USEMO
Ersatz Math Olympiad

Language: **English**

Day: **1**

*Saturday, October 22, 2022*

**Problem 1.** A *stick* is defined as a $1 \times k$ or $k \times 1$ rectangle for any integer $k \geq 1$. We wish to partition the cells of a $2022 \times 2022$ chessboard into $m$ non-overlapping sticks, such that any two of these $m$ sticks share at most one unit of perimeter. Determine the smallest $m$ for which this is possible.

**Problem 2.** A function $\psi \colon \mathbb{Z} \to \mathbb{Z}$ is said to be *zero-requiem* if for any positive integer $n$ and any integers $a_1, \ldots, a_n$ (not necessarily distinct), the sums $a_1 + a_2 + \cdots + a_n$ and $\psi(a_1) + \psi(a_2) + \cdots + \psi(a_n)$ are not both zero.

Let $f$ and $g$ be two zero-requiem functions for which $f \circ g$ and $g \circ f$ are both the identity function (that is, $f$ and $g$ are mutually inverse bijections). Given that $f + g$ is *not* a zero-requiem function, prove that $f \circ f$ and $g \circ g$ are both zero-requiem.*

**Problem 3.** Point $P$ lies in the interior of a triangle $ABC$. Lines $AP$, $BP$, and $CP$ meet the opposite sides of triangle $ABC$ at points $A'$, $B'$, and $C'$, respectively. Let $P_A$ be the midpoint of the segment joining the incenters of triangles $BPC'$ and $CPB'$, and define points $P_B$ and $P_C$ analogously. Show that if

$$AB' + BC' + CA' = AC' + BA' + CB',$$

then points $P$, $P_A$, $P_B$, and $P_C$ are concyclic.

*Recall that if $\psi_1$ and $\psi_2$ are functions from $\mathbb{Z}$ to $\mathbb{Z}$, then the composition $\psi_1 \circ \psi_2$ is defined as the function sending each $x \in \mathbb{Z}$ to $\psi_1(\psi_2(x))$, while the sum $\psi_1 + \psi_2$ is defined as the function sending each $x \in \mathbb{Z}$ to $\psi_1(x) + \psi_2(x)$.

*Time limit: 4 hours and 30 minutes.*
*Each problem is worth seven points.*

*Sunday, October 23, 2022*

**Problem 4.** Let $ABCD$ be a cyclic quadrilateral whose opposite sides are not parallel. Suppose points $P$, $Q$, $R$, $S$ lie in the interiors of segments $AB$, $BC$, $CD$, $DA$, respectively, such that

$$\angle PDA = \angle PCB, \quad \angle QAB = \angle QDC, \quad \angle RBC = \angle RAD, \quad \text{and} \quad \angle SCD = \angle SBA.$$

Let $\overline{AQ}$ intersect $\overline{BS}$ at $X$, and $\overline{DQ}$ intersect $\overline{CS}$ at $Y$. Prove that lines $\overline{PR}$ and $\overline{XY}$ are either parallel or coincide.

**Problem 5.** Let $\tau(n)$ denote the number of positive integer divisors of a positive integer $n$ (for example, $\tau(2022) = 8$). Given a polynomial $P(X)$ with integer coefficients, we define a sequence $a_1, a_2, \dots$ of nonnegative integers by setting

$$a_n = \begin{cases} \gcd\left(P(n),\ \tau(P(n))\right) & \text{if } P(n) > 0 \\ 0 & \text{if } P(n) \le 0 \end{cases}$$

for each positive integer $n$. We then say the sequence *has limit infinity* if every integer occurs in this sequence only finitely many times (possibly not at all).

Does there exist a choice of $P(X)$ for which the sequence $a_1, a_2, \dots$ has limit infinity?

**Problem 6.** Find all positive integers $k$ for which there exists a nonlinear[†] function $f\colon \mathbb{Z} \to \mathbb{Z}$ such that the equation

$$f(a) + f(b) + f(c) = \frac{f(a-b) + f(b-c) + f(c-a)}{k}$$

holds for any integers $a$, $b$, $c$ satisfying $a + b + c = 0$ (not necessarily distinct).

[†]We say $f\colon \mathbb{Z} \to \mathbb{Z}$ is nonlinear if $f(x) \neq (f(1) - f(0))x + f(0)$ for some $x \in \mathbb{Z}$.

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