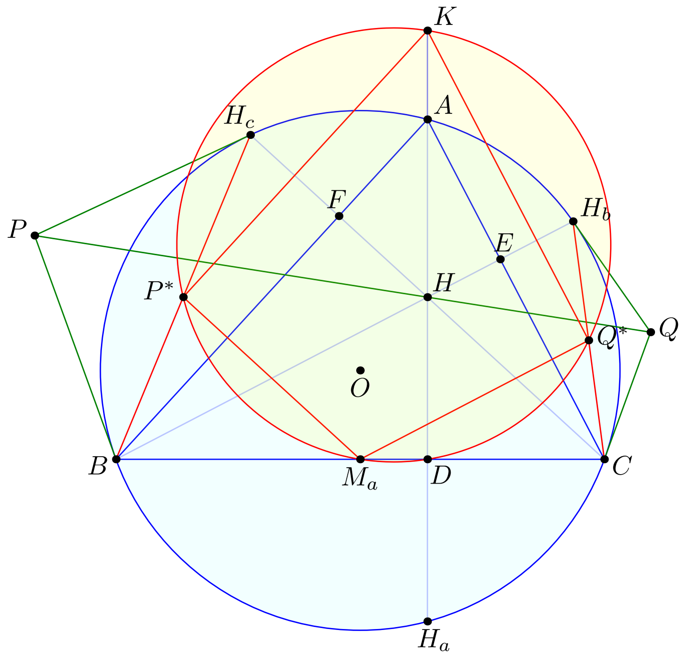
$$\left(\frac{a}{s-a} : \frac{b}{s-b} : \frac{c}{s-c}\right)$$

which is checked to be collinear with the circumcenter and incenter of $\triangle T_aT_bT_c$.

Alternate inversion approach replacing the last two claims, by Serena An After finding P and Q, it's also possible to solve the problem by using inversion. This eliminates the need to identify line EF as the radical axis of ω_A and Γ.

Inverted points are denoted with $\bullet^*$ as usual, but we will only need two points: P^*, the midpoint of $\overline{BH_c}$, and Q^*, the midpoint of $\overline{CH_b}$. Now, let $M_a = T_a^*$ denote the midpoint $\overline{BC}$ and let K be a point on ray HA with

$$KH = \frac{3}{2}AH.$$



> **Claim** — The points D, P^*, Q^* lie on the circle with diameter $\overline{KM_a}$.

Proof. Consider the homothety at H with scale factor $\frac{3}{2}$. It maps F to the midpoint of $\overline{FH_c}$ and A to K, so we find $\overline{KP^*}$ is the perpendicular bisector of $\overline{FH_c}$. As $\overline{M_aP^*} \parallel \overline{CH}$, we conclude $\angle KP^*M_a = 90^\circ$.

Similarly $\angle KQ^*M_a = 90^\circ$. And $\angle KDM_a = 90^\circ$ is given. $\square$

> **Claim** — Line $\overline{CH}$ coincides with the radical axis of ω_A^* and ω_B^*. In particular, the circles ω_A, ω_B, (COH_c) are coaxial.

Proof. Letting Γ and Γ_9 denote circumcircle and nine-point circle,

$$\begin{aligned} \operatorname{Pow}(C, \omega_A^*) &= CD \cdot CM_a = \operatorname{Pow}(C, \Gamma_9) \\ \operatorname{Pow}(H, \omega_A^*) &= HM \cdot HD = \tfrac{3}{2}HA \cdot \tfrac{1}{2}HH_a = \tfrac{3}{4}\operatorname{Pow}(H, \Gamma). \end{aligned}$$

The same calculation holds with ω_B^*. Now line CH inverts to (COH_c), as needed. $\square$

Since the circles (AOH_a), (BOH_b), (COH_c) have common radical axis equal to line OH, the problem is solved.

Remark (Nikolai Beluhov — generalization with variable ABC and fixed H)**.** Take a fixed circle Γ and a fixed point H in its interior. Then there exist infinitely many triangles ABC with orthocenter H and circumcircle Γ. In fact, for every point A on Γ we get a unique pair of B and C, determined as follows: Let line AH meet Γ again at S_A, and take B and C to be the intersection points of the perpendicular bisector of segment HS_A with Γ.

With this framework, the following generalization is true: The radical center W of ω_A, ω_B, and ω_C is the same point for all such triangles. Indeed, the Euler circle e of triangle ABC is constant because it depends only on H and Γ. Let inversion relative to Γ map e onto Ω. Then all three of T_a, T_b, and T_c lie on Ω, and so, by the solution, W is the homothety center of e and Ω. (We take the homothety center with positive ratio when triangle ABC is acute, and with negative ratio when it is obtuse. When triangle ABC is right-angled, Ω degenerates into a straight line.)

Explicitly, let Γ be the unit circle and put H on the real axis at h. Then W is also on the real axis, at $4h/(h^2+3)$.

Furthermore, it turns out the power of W with respect to ω_A, ω_B, and ω_C is constant as well. This, however, is much tougher to prove; we are not aware of a purely geometric proof at this time. Explicitly, in the setting above where Γ is the unit circle and H is on the real axis at h, the power of W with respect to ω_A, ω_B, and ω_C equals $12(h^2-1)/(h^2+3)^2$.

§3.4 Solution to USEMO4, proposed by Borislav Kirilov and Galin Totev

A function f from the set of positive real numbers to itself satisfies

$$f(x + f(y) + xy) = xf(y) + f(x + y)$$

for all positive real numbers x and y. Prove that $f(x) = x$ for all positive real numbers x.

We present two solutions.

First solution (Nikolai Beluhov) We first begin with the following observation.

> **Claim** — We must have $f(y) \geq y$ for all $y > 0$.

Proof. Otherwise, choose $0 < x < 1$ satisfying that $f(y) = (1 - x) \cdot y$. Then plugging this $P(x, y)$ gives $xf(y) = 0$, contradiction. □

Now, we make the substitution $f(x) = x + g(x)$, so that g is a function $\mathbb{R}_{>0} \to \mathbb{R}_{\geq 0}$. The given function equation reads $g(x+xy+(y+g(y)))+x+(y+g(y)) = (xy+xg(y))+(x + y + g(x + y))$, or

$$g(x + y + xy + g(y)) = (x - 1)g(y) + g(x + y). \qquad (\dagger)$$

We have to show that g is the zero function from $(\dagger)$.

> **Claim** (Injectivity for nonzero outputs) — If $g(a) = g(b)$ for $a \neq b$, then we must actually have $g(a) = g(b) = 0$.

Proof. Setting (a, b) and (b, a) in $(\dagger)$ gives $(a-1)g(b) = (b-1)g(a)$ which, since $a-1 \neq b-1$, forces $g(a) = g(b) = 0$. □

> **Claim** (g vanishes on $(1, \infty)$) — We have $g(t) = 0$ for $t > 1$.

Proof. If we set $x = 1$ in $(\dagger)$ we obtain that $g(g(y) + 2y + 1) = g(1 + y)$. As the inputs are obviously unequal, the previous claim gives $g(1 + y) = 0$ for all $y > 0$. □

Now $x = 2$ in $(\dagger)$ to get $g(y) = 0$, as needed.

Second solution (from authors) We start with the same opening of showing $f(y) \geq y$, defining $f(x) = x + g(x)$, so g satisfies $(\dagger)$. Here is another proof that $g \equiv 0$ from $(\dagger)$.

> **Claim** — If g is not the zero function, then for any constant C, we have $g(t) > C$ for sufficiently large t.

Proof. In $(\dagger)$ fix y to be any input for which $g(y) > 0$. Then

$$g\left((1 + y)x + (y + g(y))\right) \geq (x - 1)g(y)$$

so for large x, we get the conclusion. □

> **Remark.** You could phrase the lemma succinctly as "$\lim_{x\to\infty} g(x) = +\infty$". But I personally think it's a bit confusing to do so because in practice we usually talk about limits of continuous (or well-behaved) functions, so a statement like this would have the wrong connotations, even if technically correct and shorter.

On the other hand, by choosing $x = 1$ and $y = t - 1$ for $t > 1$ in (†), we get

$$g(2t + g(z) - 1) = g(t)$$

and hence one can generate an infinite sequence of fixed points: start from $t_0 = 100$, and define $t_n = 2t_{n-1} + g(t_{n-1}) - 2 > t_{n-1} + 98$ for $n \ge 1$ to get

$$g(t_0) = g(t_1) = g(t_2) = \cdots$$

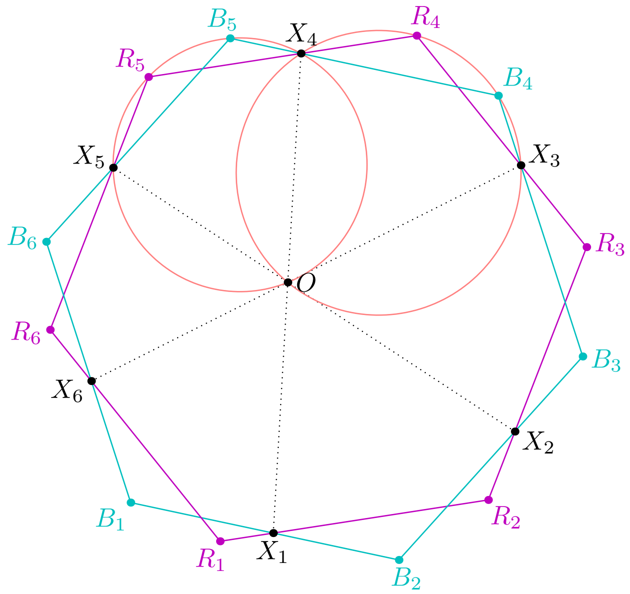
and since the t_i are arbitrarily large, this produces a contradiction if $g \not\equiv 0$.

§3.5 Solution to USEMO5, proposed by Ankan Bhattacharya

The sides of a convex 200-gon $A_1A_2 \dots A_{200}$ are colored red and blue in an alternating fashion. Suppose the extensions of the red sides determine a regular 100-gon, as do the extensions of the blue sides.

Prove that the 50 diagonals $\overline{A_1A_{101}}$, $\overline{A_3A_{103}}$, ..., $\overline{A_{99}A_{199}}$ are concurrent.

We present a diagram (with 100 replaced by 6, for simplicity).



Let $B_1 \cdots B_{100}$ and $R_1 \cdots R_{100}$ be the regular 100-gons (oriented counterclockwise), and define $X_i = A_{2i+1} = \overline{B_iB_{i+1}} \cap \overline{R_iR_{i+1}}$ for all i, where all indices are taken modulo 100. We wish to show that $\overline{X_1X_{51}}$, ..., $\overline{X_{50}X_{100}}$ are concurrent.

We now present two approaches.

First approach (by spiral similarity) Let O be the spiral center taking $B_1 \cdots B_{100} \to R_1 \cdots R_{100}$ (it exists since the 100-gons are not homothetic). We claim that O is the desired concurrency point.

Claim — $\measuredangle X_iOX_{i+1} = \frac{\pi}{50}$ for all i.

Proof. Since $\triangle OB_iB_{i+1} \overset{+}{\sim} \triangle OR_iR_{i+1}$, we have $\triangle OB_iR_i \overset{+}{\sim} \triangle OB_{i+1}R_{i+1}$, so O, X_i, B_{i+1}, R_{i+1} are concyclic. Similarly O, X_{i+1}, B_{i+1}, R_{i+1} are concyclic, so

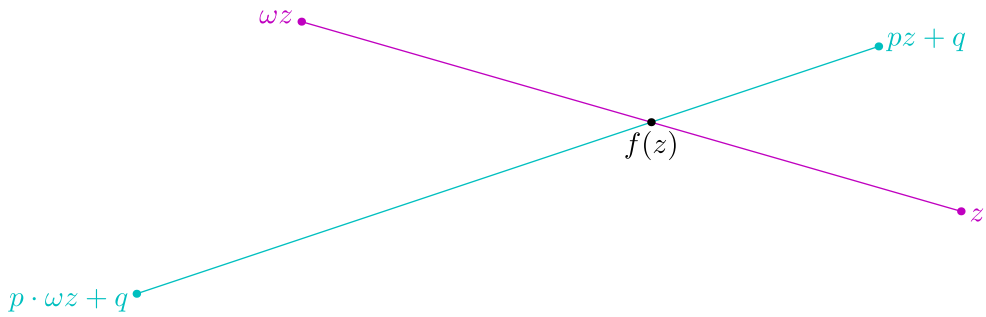
$$\measuredangle X_iOX_{i+1} = \measuredangle X_iB_{i+1}X_{i+1} = \frac{\pi}{50}$$

as wanted. $\square$

It immediately follows that O lies on all 50 diagonals $\overline{X_iX_{i+50}}$, as desired.

Second approach (by complex numbers) Let ω be a primitive 100th root of unity. We will impose complex coordinates so that $R_k = \omega^k$, while $B_k = p\omega^k + q$, where m and b are given constant complex numbers.

In general for $|z| = 1$, we will define $f(z)$ as the intersection of the line through z and ωz, and the line through $pz + q$ and $p \cdot \omega z + q$.



In particular, X_k is $f(\omega^k)$.

Claim — There exist complex numbers a, b, c such that $f(z) = a + bz + cz^2$, for every $|z| = 1$.

Proof. Since $f(z)$ and $\frac{f(z)-q}{p}$ both lie on the chord joining z to ωz we have

$$z + \omega z = f(z) + \omega z^2 \cdot \overline{f(z)}$$
$$z + \omega z = \frac{f(z)-q}{p} + \omega z^2 \cdot \frac{\overline{f(z)} - \overline{q}}{\overline{p}}.$$

Subtracting the first equation from the $\overline{p}$ times the second to eliminate $\overline{f(z)}$, we get that $f(z)$ should be a degree-two polynomial in z (where p and q are fixed constants). □

Claim — Let $f(z) = a + bz + cz^2$ as before. Then the locus of lines through $f(z)$ and $f(-z)$, as $|z| = 1$ varies, passes through a fixed point.

Proof. By shifting we may assume $a = 0$, and by scaling we may assume b is real (i.e. $\overline{b} = b$). Then the point $-\overline{c}$ works, since

$$\frac{f(z) + \overline{c}}{f(-z) + \overline{c}} = \frac{\overline{c} + bz + cz^2}{\overline{c} - bz + cz^2}$$

is real — it obviously equals its own conjugate. (Alternatively, without the assumptions $a = 0$ and $b \in \mathbb{R}$, the fixed point is $a - \frac{b\overline{c}}{\overline{b}}$.) □

Remark (We know a priori the exact coefficients shouldn't matter)**.** In fact, the exact value is

$$f(z) = \frac{-\omega \overline{q} z^2 + (1-\overline{p})(1+\omega)z - \frac{\overline{p}}{p} q}{1 - \frac{\overline{p}}{p}}.$$

Since p and q could be any complex numbers, the quantity c/b (which is all that matters for concurrence) could be made to be equal to any value. For this reason, we know *a priori* the exact coefficients should be irrelevant.

§3.6 Solution to USEMO6, proposed by Pitchayut Saengrungkongka

Prove that for every odd integer $n > 1$, there exist integers $a, b > 0$ such that, if we let $Q(x) = (x+a)^2 + b$, then the following conditions hold:

- we have $\gcd(a,n) = \gcd(b,n) = 1$;
- the number $Q(0)$ is divisible by n; and
- the numbers $Q(1)$, $Q(2)$, $Q(3)$, ... each have a prime factor not dividing n.

Let $p_1 < p_2 < \dots < p_m$ denote the odd primes dividing n and call these primes *small*. The construction is based on the following idea:

Claim — For each $i = 1, \dots, m$ we can choose a prime $q_i \equiv 1 \pmod 4$ such that

$$\left(\frac{p_j}{q_i}\right) = \begin{cases} -1 & \text{if } j = i \\ +1 & \text{otherwise.} \end{cases}$$

Proof. Fix i. By quadratic reciprocity, it suffices that $q_i \equiv 1 \pmod 4$ and that q_i is a certain nonzero quadratic residue (or not) modulo p_j for $j \neq i$.

By Chinese remainder theorem, this is a single modular condition, so Dirichlet theorem implies such primes exist. $\square$

We commit now to the choice

$$b = kq_1q_2 \dots q_m$$

where $k \geq 1$ is an integer (its value does not affect the following claim).

Claim (Main argument) — For this b, there are only finitely many integers X satisfying the equation

$$X^2 + b = p_1^{e_1} \dots p_m^{e_m} \qquad (\spadesuit)$$

where e_i are some nonnegative integers (i.e. $X^2 + b$ has only small prime factors).

Proof. In $(\spadesuit)$ the RHS is a quadratic residue modulo b. For any $i > 0$, modulo q_i we find

$$+1 = \prod_j \left(\frac{p_i^{e_i}}{q_i}\right) = (-1)^{e_i}$$

so e_i must be even. This holds for every i though! In other words all e_i are even.

Hence $(\spadesuit)$ gives solutions to $X^2 + b = Y^2$, which obviously has only finitely many solutions. $\square$

We now commit to choosing any $k \geq 1$ such that

$$k \equiv -\frac{1}{q_1q_2 \dots q_m} \pmod n$$

which in particular means $\gcd(k,n) = 1$. Now as long as $a \equiv 1 \pmod n$, we have $Q(0) \equiv 0 \pmod n$, as needed. All that remains is to take a satisfying the second claim larger than any of the finitely many bad integers in the fist claim.

Remark (Motivational comments from Nikolai Beluhov)**.** The solution I ended up with is not particularly long or complicated, but it was absurdly difficult to find. The main issue I think is that there is nothing in the problem to latch onto; no obvious place from which you can start unspooling the yarn. So what I did was throw an awful lot of different strategies at it until one stuck.

Eventually, what led me to the solution was something like this. I decided to focus on the simplest nontrivial case, when n contains just two primes. I spent some time thinking about this, and then at some point I remembered that in similar Diophantine equations I've seen before, like $2^x + 3^y = z^2$ or $3^x + 4^y = 5^z$, the main trick is first of all to prove that the exponents are even. After that, we can rearrange and factor a difference of squares. This idea turned out to be fairly straightforward to implement, and this is how I found the solution above.

Remark (The problem is OK with n even)**.** The problem works equally well for n even, but the modifications are both straightforward and annoying, so we imposed n odd to reduce the time taken in solving this problem under exam conditions.

On the other hand, for odd n, one finds that a simplified approach is possible where one proves the main argument by choosing $b \equiv 2 \pmod 4$ and then using the quadratic reciprocity argument to force the right-hand side of ($\spadesuit$) to be $1 \pmod 4$. In this case, there are no integers X at all satisfying ($\spadesuit$), and the "sufficiently large" leverage provided by the choice of a is not needed.

Remark (On the choice of conditions)**.** The equation ($\spadesuit$), and the goal to show it has finitely many solutions (or no solutions), is the heart of the problem. But the other conditions have been carefully chosen to prevent two "trivial" constructions to this.

If the condition that $\gcd(a,n) = \gcd(b,n) = 1$ or $n \mid Q(0)$ is dropped, the problem becomes much easier because one can simply ensure that $\nu_p(Q(x))$ is bounded for all $p \mid n$, by taking $b = n$ (or $b = \operatorname{rad} n$, etc.). However, these two conditions jointly together ensure that $\nu_p(Q(x))$ may be unbounded, by a Hensel-type argument.

If $b < 0$ is permitted, an easier approach to make sure that $Q(x)$ factors as the product of two polynomials by requiring b to be the negative of a perfect square. Several easier approaches become possible in this situation. For example, one can try to use Kobayashi's theorem to generate the value of a after ensuring the first two conditions are true.

Remark (Author remarks on generalization)**.** In general, *any* b satisfying $\gcd(b,n) = 1$ should still have finitely many solutions to ($\spadesuit$). The author comments that this would be a statement of Kobayashi's theorem in the ring of integers of the quadratic field $\mathbb{Q}(\sqrt{-b})$.

A confirmation this (and much more) is indeed true is given by user `Loppukilpailija` at `https://aops.com/community/p18545077`. An excerpt from this post goes:

> If there are infinitely many x which do not have this property, by the pigeonhole principle there is some integer c with $\operatorname{rad}(c) \mid n$ and $\nu_p(c) \le 2$ for all primes p such that the equation $Q(x) = cy^3$ has infinitely many solutions. This rearranges to the polynomial $cy^3 - b$ attaining infinitely many square values. Since $b \neq 0$, the roots of $cy^3 - b$ are simple. This contradicts Theorem 3 of the paper starting at page 381 of `https://www.mathunion.org/fileadmin/ICM/Proceedings/ICM1978.1/ICM1978.1.ocr.pdf`.
>
> In general, the results presented in the linked article are, together with elementary arguments, enough to characterize all polynomials which attain infinitely many perfect powers as their values. (Exercise!)

4 Marking schemes

§4.1 Rubric for USEMO1

ν_2 solutions

- **0 points** for stating that even integers are the only solutions
- **1 point** for the construction for even integers
- **1 points** for considering ν_2 of x, y and z and resolving at least one substantial case, such as $\nu_2(y)$ being maximal (additive). This point can also be awarded if done with a general prime p instead of 2.
- **7 points** for a complete solution

Factoring solutions

- **0 points** for stating that even integers are the only solutions
- **0 points** for stating WLOG $\gcd(x, y, z) = 1$
- **1 point** for the construction for even integers
- **1 points** for factoring pairwise greatest common divisors (additive)
- **7 points** for a complete solution

General ν_p solutions

- **0 points** for stating that even integers are the only solutions
- **1 point** for the construction for even integers
- **1 point** for stating $\operatorname{lcm}(x, z) \mid \operatorname{lcm}(x, y) = \operatorname{lcm}(y, z)$ (additive)
- **7 points** for a complete solution

§4.2 Rubric for USEMO2

- **1 point** for a correct construction, with the correct strategy for Hobbes which results in the correct answer (but they need not correctly calculate the answer, as long as the strategy is right).
- **2 points** for claiming a statement (no proof needed) of the form "for each integer k, Calvin can ensure a victory if the number of sets of size k is at most —" (or "the number of missing sets is at least —") and the bound listed is the correct (tight) bound.
- **3 points** for both of the above items.
- **5 points** for a correct proof of the upper bound

- **7 points** for a fully correct solution
- **-2 points** for solutions which would satisfy rubric items (3) or (4), except that the expressions given in terms of n and k are missing or not bounded correctly.

  For example: a solution might have a claim that Hobbes needs to be missing some number of sets of size k, and show how to use induction to show a bound on n, k based on n–1, k–1, but not use the correct quantities to do the bounding
- **-1 point** for a wrong answer, only for solutions that would otherwise receive a 7. Deduct no points if an answer is unsimplified but correct.

Deductions for other minor/major errors are as usual. None of the items above are additive (though see the third rubric item).

In general, for solutions that do not fit a rubric item but which you feel deserves partial credit, use your judgment to assign partial credit.

§4.3 Rubric for USEMO3

As usual, incomplete computational approaches earn partial credits only based on the amount of synthetic progress which is made.

No points are awarded for just drawing a diagram or simple observations.

Follow the notation in the typeset official solution. The following rubric items are totally additive:

(a) **1 point** for proving that P and Q are the poles of lines BH_c and CH_b.

(b) **1 point** for proving that T_a is on the radical axis of ω_B and ω_C. This point can be awarded if the proof is conditional on some reasonable description of P and Q, such as (a).

(c) **2 points** for proving that D is on the radical axis of ω_B and ω_C. This point can be awarded if the proof is conditional on some reasonable description of P and Q, such as (a).

(d) **0 points** for commenting that the homothety center of $T_aT_bT_c$ and DEF lies on the line OH.

The four rubric items above, when combined, give a perfect solution worth 7.

If none of the items above are earned: the following rubric item (not additive) is possible:

- 1 point for both claiming that P and Q are the poles of BH_c and CH_b, *and* that the radical axis of ω_B and ω_C is exactly DT_a.

§4.4 Rubric for USEMO4

General remarks

- Unlike most functional equations, this one doesn't really have that many steps of "partial progress." As such, the steps below are mainly intermediate claims, not specific equations.

- This problem does *not* ask the contestant to find all functions f that satisfy the given property. As a result, the fact that $f(x) = x$ satisfies the property does not need to be stated nor proven.
- The results below may come in many different forms, and may not be stated explicitly. As such, care must be taken to determine whether a contestant has found any of the claims below.
- Most solutions follow the following general path:
  (a) Show $f(x) \geq x$, and define $g(x) = f(x) - x$.
  (b) Find some property that shows $g(x)$ must be small or fixed in some places.
  (c) Find some property that shows $g(x)$ must be large in some places.
  (d) Combine the two to finish.

  The first claim is worth 1 point, and points (b) and (c) are meant to each be worth 3 points (non-additively), with a full solution (including finish) worth 7, of course. Solutions that make enough progress that a "standard finish" is applicable should be worth 5 points (in essence, points for both (b)-like and (c)-like things should be given). The rubric below attempts to codify most (b)-like and (c)-like progress we expect to see, but it is certainly possible that other progress exists.

Rubric items

None of these items are additive.

- **0 points** for solving the equation over some domain that is not the positive reals (e.g. reals, nonnegative reals)
- **1 point** for showing $f(x) \geq x$ for all x.
- **1 point** for showing that, if $f(x) = x$ for some x, then $f(x) = x$ for all x.
- **3 points** for showing that, if $f(x) + y = f(y) + x$, then $f(x) = x$ and $f(y) = y$.
- **3 points** for finding distinct values of x and y for which $f(x) + y = f(y) + x$.
- **3 points** for showing that $f(x) - x$ is eventually greater than any specified real.
- **5 points** for finding (or showing existence of) any value of x for which $f(x) = x$.
- **6 points** for a complete solution with a minor error that does not affect the solution.
- **7 points** for a complete solution.

§4.5 Rubric for USEMO5

Most solutions are worth 0 or 7.

- **0 points** for no progress, special cases, etc.
- **5-6 points** for any tiny slip which the contestant could have easily repaired
- **7 points** for a correct solution

For solutions which are not complete, the following items are additive:

- **1 point** for considering the spiral similarity taking $P_1 \dots P_{100}$ to $Q_1 \dots Q_{100}$ AND claiming that the center of the spiral similarity is the point of concurrency.
- **1 point** for claiming that $\angle R_i O R_{i+1} = \frac{\pi}{50}$
- **1 point** for proving that O, R_i, P_{i+1}, Q_{i+1} is concyclic
- **1 point** for further extending the above to proving that $O, R_i, R_{i+1}, P_{i+1}, Q_{i+1}$ concyclic

There is no deduction for small configuration issues (such as not using directed angles) or small typos (such as labelling points).

Usually, computational approaches which are not essentially completed are judged by any geometric content and do not earn other marks. However, the following marks (not additive with anything) are possible:

- Following the notation of the complex solution, 1 point for showing that the intersection point is quadratic AND making the general claim that $a + bz + cz^2$ is sufficient regardless of what the numbers a, b, c are.

§4.6 Rubric for USEMO6

In general, not much partial credit is expected for this problem.

The heart of the problem can be thought of as studying the equation

$$X^2 + b = p_1^{e_1} \dots p_k^{e_k}$$

where $p_1, \dots, p_k$ are a fixed set of primes, and showing that the equation cannot hold for all sufficiently large X.

- **No points** are given for steps related to the first two conditions, e.g. for the Hensel's-lemma type observation that e_i may be unbounded. This is equivalent to reducing to the case where n is squarefree.
- **No points** are given for taking specific moduli, e.g. taking the equation modulo 4.
- **No points** are given for special cases of k, such as $k = 1$.
- **1 point** is awarded for the idea to select a prime q for which $\left(\frac{p_i}{q}\right)$ is known in order to control the parity of the exponents e_i.
- There is no deduction for quoting the theorems of Dirichlet or quadratic reciprocity, or in general the quoting of any named theorems which the grader can indeed verify exists.
- 1 point is deducted if the student fails to verify the Hensel argument, but their construction holds anyways.
- 1 point is deducted if the student asserts control over $\left(\frac{p_i}{q}\right)$ with no justification whatsoever. We require the student to at least mention that quadratic reciprocity is used to get a modular condition and then use Dirichlet. (This justification may be very terse: even "by QR and Dirichlet" is accepted).

5 Statistics

This year the scoring data I received did not distinguish between blank papers and those with a score of 0, but the statistics only include students with at least one submitted file.

§5.1 Summary of Scores

N	135	1st Q	6	Max	42
μ	9.77	Median	8	Top 3	33
σ	7.99	3rd Q	14	Top 10	22

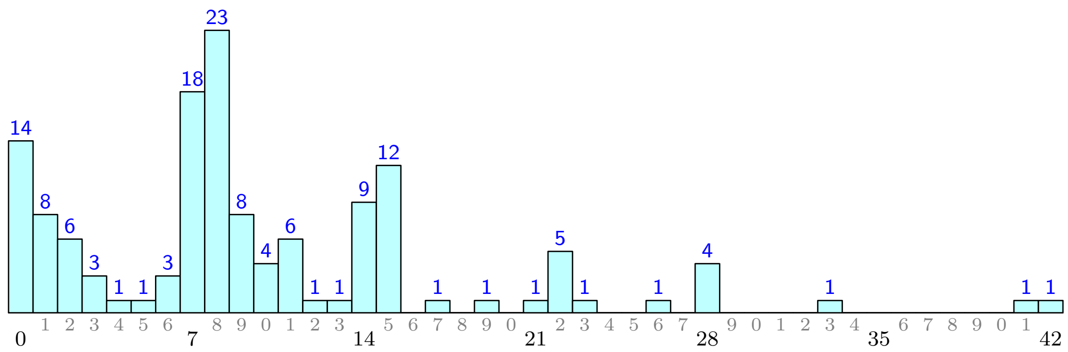
§5.2 Problem Statistics

	P1	P2	P3	P4	P5	P6
0	19	69	127	77	106	131
1	8	44	1	30	3	1
2	9	0	2	1	1	0
3	0	3	0	4	1	0
4	0	0	0	0	0	0
5	4	0	1	1	4	0
6	4	1	0	2	1	1
7	91	18	4	20	19	2
Avg	5.24	1.37	0.28	1.49	1.24	0.16

§5.3 Rankings

Sc	Num	Cu	Per	Sc	Num	Cu	Per	Sc	Num	Cu	Per
42	1	1	0.74%	28	4	7	5.19%	14	9	38	28.15%
41	1	2	1.48%	27	0	7	5.19%	13	1	39	28.89%
40	0	2	1.48%	26	1	8	5.93%	12	1	40	29.63%
39	0	2	1.48%	25	0	8	5.93%	11	6	46	34.07%
38	0	2	1.48%	24	0	8	5.93%	10	4	50	37.04%
37	0	2	1.48%	23	1	9	6.67%	9	8	58	42.96%
36	0	2	1.48%	22	5	14	10.37%	8	23	81	60.00%
35	0	2	1.48%	21	1	15	11.11%	7	18	99	73.33%
34	0	2	1.48%	20	0	15	11.11%	6	3	102	75.56%
33	1	3	2.22%	19	1	16	11.85%	5	1	103	76.30%
32	0	3	2.22%	18	0	16	11.85%	4	1	104	77.04%
31	0	3	2.22%	17	1	17	12.59%	3	3	107	79.26%
30	0	3	2.22%	16	0	17	12.59%	2	6	113	83.70%
29	0	3	2.22%	15	12	29	21.48%	1	8	121	89.63%
								0	14	135	100.00%

§5.4 Histogram



§5.5 Full stats

Rank	P1	P2	P3	P4	P5	P6	Σ
1.	7	7	7	7	7	7	42
2.	7	7	7	7	7	6	41
3.	7	7	5	7	7	0	33
4.	7	7	0	0	7	7	28
4.	7	7	0	7	7	0	28
4.	7	7	0	7	7	0	28
4.	7	7	0	7	7	0	28
8.	7	7	7	5	0	0	26
9.	7	7	2	7	0	0	23
10.	7	1	0	7	7	0	22
10.	7	1	0	7	7	0	22
10.	7	1	7	7	0	0	22
10.	7	7	0	1	7	0	22
10.	7	7	0	1	7	0	22
15.	7	7	0	0	7	0	21
16.	7	6	0	1	5	0	19
17.	7	3	0	7	0	0	17
18.	6	3	0	0	6	0	15
18.	7	0	0	1	7	0	15
18.	7	1	0	7	0	0	15
18.	7	1	0	7	0	0	15
18.	7	1	0	7	0	0	15
18.	7	1	0	7	0	0	15
18.	7	1	0	7	0	0	15
18.	7	1	0	7	0	0	15
18.	7	1	0	7	0	0	15
18.	7	7	0	1	0	0	15
18.	7	7	0	1	0	0	15
18.	7	7	0	1	0	0	15
30.	0	0	0	7	7	0	14
30.	0	0	0	7	7	0	14
30.	7	0	0	0	7	0	14
30.	7	0	0	0	7	0	14
30.	7	1	0	1	5	0	14

Rank	P1	P2	P3	P4	P5	P6	Σ
30.	7	1	0	6	0	0	14
30.	7	7	0	0	0	0	14
30.	7	7	0	0	0	0	14
30.	7	7	0	0	0	0	14
39.	7	0	0	1	5	0	13
40.	5	0	0	0	7	0	12
41.	6	0	0	0	5	0	11
41.	7	0	0	3	1	0	11
41.	7	1	0	0	3	0	11
41.	7	1	0	3	0	0	11
41.	7	1	0	3	0	0	11
41.	7	3	0	1	0	0	11
47.	7	0	0	3	0	0	10
47.	7	0	1	0	2	0	10
47.	7	1	0	1	1	0	10
47.	7	1	2	0	0	0	10
51.	7	0	0	2	0	0	9
51.	7	1	0	0	1	0	9
51.	7	1	0	1	0	0	9
51.	7	1	0	1	0	0	9
51.	7	1	0	1	0	0	9
51.	7	1	0	1	0	0	9
51.	7	1	0	1	0	0	9
51.	7	1	0	1	0	0	9
59.	0	0	0	1	7	0	8
59.	7	0	0	0	0	1	8
59.	7	0	0	1	0	0	8
59.	7	0	0	1	0	0	8
59.	7	0	0	1	0	0	8
59.	7	0	0	1	0	0	8
59.	7	0	0	1	0	0	8
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59.	7	1	0	0	0	0	8
59.	7	1	0	0	0	0	8
59.	7	1	0	0	0	0	8
59.	7	1	0	0	0	0	8
59.	7	1	0	0	0	0	8
59.	7	1	0	0	0	0	8
59.	7	1	0	0	0	0	8
59.	7	1	0	0	0	0	8
59.	7	1	0	0	0	0	8
59.	7	1	0	0	0	0	8
59.	7	1	0	0	0	0	8
59.	7	1	0	0	0	0	8
59.	7	1	0	0	0	0	8
59.	7	1	0	0	0	0	8
59.	7	1	0	0	0	0	8
82.	1	0	0	6	0	0	7
82.	5	1	0	1	0	0	7

Rank	P1	P2	P3	P4	P5	P6	Σ
82.	7	0	0	0	0	0	7
82.	7	0	0	0	0	0	7
82.	7	0	0	0	0	0	7
82.	7	0	0	0	0	0	7
82.	7	0	0	0	0	0	7
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82.	7	0	0	0	0	0	7
100.	5	0	0	1	0	0	6
100.	6	0	0	0	0	0	6
100.	6	0	0	0	0	0	6
103.	5	0	0	0	0	0	5
104.	2	1	0	1	0	0	4
105.	2	0	0	1	0	0	3
105.	2	0	0	1	0	0	3
105.	2	1	0	0	0	0	3
108.	1	0	0	1	0	0	2
108.	2	0	0	0	0	0	2
108.	2	0	0	0	0	0	2
108.	2	0	0	0	0	0	2
108.	2	0	0	0	0	0	2
108.	2	0	0	0	0	0	2
114.	0	1	0	0	0	0	1
114.	0	1	0	0	0	0	1
114.	1	0	0	0	0	0	1
114.	1	0	0	0	0	0	1
114.	1	0	0	0	0	0	1
114.	1	0	0	0	0	0	1
114.	1	0	0	0	0	0	1
114.	1	0	0	0	0	0	1
122.	0	0	0	0	0	0	0
122.	0	0	0	0	0	0	0
122.	0	0	0	0	0	0	0
122.	0	0	0	0	0	0	0
122.	0	0	0	0	0	0	0
122.	0	0	0	0	0	0	0
122.	0	0	0	0	0	0	0
122.	0	0	0	0	0	0	0
122.	0	0	0	0	0	0	0
122.	0	0	0	0	0	0	0
122.	0	0	0	0	0	0	0

Rank	P1	P2	P3	P4	P5	P6	Σ
122.	0	0	0	0	0	0	0
122.	0	0	0	0	0	0	0
122.	0	0	0	0	0	0	0