The 1st US Ersatz Math Olympiad

Solutions and Results

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Contents

1 Summary

The first USEMO was held on May 23, 2020 and May 24, 2020. We had a total of approximately 239 contestants who started the contest.

We held true to our word in producing what we think was a genuinely IMO-level competition, and in fact we overshot the mark on the first day, with no 7's awarded on Problem 2 and no points awarded on Problem 3. The second day seems closer to true IMO difficulty.

We hope no participants are discouraged by not making much progress on the problems. Given the difficulty of the competition, solving any single problem is a fine achievement. I made the deliberate decision to not "water down" the exam, despite the fact that it is open to everyone, because I believe the students willing to dedicate nine hours of a weekend to an exam like this will have the courage and determination to overcome initial failures and great challenges.

The grading was strenuous, but the team of graders eventually completed the task with expertise and grace. But that said, I think until we have a larger base of repeat volunteers, we are unlikely to offer a multi-division contest next time, despite there being many such requests. I remain hopeful in a few years that our presence may grow enough to produce enough volunteers to do such an expansion.

Looking forward, I am hoping that we may run the next USEMO in the fall of 2020.

§1.1 Thanks

I am indebted to many individuals for the formulation of this contest.

I would like to thank the Art of Problem Solving for offering the software and platform for us to run the competition. Special thanks to Amanda Reilly who led the development of the software. Other names I encountered from the office include Jeremy Copeland, Andres Lebbos, Eric Olson, Shannon Rogers, Richard Rusczyk, and Deven Ware. Surely there are others who worked behind the scenes who I did not even get to see, and I am thankful for their time as well.

I'd like to extend a thanks to Andrew Gu, Ankan Bhattacharya, Brice Huang, Carl Schildkraut, David Altizio, Evan Chen, Jeffery Li, Michael Diao, Nikolai Beluhov, Robin Son, Tristan Shin, Varun K-pati, Yannick Yao for suggesting at least one problem for the competition, even if the problem was ultimately not selected.

The review of problems was carried out by Alex Rudenko, Anant Mudgal, Andrew Gu, Ankan Bhattacharya, Brice Huang, Carl Schildkraut, David Altizio, Evan Chen, James Lin, Michael Ren, Mihir Singhal, Milan Haiman, Nikolai Beluhov, Tristan Shin, Vincent Huang, Yang Liu, and Zack Chroman. Many thanks for your time in helping select the exam.

Last but certainly not least, I would also like to thank everyone who offered to help grade the USEMO (even if real life got in the way — as it does in these challenging COVID-19 times — and made it impossible for you to follow through on the offer). These are Abrar Fiaz, Alex Rudenko, Anant Mudgal, Anders Olsen, Andrew Gu, Ankan Bhattacharya, Arman Raayatsanati, Aron Thomas, Ashwin Sah, Bobby Shen, Brandon Wang, Brian Chen, Brice Huang, Carl Schildkraut, Cathy Ye, Colin Tang, Danielle Wang, Dominick Joo, Eric Zhang, Evan Chen, Farrell Wu, Foyez Alauddin, Hadyn Tang, Henry Weng, James Lin, Jeck Lim, Jeffery Li, Jennifer Wang, Jit Wu Yap, Kevin Sun, Mehmet Kaysi, Michael Ren, Mihir Singhal, Milan Haiman, Nikolai Beluhov, Orlin Kuchumbov, Rohan Goyal, Tahmid Hameem Chowdhury Zarif, Tom Luo, Tristan Shin, Valentio Iverson, Victor Wang, Vincent Huang, Yang Liu, Yannick Yao, Yundi Duan, Zack Chroman, and Zhou Li.

2 Results

If you won one of the seven awards, please reach out to `usemo@evanchen.cc` to claim your prize!

§2.1 Top Scores

Congratulations to the top three scorers, who win the right to propose problems to future contests.

1st place Jeffrey Kwan (33 points)

2nd place Jaedon Whyte (30 points)

3rd place Luke Robitaille (29 points)

§2.2 Special awards

See the Rules for a description of how these are awarded. Ties are broken by elegance of solution (obviously subjective); when this occurs, runner-ups are noted below as well.

Top female Ali Cy (20 points)

Youth prize Ethan Liu (22 points)

Top day 1 Ankit Bisain (runner-up: Gopal Goel)

Top day 2 Noah Walsh

§2.3 Honorable mentions

This year we award Honorable Mention to anyone scoring at least 22 points. The HM's are listed below in alphabetical order.

Daniel Hong

Eddie Chen

Ethan Liu

Grant Yu

Noah Walsh

Sean Jinxiang Li

§2.4 Distinction

We award Distinction to anyone scoring at least 14 points (two fully solved problems). The Distinction awards are listed below in alphabetical order.

Akash Das

Alex Xu

Ali Cy

Andrew Gu

Andrew Wen

Andrew Yuan

Ankit Bisain

Bradley Guo

Brandon Chen

Brian Liu

Charley Cheng

Daniel Xu

Derek Liu

Easton Singer

Edward Yu

Espen Slettnes

Ethan Zhou

Gopal K. Goel

Grace Wang

Holden Mui

Isaac Zhu

Jason Cheah

Jeffrey Chen

Jeffrey Liu

Jeffrey Lu

Justin Yu

Karthik Seetharaman

Karthik Vedula

Kevin Wu

Kristie Sue

Luke Choi

Mason Fang

Maximus Lu

Maxwell Sun

Nicholas Song

Nilay Mishra

Niyanth Rao

Paul Hamrick

Rafael

Reagan Choi

Rich Wang

Rishabh Das

Ryan Li

Samuel Wang

Sanjana Das

Serena An

Shreyas Ramamurthy

Srinath Mahankali

Sumith Nalabolu

Vittal Thirumalai

William Wang

William Yue

Yunseo Choi

3 Solutions to the Problems

§3.1 Solution to USEMO1, by Robin Son

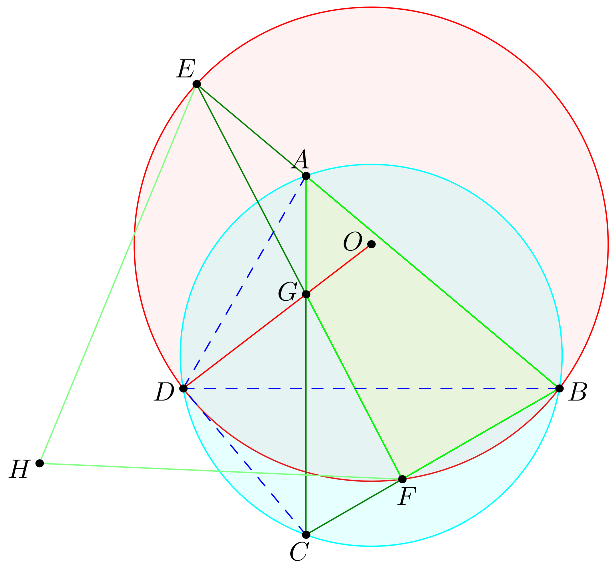
Let $ABCD$ be a cyclic quadrilateral. A circle centered at O passes through B and D and meets lines BA and BC again at points E and F (distinct from A, B, C). Let H denote the orthocenter of triangle DEF. Prove that if lines AC, DO, EF are concurrent, then triangles ABC and EHF are similar.

Define G as the intersection of $\overline{AC}$ and $\overline{EF}$.

Claim — Quadrilateral $DCGF$ is cyclic.

First proof. Because $\measuredangle DCG = \measuredangle DCA = \measuredangle DBA = \measuredangle DBE = \measuredangle DFE = \measuredangle DFG$ □

Second proof. Follows since D is Miquel point of $GABF$. □



Claim — If G lies on line $\overline{DO}$, we have $\overline{AC} \perp \overline{BD}$.

Proof. We have

$$\begin{aligned}\measuredangle BDG = \measuredangle BDO &= 90^\circ - \measuredangle DEB = 90^\circ - \measuredangle DFB \\ &= 90^\circ - \measuredangle DFC = 90^\circ - \measuredangle DGC.\end{aligned}$$

□

To finish,

$$\measuredangle HEF = 90^\circ - \measuredangle EFD = 90^\circ - \measuredangle EBD = 90^\circ - \measuredangle ABD = \measuredangle CAB.$$

Similarly $\measuredangle HFE = \measuredangle ACB$ and the proof is done.

Remark. The original version of this problem was in the converse direction: showing that $\overline{AC} \perp \overline{BD}$ implied the concurrence. Unfortunately, this turns out to be susceptible to Cartesian coordinates by setting the x and y axes along these lines, as well as complex methods.

Interestingly, it does not appear to be easy to show directly that the converse of the problem implies the original statement (other than actually solving the problem, and adapting the proof). Note in particular that the case where $E = A$ and $F = C$ is a counterexample to the converse direction as stated.

§3.2 Solution to USEMO2, by Carl Schildkraut

Let $\mathbb{Z}[x]$ denote the set of single-variable polynomials in x with integer coefficients. Find all functions $\theta \colon \mathbb{Z}[x] \to \mathbb{Z}[x]$ (i.e. functions taking polynomials to polynomials) such that

- for any polynomials $p, q \in \mathbb{Z}[x]$, $\theta(p+q) = \theta(p) + \theta(q)$;
- for any polynomial $p \in \mathbb{Z}[x]$, p has an integer root if and only if $\theta(p)$ does.

The answer is that

$$\theta(x) = r(x) \cdot p(\pm x + c)$$

for any choice of $c \in \mathbb{Z}$, $r(x)$ without an integer root, with the choice of sign fixed. For the converse direction we present two approaches.

First solution It's clear that this works, so we prove it is the only one. Let $r(x) = \theta(1)$, which has no integer root since the constant 1 has no roots at all.

Part 1. We fix a positive integer n and start by determining $\theta(x^n)$ which is the bulk of the problem. Let $f(x) = \theta(x^n)$. We look at

$$\theta(ax^n + b) = a \cdot f(x) + b \cdot r(x).$$

Let $g(x) = f(x)/r(x)$, a quotient of two polynomials whose denominator never vanishes. By using the problem condition in both directions, varying $x \in \mathbb{Z}$ and $-b/a \in \mathbb{Q}$, we find that

$$\frac{f(x)}{r(x)} \text{ takes on exactly the values } \dots, (-2)^n, (-1)^n, 0^n, 1^n, 2^n, 3^n, \dots \text{ for } x \in \mathbb{Z}$$

So let $g(x) = f(x)/r(x)$ now.

Claim (Rational functions can't be integer-valued forever) — Since g maps integers to integers, it must actually be a polynomial with rational coefficients.

Proof. We will only need the condition that g maps integers to integers.

If not, then by the division algorithm, we have $g(x) = d(x) + \frac{f_1(x)}{f_2(x)}$ for some polynomials $d(x), f_1(x), f_2(x)$ in $\mathbb{Q}[x]$ with $\deg f_2 > \deg f_1 \geq 0$. There exists an integer D such that $D \cdot d(x) \in \mathbb{Z}[x]$ (say the lcm of the denominators of the coefficients of g).

But for large enough integers x the value of $\frac{f_1(x)}{f_2(x)}$ is a nonzero and has absolute value less than $\frac{1}{D}$. This is a contradiction. □

Remark. You can't drop the condition that g has rational (rather than integer) coefficients in the proof above; consider $g(x) = \frac{1}{2}x(x+1)$ for example.

A common wrong approach is to try to use the same logic on $\theta(x^n)/\theta(x^{n-1})$ for $n \geq 2$. This doesn't work since $\theta(x^n)$ and $\theta(x^{n-1})$ could have a common root for $n \geq 2$ and therefore the problem condition essentially says nothing.

Let C be an integer divisible by every denominator in the coefficients of g. Then apparently

$$h(x) = C^n \cdot g(x)$$

is a polynomial which only takes only nth powers as $x \in \mathbb{Z}$.

Claim (Polya and Szego) — Since h is a polynomial with integer coefficients whose only values are nth powers, it must itself be the nth power of a polynomial.

Proof. This is a classical folklore problem, but we prove it for completeness.

Decompose h into irreducible factors as

$$h(x) = c \cdot f_0(x)^{e_0} \cdot f_1(x)^{e_1} \cdot f_2(x)^{e_2} \cdot f_3(x)^{e_3} \cdot \dots \cdot f_m(x)^{e_m}$$

where the f_i are nonconstant and c is an integer, and $e_i > 0$ for all $i > 0$. We also assume $m > 0$.

We use the following facts:

- In general, if $A(x), B(x) \in \mathbb{Z}[x]$ are coprime, then $\gcd(A, B)$ is bounded by some constant $C_{A,B}$. This follows by Bezout lemma.
- If $A(x) \in \mathbb{Z}[x]$ is a nonconstant polynomial, then there are infinitely many primes dividing some element in the range of A. This is called Schur's theorem.
- Let $A(x) \in \mathbb{Z}[x]$ be an irreducible polynomial, and let $A'(x)$ be its derivative. Then if p is prime and $p > C_{A,A'}$, and p has root in $\mathbb{F}_p$, then there exists x with $\nu_p(A(x)) = 1$. This follows by Hensel lemma.

Now for the main proof. By the above facts and the Chinese remainder theorem (together with Dirichlet theorem), we can select enormous primes $p_1 < p_2 < \dots < p_m < q$ (exceeding c, e, $\max e_i$, $\max C_{f_i,x}$, $\max C_{f_i,f_j}$ for all i and j) and a single integer N satisfying the following constraints:

- $\nu_{p_i}(f_i(N)) = 1$ for all $i = 1, \dots, m$, by requiring $N \equiv t_i \pmod{p_i^2}$ for suitable constant t_i not divisible by p_i (because of Hensel lemma);
- $p_i \nmid f_j(N)$ whenever $i \neq j$; this follows by the fact that $p_i > C_{f_i,f_j}$;

Now look at the value of $f(N)$. It has

$$\begin{aligned}
\nu_{p_1}(f(N)) &= e_1 \\
\nu_{p_2}(f(N)) &= e_2 \\
&\vdots \\
\nu_{p_m}(f(N)) &= e_m.
\end{aligned}$$

Now $f(N)$ is a nth power so n divides all of $e_1, \dots, e_m$. Finally c must be an nth power too. □

So $h(x)$ is an nth power; thus so is $g(x)$. Let's write $g(x) = p(x)^n$ then; so we find that the range of $p(x)$ contains either k or $-k$, for every integer k. For density reasons, this forces p to be linear, and actually of the form $p(x) = \pm x + c$ for some constant c.

Part 2. We have now shown $\theta(x^n) = (\pm x + c)^n r(x)$, for every n, for some sign and choice of c depending possibly on n. It remains to show that the choices of signs and constants are compatible across the different values of n. So let's verify this.

By applying a suitable transformation on x let's assume $\theta(x) = x$ for simplicity. Then look at $\theta(x^n + ax) = (\pm x + c)^n + ax$ for choices of integers a. This is apparently supposed to have a root for each choice of a, but if $c \neq 0$, this means $\frac{1}{x}(\pm x + c)^n$ can take any integer value, which is obviously not true for density reasons. This means $c = 0$, so it shows $\theta(x^n) = \pm x^n$ for any integer n.

Finally, by considering $\theta(x^n + x - 2) = \pm x^n - x + 2$, we see the sign must be $+$ for the RHS to have an integer root. This finishes the proof.

Second solution, outline (by contestants) The solution is like the previous one, but replaces the high-powered Polya and Szego with the following simpler result.

Claim (Odd-degree polynomials are determined by their range) — Let $P(x) \in \mathbb{Z}[x]$ be an odd-degree polynomial. Let $Q(x)$ be another polynomial with the same range as P over $\mathbb{Z}$. Then $P(x) = Q(\pm x + c)$ for some $\pm$ and c.

Proof. First, Q also has odd degree since it must be unbounded in both directions. By negating if needed, assume Q has positive leading coefficient.

Take a sufficiently large integer n_0 such that $P(x)$ and $Q(x)$ are both strictly increasing for $x \geq n_0$, and moreover $P(n_0) > \max_{x<n_0} P(x)$, $Q(n_0) > \max_{x<n_0} P(x)$. Then take an even larger integer $n_1 > n_0$ such that $\min(P(n_1), Q(n_1)) > \max(P(n_0), Q(n_0))$. Choose $n_2 > n_0$ such that $P(n_1) = Q(n_2)$. We find that this implies

$$\begin{aligned} P(n_1) &= Q(n_2) \\ P(n_1+1) &= Q(n_2+1) \\ P(n_1+2) &= Q(n_2+2) \\ P(n_1+3) &= Q(n_2+3) \end{aligned}$$

and so on. So P is a shift of Q as needed. □

This is enough to force $\theta(x^n) = (\pm x + c)^n r(x)$ when n is odd. When n is even, for each integer k one can consider

$$\theta(kx^{n+3} + x^n) = k\theta(x^{n+3}) + \theta(x^n)$$

and use the claim on $\theta(x^{n+3})$ and $\theta(kx^{n+3} + x^n)$ to pin down $\theta(x^n)$.

Third solution (from author) The answers are as before and we prove only the converse direction.

Lemma

Given two polynomials $P, Q \in \mathbb{Z}[x]$, if $P + nQ$ has an integer root for all n, then either P and Q share an integer root or $P(x) = \binom{x+m}{k} Q(x)$ for some integers m, k with $k \neq 0$.

Proof. Let $d = \gcd(P(0), Q(0))$ so $P(0) = dr$ and $Q(0) = ds$. Now, for an integer root k_n of $P + nQ$,

$$k_n | P(0) + nQ(0) = dr + nds = d(r + ns).$$

Let p be a prime $\equiv r \bmod s$, of which there are infinitely many by Dirichlet's theorem. Now, for $n = \frac{p-r}{s}$, we have

$$k_n | dp.$$

As the divisors of dp are exactly those of d times 1 or p, there exists a (not necessarily positive) divisor j of d and a $t \in \{1, p\}$ so that $k_n = dt$ for infinitely many n. In the first case, we have that $P(j) + nQ(j) = 0$ for infinitely many n and some fixed j, which implies that j is a root of both P and Q. In the second case, we have, noting $p = r + ns$, that

$$P(j(r + ns)) + nQ(j(r + ns)) = 0.$$

As this holds for infinitely many n, we may rewrite it as a polynomial equation

$$P(x) = (ax+b)Q(x)$$

for some rational a, b. Now, we know that $(ax+b+n)Q(x)$ has a rational root for all $n \in \mathbb{Z}$. If Q has an integer root then P does as well and we are in our first case; otherwise, $\frac{n+b}{a} \in \mathbb{Z}$ for all $n \in \mathbb{Z}$. This implies that $1/a \in \mathbb{Z}$, let it be k. Then $b/a \in \mathbb{Z}$; let it be m. This finishes the proof. $\square$

Now, let $P_n(x) = f(x^n)$. We claim that $P_1(x) = (\pm x + t)P_0(x)$ for some $t \in \mathbb{Z}$. Indeed, $P_1 + nP_0$ has an integer root for all n, so either P_1 and P_0 share an integer root or $P_1(x) = \left(\frac{x+m}{k}\right) P_0(x)$ for some $m, k \in \mathbb{Z}$. They clearly cannot share a root, since $P_0(x)$ cannot have any integer roots. Now,

$$kP_1(x) + P_0(x) = (x+m+k)P_0(x)$$

has an integer root, so $kx+1$ must as well, and thus $k = \pm 1$, as desired. Now, we see that

$$\theta\left(a(x^n - c^n) + b(x-c)\right) = a\big(P_n(x) - c^n P_0(x)\big) + b\big(P_1(x) - cP_0(x)\big)$$

has an integer root for any c, a, b. Let $Q = P_n - c^n P_0$ and $R = P_1 - cP_0$. Since $aQ + bR$ has an integer root for all $a, b \in \mathbb{Z}$, we can apply our lemma on both the pair (Q, R) and (R, Q); if they do not share an integer root, then Q must be a linear polynomial times R and R must be a linear times Q, a contradiction unless they are both 0 (in which case they share any integer root). So, Q and R share an integer root. We have

$$R(x) = P_1(x) - cP_0(x) = (\pm x + t - c)P_0(x),$$

and P_0 has no integer root as 1 has no integer root, so we have that $\pm(c-t)$ is the only integer root of R and is thus also a root of Q; in particular

$$P_n(\pm(c-t)) = c^n P_0(\pm(c-t))$$

for all $c \in \mathbb{Z}$. This is a polynomial equation that holds for infinitely many c so we must have that

$$P_n(\pm(x-t)) = x^n P_0(\pm(x-t)) \implies P_n(x) = (\pm x + t)^n P_0(x).$$

Thus, if $Q(x) = \sum_{i=0}^d a_i x^i$,

$$\theta(Q(x)) = \theta\left(\sum_{i=0}^d a_i x^i\right) = \sum_{i=0}^d a_i (\pm x + t)^i P_0(x) = P_0(x)Q(\pm x + t),$$

finishing the proof.

§3.3 Solution to USEMO3, by Nikolai Beluhov

Consider an infinite grid $\mathcal{G}$ of unit square cells. A *chessboard polygon* is a simple polygon (i.e. not self-intersecting) whose sides lie along the gridlines of $\mathcal{G}$.

Nikolai chooses a chessboard polygon F and challenges you to paint some cells of $\mathcal{G}$ green, such that any chessboard polygon congruent to F has at least 1 green cell but at most 2020 green cells. Can Nikolai choose F to make your job impossible?

The answer is YES, the task can be made impossible.

The solution is split into three parts. First, we describe a "polygon with holes" F. In the second part we prove that this F works. Finally, we show how to take care of the holes to obtain a true polygon.

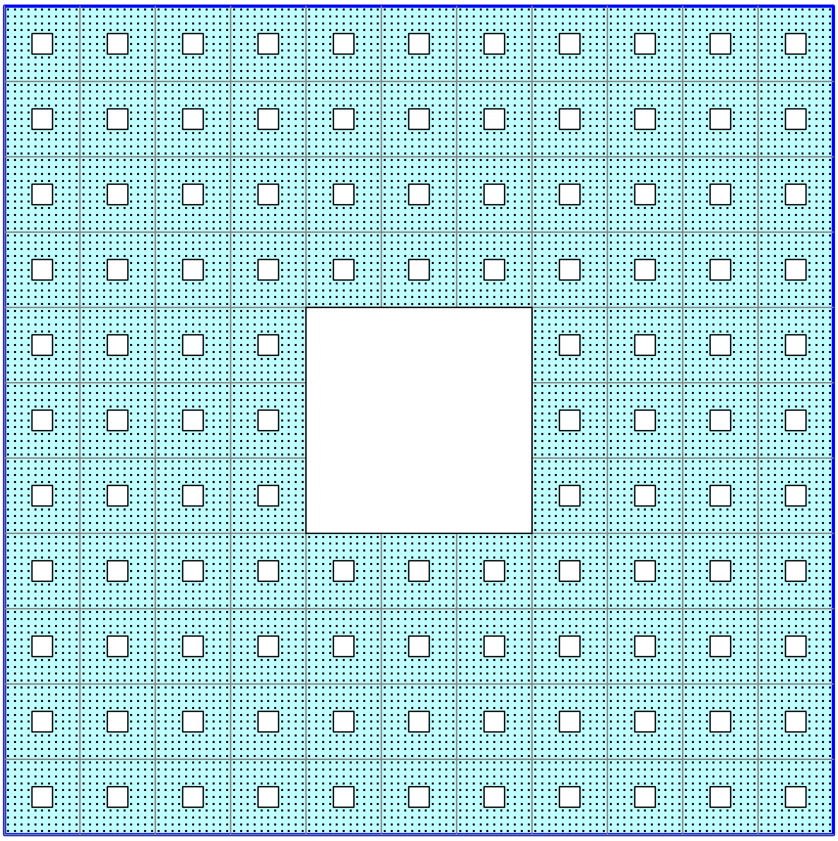
Part 1. Construction. Choose large integers $m \geq 5$, $n \geq 1$ with m odd. We will let

$$s = m^n$$

throughout the solution.

Let F_0 be a square of side length s. Starting from F_0, iterate the following procedure. Divide F_i into squares of side s/m^i and poke a square hole of side $3s/m^{i+1}$ (a *phase-i* hole) in the center of each such square to obtain F_{i+1}. Finally, let $F = F_n$.

The output is shown below for $m = 11$ and $n = 3$. The claim is that for a suitable choice of (m, n) this will serve as the desired example.



Part 2. Proof of this example. Suppose we have a green coloring as described. For every green cell, the *standard copy* of F is a copy of F centered at the green cell.

Claim — The standard copies of F completely cover the plane.

Proof. This is equivalent to every copy of F having at least one green cell, owing to symmetry of F. □

Claim — In any square C with side length $5s$, there are at least $n+1$ standard copies of F.

Proof. Let C_0 be a square of side $3s$ cocentric with C. Starting from C_0, iterate the following procedure for $1 \le i \le n+1$:

- Let S_{i-1} be one of the standard copies of F that cover the center of C_{i-1},
- If $i \neq n+1$, let C_i be a phase-i hole in S_{i-1} that lies in the interior of C_{i-1}.

Since C_0 and the holes $C_1, C_2, \dots, C_n$ are nested, the standard copies $S_0, S_1, S_2, \dots, S_n$ of F are distinct. It follows that C contains at least $n+1$ standard copies of F. □

However, the area of F is $s^2\left(1-\frac{9}{m^2}\right)^n$. Therefore, at least one cell c within C is covered by at least

$$k = \frac{n+1}{25}\left(1-\frac{9}{m^2}\right)^n$$

standard copies of F. The copy of F centered at c then contains at least k green cells.

When $n = 25 \cdot 2020$ and m is sufficiently large, however, we get $k > 2020$.

Part 3. Handling the holes. We are left to show how to repair the above construction so that F becomes a true polygon.

Let D be any sufficiently large positive integer. Consider a homothetic copy F_D of F scaled by a factor of D. Cut several canals of unit width into F_D so that F_D continues to be connected and every hole in F_D is joined by a canal to the boundary of F_D. (Canals do not need to be straight; they may go around holes.) When F_D is repaired in this way, it becomes a true polygon F'_D.

Since the total area of all canals is proportional to D and the total area of F_D is proportional to D^2, when D becomes arbitrarily large the ratio of the area of F'_D to the area of F_D becomes arbitrarily close to one. Therefore, for all sufficiently large D our proof that F is in fact a counterexample goes through for F'_D as well, with straightforward adjustments.

The solution is complete.

Remark (Author comments)**.** Some time after I came up with this problem, Ilya Bogdanov pointed out to me that it is similar to problem 3.6 in Ilya Bogdanov and Grigory Chelnokov, Pokritiya Kletchatimi Figurkami, Summer Conference of the Tournament of Towns, 2007, https://www.turgor.ru/lktg/2007/3/index.php. The USEMO directors agreed that the two problems are different enough that mine was suitable for the contest.

§3.4 Solution to USEMO4, by Robin Son

Prove that for any prime p, there exists a positive integer n such that

$$1^n + 2^{n-1} + 3^{n-2} + \cdots + n^1 \equiv 2020 \pmod{p}.$$

The idea is to pick $n = c \cdot p \cdot (p-1)$ for suitable integer c. In what follows, everything is written modulo p.

Claim — When $n = c \cdot p \cdot (p-1)$, the left-hand side is equal to

$$\begin{aligned} c \cdot \sum_{a=0}^{p-2} \sum_{b=1}^{p-1} b^a = c \cdot \big[& 1^0 + 2^0 + \cdots + (p-1)^0 \\ & + 1^1 + 2^1 + \cdots + (p-1)^1 \\ & + 1^2 + 2^2 + \cdots + (p-1)^2 \\ & + \dots \\ & + 1^{p-2} + 2^{p-2} + \cdots + (p-1)^{p-2} \big]. \end{aligned}$$

Proof. In the original sum, we discard all the terms divisible by p, reduce all the bases modulo p, and reduce all the exponents modulo $p-1$ (by Fermat's little theorem). Then each block of $p(p-1)$ terms equals

$$\begin{aligned} & 1^0 + 2^{p-2} + 3^{p-3} + \cdots + (p-1)^1 \\ & + 1^{p-2} + 2^{p-3} + 3^{p-4} + \cdots + (p-1)^0 \\ & + 1^{p-3} + 2^{p-4} + 3^{p-5} + \cdots + (p-1)^{p-2} \\ & + \dots \\ & + 1^1 + 2^0 + 3^{p-2} + \cdots + (p-1)^2 \end{aligned}$$

which rearranges to the desired sum. □

Claim — We have

$$\sum_{a=0}^{p-2} \sum_{b=1}^{p-1} b^a \equiv -1 \pmod{p}.$$

First proof. By the geometric series formula

$$\sum_{a=0}^{p-2} b^a = \frac{b^{p-1}-1}{b-1} = 0 \qquad \forall\, b = 2, 3, \dots p-1.$$

The terms with $b = 1$ contribute $1^0 + 1^1 + \cdots + 1^{p-2} = p-1$ and done. □

Second proof. In fact, it's a classical lemma (proved in the same way, using primitive roots) that

$$\sum_{b=1}^{p-1} b^a \equiv \begin{cases} -1 & p-1 \mid a \\ 0 & p-1 \nmid a \end{cases} \pmod{p}$$

so this is immediate. □

Thus we simply need to select $c \equiv -2020 \pmod{p}$ and win (and $c > 0$).

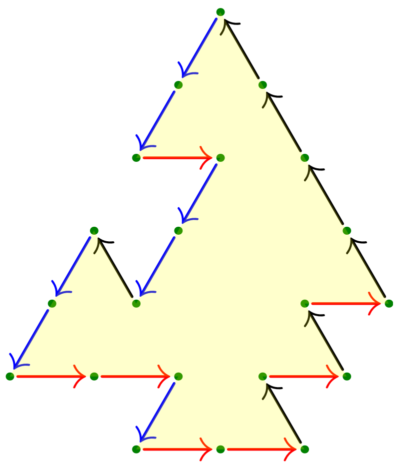
§3.5 Solution to USEMO5, by Ankan Bhattacharya

Let $\mathcal{P}$ be a regular polygon, and let $\mathcal{V}$ be the set of its vertices. Each point in $\mathcal{V}$ is colored red, white, or blue. A subset of $\mathcal{V}$ is *patriotic* if it contains an equal number of points of each color, and a side of $\mathcal{P}$ is *dazzling* if its endpoints are of different colors.

Suppose that $\mathcal{V}$ is patriotic and the number of dazzling edges of $\mathcal{P}$ is even. Prove that there exists a line, not passing through any point of $\mathcal{V}$, dividing $\mathcal{V}$ into two nonempty patriotic subsets.

We prove the contrapositive: if there is no way to split $\mathcal{V}$ into two patriotic sets, then the number of dazzling edges is odd.

Let $\zeta = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$ be a root of unity. Read the n vertices of the polygon in order starting from any point. In the complex plane, start from the origin and, corresponding to red, white, or blue, move by 1, ζ, or ζ^2, respectively, to get a path. The diagram below shows an example (where black stands in for white, for legibility reasons).



Note that:

- The path we get is actually a closed loop, since $\mathcal{V}$ was assumed to be patriotic.
- Because there is no nontrivial patriotic subset, this closed loop does not intersect itself, so it corresponds to some polygon $\mathcal{Q}$.

We have to show the number m of vertices of $\mathcal{Q}$ (corresponding to dazzling edges) is odd. Let x and y denote the number of 60° and 300° angles, so $60x+300y = 180(x+y-2)$. This gives $x - y = 3$ so $x + y$ is odd.

§3.6 Solution to USEMO6, by Ankan Bhattacharya

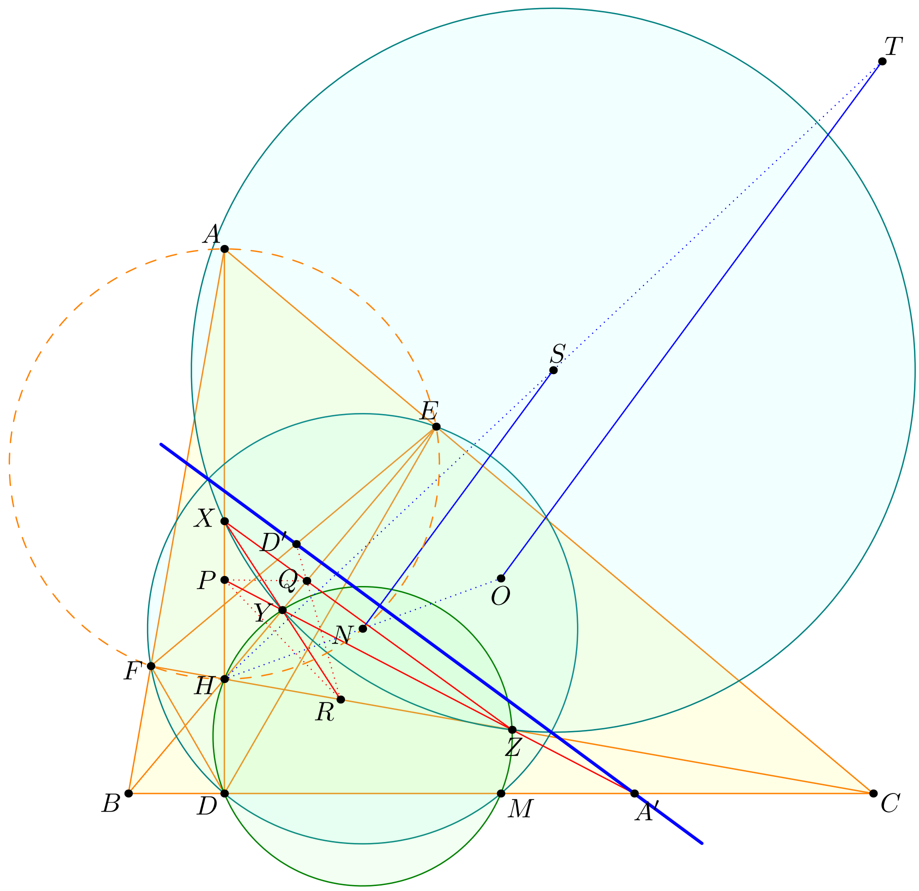
Let ABC be an acute scalene triangle with circumcenter O and altitudes $\overline{AD}$, $\overline{BE}$, $\overline{CF}$. Let X, Y, Z be the midpoints of $\overline{AD}$, $\overline{BE}$, $\overline{CF}$. Lines AD and YZ intersect at P, lines BE and ZX intersect at Q, and lines CF and XY intersect at R.

Suppose that lines YZ and BC intersect at A', and lines QR and EF intersect at D'. Prove that the perpendiculars from A, B, C, O to the lines QR, RP, PQ, $A'D'$, respectively, are concurrent.

We present two solutions.

Radical axis approach (author's solution) The main idea is to show that (DEF) and (XYZ) has radical axis $\overline{A'D'}$.

Let H be the orthocenter of $\triangle ABC$. We'll let (AH), (BH), (CH) denote the circles with diameters $\overline{AH}$, $\overline{BH}$, $\overline{CH}$.



Claim — Points H, D, Y, Z are cyclic.

Proof. Let M be the midpoint of $\overline{BC}$. We claim they lie on a circle with $\overline{HM}$.

Clearly $\angle HDM = 90°$. The segment $\overline{YM}$ is the B-midline of $\triangle BEC$, so $\overline{YM} \parallel \overline{EC} \perp \overline{HY}$: thus $\angle HYM = 90°$. Similarly $\angle HZM = 90°$. $\square$

Claim — The point P is the radical center of (HB), (HC), (XYZ), $(HYZD)$. Also, QR is the radical axis of (HA) and (XYZ).

Proof. First part since $PH \cdot PD = PY \cdot PZ$; second part by symmetric claims. $\square$

We are now ready for the key claim.

Claim (Key claim) — The points A' and D' lie on the radical axis of (DEF) and (XYZ).

Proof. The radical center of (DEF), (XYZ), $(HYZD)$ is $A' = \overline{YZ} \cap \overline{BC}$, and the radical center of (DEF), (XYZ), (HA) is $D' = \overline{EF} \cap \overline{QR}$, so we're done. $\square$

Let S be the center of (XYZ) and T the reflection of H over S. Let N denote the nine-point center.

Claim (Concurrence) — The point T is the concurrency point in the problem.

Proof. The line through the centers of (HA) and (XYZ) is perpendicular to the radical axis $\overline{QR}$. Now, a homothety with center H and scale 2 sends these centers to A and T, so $\overline{AT} \perp \overline{QR}$. Similarly, $\overline{BT} \perp \overline{RP}$ and $\overline{CT} \perp \overline{PQ}$.

Similarly from $\overline{NS} \perp \overline{A'D'}$, a dilation at H by a factor of 2 shows $\overline{OT} \perp \overline{A'D'}$, as desired. $\square$

Remark (Author comments on problem creation)**.** The main goal was to create a problem to showcase the midpoints of the altitudes: while they arise due to the midpoint of altitude lemma (Lemma 4.14 in EGMO), I have rarely seen them studied in their own right. This problem strives to be a synthesis of properties relating to the midpoints of altitudes.

Remark. An original, more long-winded version of the problem asks to show that if B', C', E', F' are defined similarly, then all six points are collinear and perpendicular to $\overline{OT}$. The second approach below proves this.

Orthology approach (from contestants) Define B', C', E', F' in an analogous fashion,

Claim — Points A', B', C', D', E', F' are collinear.

Proof. Three applications of Desargue:

- ABC and XYZ are perspective at H so A', B', C' are collinear.
- DEF and PQR are perspective at H so D', E', F' are collinear.
- $C'FR$ and $B'EQ$ are perspective through A-altitude so $B'C'$, EF, QR are concurrent (at D').

$\square$

Claim — The perpendiculars from A, B, C to $\overline{QR}$, $\overline{RP}$, $\overline{PQ}$ are concurrent.

Proof. This follows from the fact that $\triangle ABC$ and $\triangle PQR$ are orthologic with one orthology center at O. $\square$

Claim — The perpendiculars from A, O, C to $\overline{QR}$, $\overline{D'F'}$, $\overline{PQ}$ are concurrent.

Proof. This follows from the fact that $\triangle D'F'Q$ and $\triangle AOC$ are orthologic with one orthology center at E (note that $\overline{AO} \perp \overline{ED'F}$). $\square$

Remark. This solution does not even use the fact that X, Y, Z were the midpoints of the altitudes!

4 Marking schemes

§4.1 Marking scheme for problem 1

Most solutions are worth 0 or 7. The following partial items are available but *not additive*:

- 5 points for proving if AC, DO, EF are concurrent implies $AC \perp BD$ but not finishing.
- 4 points for solutions for which spiral similarity justification is entirely absent, but which would be complete if these details were supplied correctly.
- 1 point for proving that $AC \perp BD$ is equivalent to the problem.

There is no deduction for configuration issues (Such as not using directed angles) or small typos in angle chasing.

No points awarded for noting $\angle ABC = \angle EHF$, or proving/noting that D is the Miquel point of $AEFC$ but not making further progress on the problem.

Computational approaches which are not completed are judged by any geometric content and do not earn other marks.

§4.2 Marking scheme for problem 2

For solutions which are not complete, the following items are available but *not additive*:

- 0 points for the correct answer.
- 1 point for proving $\theta(1)$ divides $\theta(P)$ over $\mathbb{Q}[x]$
- 1 point for showing $\theta(x)$, $\theta(x^2)$, ... all have a common integer root, or that every pair does.
- 1 point for showing $\theta(x)/\theta(1)$ is a linear polynomial with rational coefficients.
- 2 points are awarded for a solution that starts to make progress on $\theta(x^n)$ for $n \geq 2$, by proving some main lemma or claim. Showing that $\theta(x)$ is a linear multiple of $\theta(1)$ does not earn this point.

  (A common wrong approach is to claim that the rational function $\frac{\theta(x^n)}{\theta(x^{n-1})}$ takes on every integer value; this does not work since $\theta(x^n)$ and $\theta(x^{n-1})$ could have a common root for $n \geq 2$.)

For solutions which are complete with errors, the following deductions apply, and all deductions are additive:

- -1 point for an incorrect answer. This may include forgetting the ± 1, for example.
- -1 point for a minor error. This most commonly applies to students who took the quotient of two integer polynomials and assumed the coefficients were integers when in fact they could be rational numbers, but the solution can be easily patched once this is pointed out.
- -2 points for a more significant error that is easily fixable.

§4.3 Marking scheme for problem 3

- 0 points for the correct answer
- 6 points for a solution that uses a set of grid cells which do not form a polygon, but is otherwise correct (this includes sets which are not connected)
- 7 points for a correct solution

Any partial credit is done on case-by-case basis.

§4.4 Marking scheme for problem 4

For solutions which are not complete, the following items apply but are not additive.

- 0 points for no progress.
- 0 points for considering $n = cp(p-1)$ and nothing else.
- 0 points for partial steps in computing this sum, e.g. using primitive roots or the geometric series but not tying it together.
- 1 point for showing the sum when $n = cp(p-1)$ is c times the sum when $n = p(p-1)$, but not evaluating the latter sum.
- 1 point for evaluating the sum when $n = p(p-1)$ but not finishing the problem.

For essentially complete solutions, the following deductions could apply and are additive:

- -1 point: the student shows how to solve the problem for any negative number in place of 2020, but doesn't realize you can wrap around.
- -1 point: the student calculates the sum at $n = p(p-1)$ and from this assumes that the sum of any n consecutive terms is $-1 \bmod p$.
- -1 point: the student states with no proof that the sum of x^k as x varies across a residue system mod p is -1 if $p-1$ divides k and 0 otherwise. (Mentioning primitive roots is OK.)
- -1 point: calculation error leading to wrong final answer

Do not deduct for the following errors:

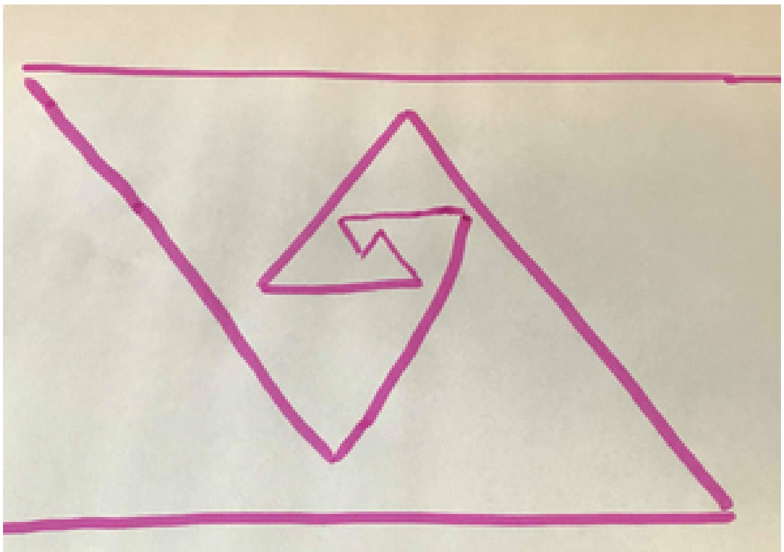
- The student forgets things like $0^{p-1} = 0$ instead of 1, as long as the errors do not change the final result.
- The student uses negative exponents on multiples of p, as long as the solution would be correct if the negative powers were replaced with "corresponding" positive powers

§4.5 Marking scheme for problem 5

- 0 points for an incorrect solution.
- 0 points for a 1-dimensional tracker argument, to prove e.g. the problem with two colors instead of three.
- 1 point for making a broken-line polygon (with any set of angles) and not finishing. This includes both:
  - Making a polygon using 1, ω, ω^2 vectors and not attempting the angle-sum argument.
  - Making a polygon with a different set of vectors summing to 0, such that finishing with the angle-sum argument is hard.
- 6 points for a correct solution with a minor error.

  The most common form of this will likely be the official solution with a different set of vectors, with a mistake in the resulting algebra. The mistake must be easily fixable.
- 7 points for a correct solution.

A number of solutions make a polygon with 1, ω, ω^2 vectors and then try to "smooth away" 300 degree angles. For this to be graded 7^-, the smoothing argument must be *very explicit*. It should be able to handle extremely concave shapes, like the one below. Otherwise, just award the 1 point for considering the broken-line polygon.



§4.6 Marking scheme for problem 6

As this problem is difficult, there are not many correct solutions, so we may manually flag any unusual cases to review as a group. Hence, this rubric is very sparse on details, outlining only the common 0^+ or 7^- cases.

The following partial items are not additive.

- 0 points for proving just the concurrence of the first three perpendiculars: this is a known follows from the fact that ABC and PQR are orthologic

- 0 points for H, Y, Z, D cyclic
- 1 point for proving $A'B'C'D'E'F'$ are collinear; no points for just conjecturing this.
- 1 point for realizing that $A'D'$ is the radical axis of the two relevant circles, even without proof.
- 1 point for realizing that the concurrence point is the reflection of H across the center of (XYZ), even without proof.
- 2 points for noticing $D'QF'$ and ACO are orthologic (or analogous).

5 Statistics

A large number of students started the contest but submitted no files. This skews the statistics a lot, but there isn't a real way for me to discard them without losing some legitimate zeros as well. Thus the difficulty of the competition is somewhat exaggerated.

§5.1 Summary of Scores

N	239	1st Q	0	Max	33
μ	6.36	Median	2	Top 3	29
σ	7.49	3rd Q	14	Top 10	21

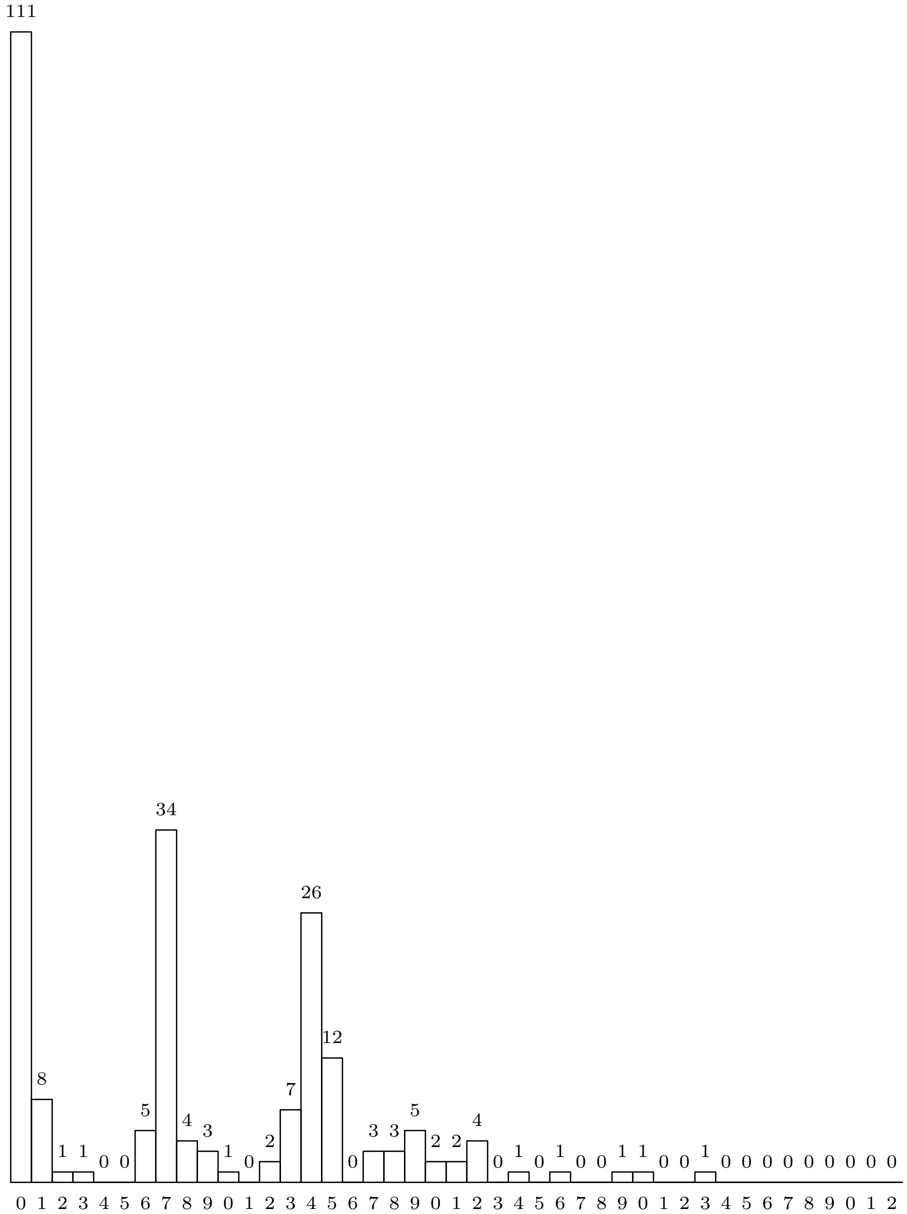
§5.2 Problem Statistics

	P1	P2	P3	P4	P5	P6
0	132	202	239	140	217	233
1	11	20	0	5	7	2
2	0	2	0	3	0	0
3	0	1	0	0	1	0
4	2	8	0	0	0	0
5	1	3	0	2	1	1
6	0	3	0	24	0	0
7	93	0	0	65	13	3
Avg	2.82	0.38	0.00	2.59	0.44	0.12

§5.3 Rankings

Sc	Num	Cu	Per	Sc	Num	Cu	Per	Sc	Num	Cu	Per
42	0	0	0.00%	28	0	3	1.26%	14	26	62	25.94%
41	0	0	0.00%	27	0	3	1.26%	13	7	69	28.87%
40	0	0	0.00%	26	1	4	1.67%	12	2	71	29.71%
39	0	0	0.00%	25	0	4	1.67%	11	0	71	29.71%
38	0	0	0.00%	24	1	5	2.09%	10	1	72	30.13%
37	0	0	0.00%	23	0	5	2.09%	9	3	75	31.38%
36	0	0	0.00%	22	4	9	3.77%	8	4	79	33.05%
35	0	0	0.00%	21	2	11	4.60%	7	34	113	47.28%
34	0	0	0.00%	20	2	13	5.44%	6	5	118	49.37%
33	1	1	0.42%	19	5	18	7.53%	5	0	118	49.37%
32	0	1	0.42%	18	3	21	8.79%	4	0	118	49.37%
31	0	1	0.42%	17	3	24	10.04%	3	1	119	49.79%
30	1	2	0.84%	16	0	24	10.04%	2	1	120	50.21%
29	1	3	1.26%	15	12	36	15.06%	1	8	128	53.56%
								0	111	239	100.00%

§5.4 Histogram



§5.5 Full stats

Rank	P1	P2	P3	P4	P5	P6	Σ
1.	7	5	0	7	7	7	33
2.	7	4	0	7	7	5	30
3.	7	1	0	7	7	7	29
4.	7	4	0	7	7	1	26
5.	7	4	0	6	0	7	24
6.	7	1	0	7	7	-	22
6.	7	1	-	7	7	0	22
6.	7	1	-	7	7	-	22
6.	7	1	-	7	7	-	22
10.	7	0	-	7	7	-	21
10.	7	6	0	7	1	0	21
12.	7	6	0	7	0	0	20
12.	7	6	0	7	0	0	20
14.	7	0	-	5	7	0	19
14.	7	1	-	6	5	-	19
14.	7	4	-	7	1	-	19
14.	7	5	0	7	0	-	19
14.	7	5	-	7	0	-	19
19.	7	4	0	6	1	0	18
19.	7	4	-	7	0	0	18
19.	7	4	-	7	0	-	18
22.	7	1	0	6	3	0	17
22.	7	3	0	7	0	0	17
22.	7	4	-	6	0	-	17
25.	1	-	-	7	7	-	15
25.	7	0	-	7	1	0	15
25.	7	0	-	7	-	1	15
25.	7	1	0	7	0	0	15
25.	7	1	0	7	0	-	15
25.	7	1	0	7	-	0	15
25.	7	1	0	7	-	-	15
25.	7	1	-	7	0	0	15
25.	7	1	-	7	0	-	15
25.	7	1	-	7	-	0	15
25.	7	1	-	7	-	-	15
25.	7	2	-	6	0	-	15
37.	0	-	-	7	7	0	14
37.	7	0	0	6	1	-	14
37.	7	0	0	7	0	0	14
37.	7	0	0	7	0	0	14
37.	7	0	0	7	0	0	14
37.	7	0	0	7	0	0	14
37.	7	0	0	7	0	0	14
37.	7	0	0	7	0	0	14
37.	7	0	0	7	0	0	14
37.	7	0	0	7	0	-	14
37.	7	0	0	7	-	0	14

Rank	P1	P2	P3	P4	P5	P6	Σ
37.	7	0	0	7	-	-	14
37.	7	0	0	7	-	-	14
37.	7	0	-	7	0	0	14
37.	7	0	-	7	0	0	14
37.	7	0	-	7	0	0	14
37.	7	0	-	7	0	-	14
37.	7	0	-	7	-	-	14
37.	7	0	-	7	-	-	14
37.	7	0	-	7	-	-	14
37.	7	0	-	7	-	-	14
37.	7	1	0	6	0	0	14
37.	7	1	0	6	-	0	14
37.	7	1	-	6	-	0	14
37.	7	-	-	7	-	-	14
37.	7	-	-	7	-	-	14
63.	0	0	-	6	7	-	13
63.	7	0	0	6	0	0	13
63.	7	0	0	6	-	-	13
63.	7	0	0	6	-	-	13
63.	7	0	-	6	0	0	13
63.	7	-	-	6	0	0	13
63.	7	-	-	6	-	-	13
70.	5	0	0	7	0	0	12
70.	7	0	-	5	-	-	12
72.	4	0	-	6	0	-	10
73.	7	0	-	2	-	0	9
73.	7	0	-	2	-	-	9
73.	7	1	-	1	-	-	9
76.	1	1	-	6	0	-	8
76.	1	-	-	7	-	-	8
76.	7	0	0	0	1	-	8
76.	7	0	-	1	-	-	8
80.	0	0	0	7	0	0	7
80.	0	0	0	7	0	-	7
80.	0	0	0	7	0	-	7
80.	0	0	-	7	-	-	7
80.	0	0	-	7	-	-	7
80.	1	0	-	6	0	-	7
80.	7	0	0	0	0	0	7
80.	7	0	0	0	0	0	7
80.	7	0	0	0	-	0	7
80.	7	0	0	0	-	-	7
80.	7	0	0	0	-	-	7
80.	7	0	0	-	-	0	7
80.	7	0	0	-	-	-	7
80.	7	0	0	-	-	-	7
80.	7	0	0	-	-	-	7
80.	7	0	-	0	0	0	7
80.	7	0	-	0	0	-	7

Rank	P1	P2	P3	P4	P5	P6	Σ
80.	7	0	-	0	-	0	7
80.	7	0	-	0	-	-	7
80.	7	0	-	0	-	-	7
80.	7	0	-	-	0	-	7
80.	7	0	-	-	-	-	7
80.	7	0	-	-	-	-	7
80.	7	-	-	0	-	-	7
80.	7	-	-	0	-	-	7
80.	7	-	-	-	0	0	7
80.	7	-	-	-	-	-	7
80.	-	0	-	7	-	-	7
80.	-	-	-	7	0	-	7
80.	-	-	-	7	-	-	7
80.	-	-	-	7	-	-	7
80.	-	-	-	7	-	-	7
80.	-	-	-	7	-	-	7
80.	-	-	-	7	-	-	7
114.	0	0	0	6	-	-	6
114.	0	-	-	6	-	-	6
114.	4	2	-	-	-	-	6
114.	-	0	-	6	0	-	6
114.	-	-	-	6	-	-	6
119.	1	0	-	1	1	-	3
120.	-	-	-	2	-	-	2
121.	1	0	0	0	0	0	1
121.	1	0	0	0	0	0	1
121.	1	-	0	-	-	-	1
121.	1	-	-	-	-	-	1
121.	1	-	-	-	-	-	1
121.	1	-	-	-	-	-	1
121.	-	0	-	1	-	-	1
121.	-	-	-	1	-	-	1
129.	0	0	0	0	0	0	0
129.	0	0	0	0	0	0	0
129.	0	0	0	0	0	0	0
129.	0	0	0	0	0	0	0
129.	0	0	0	0	-	0	0
129.	0	0	0	0	-	-	0
129.	0	0	0	-	-	-	0
129.	0	0	0	-	-	-	0
129.	0	0	0	-	-	-	0
129.	0	0	-	0	0	0	0
129.	0	0	-	0	0	0	0
129.	0	0	-	0	-	-	0
129.	0	0	-	-	-	-	0
129.	0	0	-	-	-	-	0
129.	0	-	-	0	0	0	0
129.	0	-	-	-	-	-	0
129.	0	-	-	-	-	-	0

Rank	P1	P2	P3	P4	P5	P6	Σ
129.	0	-	-	-	-	-	0
129.	-	0	0	0	0	-	0
129.	-	0	-	0	-	-	0
129.	-	0	-	-	-	-	0
129.	-	0	-	-	-	-	0
129.	-	0	-	-	-	-	0
129.	-	-	-	0	0	-	0
129.	-	-	-	0	-	0	0
129.	-	-	-	0	-	0	0
129.	-	-	-	0	-	-	0
129.	-	-	-	0	-	-	0
129.	-	-	-	-	-	-	0
129.	-	-	-	-	-	-	0
129.	-	-	-	-	-	-	0
129.	-	-	-	-	-	-	0
129.	-	-	-	-	-	-	0
129.	-	-	-	-	-	-	0
129.	-	-	-	-	-	-	0
129.	-	-	-	-	-	-	0
129.	-	-	-	-	-	-	0
129.	-	-	-	-	-	-	0
129.	-	-	-	-	-	-	0
129.	-	-	-	-	-	-	0
129.	-	-	-	-	-	-	0
129.	-	-	-	-	-	-	0
129.	-	-	-	-	-	-	0
129.	-	-	-	-	-	-	0
129.	-	-	-	-	-	-	0
129.	-	-	-	-	-	-	0
129.	-	-	-	-	-	-	0
129.	-	-	-	-	-	-	0
129.	-	-	-	-	-	-	0
129.	-	-	-	-	-	-	0
129.	-	-	-	-	-	-	0
129.	-	-	-	-	-	-	0
129.	-	-	-	-	-	-	0
129.	-	-	-	-	-	-	0
129.	-	-	-	-	-	-	0
129.	-	-	-	-	-	-	0
129.	-	-	-	-	-	-	0
129.	-	-	-	-	-	-	0
129.	-	-	-	-	-	-	0
129.	-	-	-	-	-	-	0
129.	-	-	-	-	-	-	0
129.	-	-	-	-	-	-	0
129.	-	-	-	-	-	-	0
129.	-	-	-	-	-	-	0
129.	-	-	-	-	-	-	0
129.	-	-	-	-	-	-	0
129.	-	-	-	-	-	-	0

Rank	P1	P2	P3	P4	P5	P6	Σ
129.	-	-	-	-	-	-	0
129.	-	-	-	-	-	-	0
129.	-	-	-	-	-	-	0
129.	-	-	-	-	-	-	0
129.	-	-	-	-	-	-	0
129.	-	-	-	-	-	-	0
129.	-	-	-	-	-	-	0
129.	-	-	-	-	-	-	0
129.	-	-	-	-	-	-	0
129.	-	-	-	-	-	-	0
129.	-	-	-	-	-	-	0
129.	-	-	-	-	-	-	0
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