

Day: 1

Saturday, October 22, 2022

**Problem 1.** A *stick* is defined as a  $1 \times k$  or  $k \times 1$  rectangle for any integer  $k \ge 1$ . We wish to partition the cells of a  $2022 \times 2022$  chessboard into m non-overlapping sticks, such that any two of these m sticks share at most one unit of perimeter. Determine the smallest m for which this is possible.

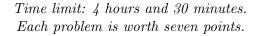
**Problem 2.** A function  $\psi: \mathbb{Z} \to \mathbb{Z}$  is said to be *zero-requiem* if for any positive integer n and any integers  $a_1, \ldots, a_n$  (not necessarily distinct), the sums  $a_1 + a_2 + \cdots + a_n$  and  $\psi(a_1) + \psi(a_2) + \cdots + \psi(a_n)$  are not both zero.

Let f and g be two zero-requiem functions for which  $f \circ g$  and  $g \circ f$  are both the identity function (that is, f and g are mutually inverse bijections). Given that f + g is not a zero-requiem function, prove that  $f \circ f$  and  $g \circ g$  are both zero-requiem.<sup>\*</sup>

**Problem 3.** Point P lies in the interior of a triangle ABC. Lines AP, BP, and CP meet the opposite sides of triangle ABC at points A', B', and C', respectively. Let  $P_A$  be the midpoint of the segment joining the incenters of triangles BPC' and CPB', and define points  $P_B$  and  $P_C$  analogously. Show that if

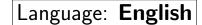
$$AB' + BC' + CA' = AC' + BA' + CB',$$

then points P,  $P_A$ ,  $P_B$ , and  $P_C$  are concyclic.





<sup>\*</sup>Recall that if  $\psi_1$  and  $\psi_2$  are functions from  $\mathbb{Z}$  to  $\mathbb{Z}$ , then the composition  $\psi_1 \circ \psi_2$  is defined as the function sending each  $x \in \mathbb{Z}$  to  $\psi_1(\psi_2(x))$ , while the sum  $\psi_1 + \psi_2$  is defined as the function sending each  $x \in \mathbb{Z}$  to  $\psi_1(x) + \psi_2(x)$ .



Day: 2

Sunday, October 23, 2022

**Problem 4.** Let ABCD be a cyclic quadrilateral whose opposite sides are not parallel. Suppose points P, Q, R, S lie in the interiors of segments AB, BC, CD, DA, respectively, such that

$$\angle PDA = \angle PCB, \quad \angle QAB = \angle QDC, \quad \angle RBC = \angle RAD, \text{ and } \angle SCD = \angle SBA.$$

Let  $\overline{AQ}$  intersect  $\overline{BS}$  at X, and  $\overline{DQ}$  intersect  $\overline{CS}$  at Y. Prove that lines  $\overline{PR}$  and  $\overline{XY}$  are either parallel or coincide.

**Problem 5.** Let  $\tau(n)$  denote the number of positive integer divisors of a positive integer n (for example,  $\tau(2022) = 8$ ). Given a polynomial P(X) with integer coefficients, we define a sequence  $a_1, a_2, \ldots$  of nonnegative integers by setting

$$a_n = \begin{cases} \gcd\left(P(n), \ \tau(P(n))\right) & \text{if } P(n) > 0\\ 0 & \text{if } P(n) \le 0 \end{cases}$$

for each positive integer n. We then say the sequence has limit infinity if every integer occurs in this sequence only finitely many times (possibly not at all).

Does there exist a choice of P(X) for which the sequence  $a_1, a_2, \ldots$  has limit infinity?

**Problem 6.** Find all positive integers k for which there exists a nonlinear<sup>†</sup> function  $f: \mathbb{Z} \to \mathbb{Z}$  such that the equation

$$f(a) + f(b) + f(c) = \frac{f(a-b) + f(b-c) + f(c-a)}{k}$$

holds for any integers a, b, c satisfying a + b + c = 0 (not necessarily distinct).

<sup>†</sup>We say  $f: \mathbb{Z} \to \mathbb{Z}$  is nonlinear if  $f(x) \neq (f(1) - f(0))x + f(0)$  for some  $x \in \mathbb{Z}$ .

Time limit: 4 hours and 30 minutes. Each problem is worth seven points.

