USEMO
Ersatz Math Olympiad

Language: **English**

Day: **1**

October 21, 2023

Problem 1. A positive integer n is called *beautiful* if, for every integer $4 \le b \le 10000$, the base-b representation of n contains the consecutive digits 2, 0, 2, 3 (in this order, from left to right). Determine whether the set of all beautiful integers is finite.

Problem 2. Each point in the plane is labeled with a real number. Show that there exist two distinct points P and Q whose labels differ by less than the distance from P to Q.

Problem 3. Canmoo is trying to do constructions, but doesn't have a ruler or compass. Instead, Canmoo has a device that, given four distinct points A, B, C, P in the plane, will mark the isogonal conjugate of P with respect to triangle ABC, if it exists. Show that if two points are marked on the plane, then Canmoo can construct their midpoint using this device, a pencil for marking additional points, and no other tools.

(Recall that the *isogonal conjugate* of P with respect to triangle ABC is the point Q such that lines AP and AQ are reflections around the bisector of $\angle BAC$, lines BP and BQ are reflections around the bisector of $\angle CBA$, lines CP and CQ are reflections around the bisector of $\angle ACB$. Additional points marked by the pencil can be assumed to be in general position, meaning they don't lie on any line through two existing points or any circle through three existing points.)

Time limit: 4 hours and 30 minutes.
Each problem is worth seven points.

October 22, 2023

Problem 4. Let ABC be an acute triangle with orthocenter H. Points A_1, B_1, C_1 are chosen in the interiors of sides BC, CA, AB, respectively, such that $\triangle A_1B_1C_1$ has orthocenter H. Define $A_2 = \overline{AH} \cap \overline{B_1C_1}$, $B_2 = \overline{BH} \cap \overline{C_1A_1}$, and $C_2 = \overline{CH} \cap \overline{A_1B_1}$.

Prove that triangle $A_2B_2C_2$ has orthocenter H.

Problem 5. Let $n \geq 2$ be an integer. A cube of size $n \times n \times n$ is dissected into n^3 unit cubes. A nonzero real number is written at the center of each unit cube so that the sum of the n^2 numbers in each slab of size $1 \times n \times n$, $n \times 1 \times n$, or $n \times n \times 1$ equals zero. (There are a total of $3n$ such slabs, forming three groups of n slabs each such that slabs in the same group are parallel and slabs in different groups are perpendicular.)

Could it happen that some plane in three-dimensional space separates the positive and the negative written numbers? (The plane should not pass through any of the numbers.)

Problem 6. Let $n \geq 2$ be a fixed integer.

(a) Determine the largest positive integer m (in terms of n) such that there exist complex numbers $r_1, \dots, r_n$, not all zero, for which

$$\prod_{k=1}^{n}(r_k + 1) = \prod_{k=1}^{n}(r_k^2 + 1) = \dots = \prod_{k=1}^{n}(r_k^m + 1) = 1.$$

(b) For this value of m, find all possible values of

$$\prod_{k=1}^{n}(r_k^{m+1} + 1).$$

Time limit: 4 hours and 30 minutes.
Each problem is worth seven points.