USEMO
Ersatz Math Olympiad

Language: **English**

Day: **1**

Saturday, October 26, 2024

Problem 1. There are 1001 stacks of coins $S_1, S_2, \ldots, S_{1001}$. Initially, stack S_k has k coins for each $k = 1, 2, \ldots, 1001$. In an operation, one selects an ordered pair (i, j) of indices i and j satisfying $1 \le i < j \le 1001$ subject to two conditions:

- The stacks S_i and S_j must each have at least one coin.
- The ordered pair (i, j) must *not* have been selected in any previous operation.

Then, if S_i and S_j have a coins and b coins, respectively, one removes $\gcd(a, b)$ coins from each stack.

What is the maximum number of times this operation could be performed?

Problem 2. Let k be a fixed positive integer. For each integer $1 \le i \le 4$, let x_i and y_i be positive integers such that their least common multiple is k. Suppose that the four points (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4) are the vertices of a non-degenerate rectangle in the Cartesian plane. Prove that $x_1x_2x_3x_4$ is a perfect square.

Problem 3. Let ABC be an acute triangle with incenter I. Two distinct points P and Q are chosen on the circumcircle of ABC such that

$$\angle API = \angle AQI = 45^\circ.$$

Lines PQ and BC meet at S. Let H denote the foot of the altitude from A to BC. Prove that $\angle AHI = \angle ISH$.

The organizers of the USEMO are grateful to be sponsored this year by the CoRe Lab, Institute of Artificial Intelligence, Peking University.

Time limit: 4 hours and 30 minutes.
Each problem is worth seven points.

USEMO
Ersatz Math Olympiad

Language: **English**

Day: **2**

Sunday, October 27, 2024

Problem 4. Find all sequences a_1, a_2, ... of nonnegative integers such that for all positive integers n, the polynomial

$$1 + x^{a_1} + x^{a_2} + \cdots + x^{a_n}$$

has at least one integer root. (Here $x^0 = 1$.)

Problem 5. Let ABC be a scalene triangle whose incircle is tangent to BC, CA, AB at D, E, F respectively. Lines BE and CF meet at G. Prove that there exists a point X on the circumcircle of triangle EFG such that the circumcircles of triangles BCX and EFG are tangent, and

$$\angle BGC = \angle BXC + \angle EDF.$$

Problem 6. Let n be an odd positive integer and consider an $n \times n$ chessboard of n^2 unit squares. In some of the cells of the chessboard, we place a knight. A knight in a cell c is said to *attack* a cell c' if the distance between the centers of c and c' is exactly $\sqrt{5}$ (in particular, a knight does not attack the cell which it occupies).

Suppose each cell of the board is attacked by an even number of knights (possibly zero). Show that the configuration of knights is symmetric with respect to all four axes of symmetry of the board (i.e. the configuration of knights is both horizontally and vertically symmetric, and also unchanged by reflection along either diagonal of the chessboard).

The organizers of the USEMO are grateful to be sponsored this year by the CoRe Lab, Institute of Artificial Intelligence, Peking University.

Time limit: 4 hours and 30 minutes.
Each problem is worth seven points.